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ELEMENTARY MECHANICS

WITHOUT HYDROSTATICS

By D. HUMPHREY, B.A., B.Sc.

REVISION MATHEMATICS FOR SCHOOL CERTIFICATE

INTERMEDIATE MECHANICS

Vol. I. Dynamics

Vol. II. Statics and Hydrostatics

By D. HUMPHREY, B.A., B.Sc., and E. A. BAGGOTT, M.Sc., Ph.D.

ELEMENTARY MECHANICS
WITHOUT HYDROSTATICS

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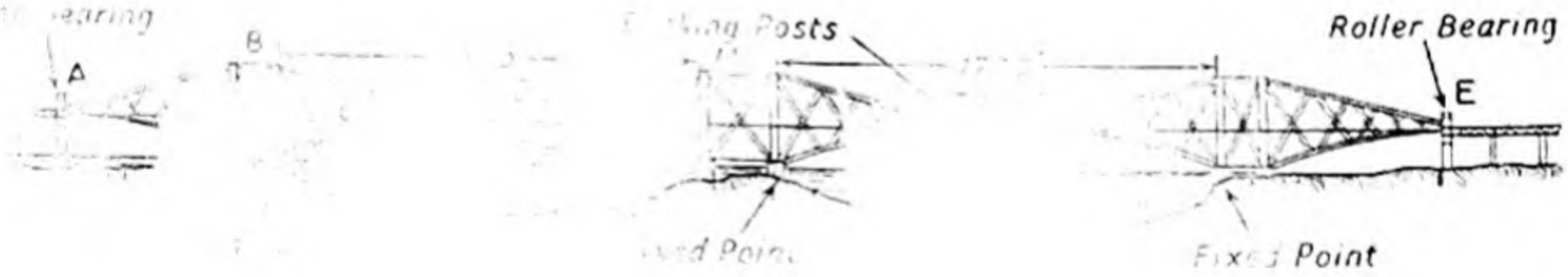
By D. HUMPHREY, B.A., B.Sc., and J. TOPPING, M.Sc., Ph.D., D.I.C.

A SHORTER INTERMEDIATE MECHANICS

2944

[*Fox Photos*]



THE FORTH BRIDGE

The Forth Bridge, designed by Sir John Fowler and Sir Benjamin Baker, spans the Firth of Forth. It consists of three double cantilevers, as A B C, which support two spans each 675 feet long. Each cantilever, carrying a full load, exerts a pressure of 4,000 tons when loaded. To counterpoise this load the land arms at A and E are carried on rocking posts at C and D. The girders are carried on rocking posts at two places, so that any variation in the lengths of the girders and cantilevers may be made for change of temperature.

ELEMENTARY MECHANICS

WITHOUT HYDROSTATICS

BY

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PREFACE

THE object of this book is to provide an elementary course in Mechanics suitable for all types of science and engineering students. It is not written with a view to any particular examination, but the work covers all the requirements of the various School Certificates, and it is hoped that the book will prove equally useful as a first course to both Secondary and Technical students.

As the whole subject is based on the results of experiment and observation, this side of the work has been rather stressed, particularly in the earlier parts. Frequent references have been made to the practical applications of the subject.

The authors feel that if the distinction between "mass" and "weight" is appreciated, the choice between "absolute" and "gravitational" units of force is not of fundamental importance, and both systems are used. It is worthy of notice that in the examination papers for Associate Membership of two of the leading Engineering Institutions, "M" occurs quite as frequently as "W"/g.

A considerable number of worked examples and nearly one thousand to be worked by the student are included. From long experience the authors are of opinion that in this subject, more than in any other branch of mathematics, ability to apply principles to examples is of the utmost importance.

Acknowledgments are due to the Cambridge Local Examinations Syndicate, the Central Welsh Board, the Northern Universities Joint Matriculation Board, the Senate of the University of London, the Delegates of the Oxford Local Examination, the Oxford and Cambridge Schools Examinations Board and the Union of Lancashire and Cheshire Institutes for permission to include examples set in their examination.

D. HUMPHREY.
E. A. BAGGOTT.

March 1939.

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THE following abbreviations are used to denote the sources from which examples are taken :—

C.	Cambridge School Certificate.
C.W.B.	Central Welsh Board School Certificate.
G.S.	London University General School Certificate.
J.	Northern Universities Joint Matriculation Board.
L.M.	London Matriculation.
O.	Oxford School Certificate.
O.C.	Oxford and Cambridge School Certificate.

MECHANICS

CHAPTER I

INTRODUCTION

1. Introduction.

Many of the subjects which are considered in the following pages have familiar names. Force, pressure, speed, energy and work are a few of them. It will be found, however, that when these words are used more definite meanings must be given to them than is usual in everyday language. As information is gained about **force,** for example, it is necessary at a certain stage to describe the special meaning to be given to the word as accurately as possible. Such a description of the meaning of a word is called a **definition.** It will be appreciated that to give a definition of what we are to understand by mechanics at the present stage is difficult, but it is hoped that the following will give some idea of the science we are about to study.

Mechanics deals with bodies which are in states of rest or of motion and the forces acting upon them which produce or alter these states.

Three of the subdivisions of mechanics which are to be treated are :—

1. Dynamics, which deals with bodies in motion.
2. Statics, which deals with bodies at rest.
3. Hydrostatics, which deals with liquids at rest.

2. Laws in Mechanics.

Suppose the cost of a length of carpet is 10s. per yard, then

the law connecting the "cost" and "length" of carpet may be stated :

Cost of a length of carpet (in shillings) = length of carpet (in yards) × 10.

If a stone is dropped over the edge of a cliff and falls freely, *i.e.*, without striking other bodies, it will be shown later that we can calculate the distance through which it will fall in one second, three seconds, or, in fact, any given time. This arises from the fact that the distance through which the stone falls depends upon the length of time it has been falling. In mechanics we should say that there is a "law" connecting the distance fallen and the time taken by the stone to fall through that distance.

There is a very important difference between the laws of mechanics and laws such as that connecting cost and length of carpet, for it is only on condition that the resistance of the air produces a negligible effect upon the motion of the stone through it that we are able to state the law connecting distance fallen and time taken. In these days of aeroplanes, streamlined cars and locomotives, we are most of us aware of the fact that the resistance of the air to a body moving *rapidly* through it is far from negligible in its effect. Many of the laws of mechanics are not strictly true of ordinary bodies, for they often describe the behaviour of bodies which are supposed to possess very special properties, which in fact are not possessed by any ordinary body. We may, for example, find it convenient to suppose the bearings of a wheel to be perfectly smooth, *i.e.*, to offer no resistance to the turning of the wheel. When bodies are assumed to possess such special properties it must be understood that the laws which are being tested or discovered concerning these bodies are not affected by these assumptions. Nevertheless, these laws are useful because they describe fairly accurately what happens in practice. Thus if a stone falls freely for one second its motion is not appreciably affected by air resistance. Again, if we suppose a wheel to have well-lubricated ball-bearings, the resistance to the turning of the wheel may be so small as to be negligible in a particular experiment which is being performed with the wheel.

3. Experimental Work.

When testing any of the laws of mechanics, or in attempting to discover new laws, the following are a number of suggestions which, if carried out, will assist in obtaining satisfactory results from the experiments to be performed :—

(1) be thorough in your work, and on no account hurry through an experiment, for by so doing you may miss observing important details.

(2) Your apparatus is capable of giving a certain degree of accuracy which is, *in general*, that of the least accurate of the measures of, say, length or weight used in the experiment. Thus, if a length is at best correct to 1 part in 100 parts (*e.g.*, $0{\cdot}050 \pm 0{\cdot}0005$ cm. or $50 \pm 0{\cdot}5$ in.), results depending upon the use of this measure can at best be correct to 1 in 100 (see also § 8).

(3) Make a written note of all readings before proceeding to a new set or to a new experiment.

(4) When an experiment is to be repeated a number of times *make a table of the readings.*

4. The Measurement of Length.

The success of many experiments in mechanics depends to a considerable extent upon the accurate measurement of length. We shall therefore consider the units of length commonly used and the practical problem of measuring lengths by comparing them with scales subdivided into multiples or fractions of these units.

The Imperial Yard.—The British unit of length is the Imperial Yard, and is the distance between two lines on gold plugs fitted into a bronze bar when the temperature of the bar is 62° F. Other units of length such as mile, furlong, foot, inch, etc., are obtained by taking multiples or fractions of the Imperial Yard.

1 yard = 3 feet = 36 inches.
1,760 yards = 8 furlongs = 1 mile.

5. The Metric Unit of Length.

At the time of the French Revolution, French scientists decided to take as their unit of length one forty-millionth (1/40,000,000) part of a great circle of the earth which passes through the poles, *i.e.*, a circle of longitude. This length, called *one metre*, is the distance between two lines marked on a bar which is an alloy of platinum and iridium at the temperature of pure melting ice. It is unfortunate that more recent calculations of the length of a circle of longitude give a slightly different result from that used in fixing the length of 1 metre. Since it is out of the question that the unit of length should be altered whenever a more accurate estimate of the length of a great circle of longitude is made, the original unit is retained.

1 metre = 100 centimetres = 1,000 millimetres.
1 kilometre = 1,000 metres.

6. Relations between Metric and British Units.

1 inch = 2·540 centimetres.
1 yard = 0·9144 metres.
1 metre = 39·37 inches.
1 kilometre = 1,094 yards (½ mile = 1,100 yards)

7. The Error due to Parallax.

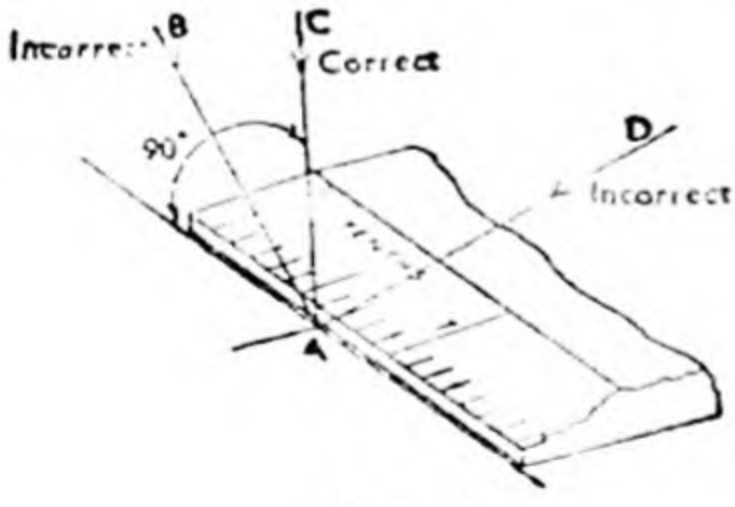

FIG. 1.

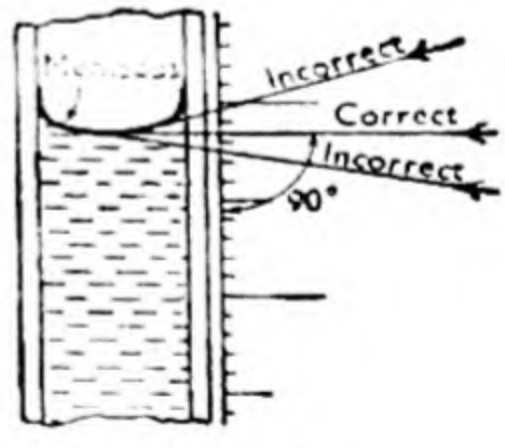

FIG. 2.

The two figures above illustrate the error due to parallax in the measurement of lengths, an error which is commonly made by beginners in experimental work. When the position of a

point on a line is being read on a scale it is essential that the line joining the eye to the point should be perpendicular to the edge of the scale. The left-hand figure shows clearly how great an error can be made if this precaution is not taken. A thick ruler has been drawn, and it should be clear that the closer the scale on the ruler to the line whose length is being measured the less chance will there be of making a large error. It is usually best to close one eye when reading the position of a point on a scale, but the figure illustrated on the right shows a case in which both eyes can be used. When the height of a vertical column of liquid is being measured the position of the lowest point of the meniscus should be noted, and for this purpose the eyes must be at the same level as the point. In the case of mercury the position of the top point of the meniscus should be noted.

8. Degree of Accuracy in Measurements.

When a number such as 20·4 is the number of units of length in a line, it is usually to be understood that the true length of the line is nearer to 20·4 units than to either 20·3 or 20·5 units. The number 20·4 is then correct to the third significant figure.

If the length of a line is 20·4 cm. correct to the third significant figure, its value to the same degree of accuracy may be expressed as 0·204 m., 204 mm., or 0·000204 km. The number of decimal places used in expressing the magnitude of a quantity is no indication of the accuracy with which the number represents the magnitude; it is the number of *significant figures* used which shows the accuracy. Thus,

3·5 mm., error less than 0·05 mm., is correct to an accuracy of the order 1 part in 70 parts.

4·95 in., error less than 0·005 in., is correct to an accuracy of the order 1 part in 990 parts.

235 ft., error less than 0·5 ft., is correct to an accuracy of the order 1 part in 470 parts.

0·00847 mile, error less than 0·000005 mile, is correct to an accuracy of the order 1 part in 1,694 parts.

It is a common saying that the strength of a chain is that of its weakest link; if a calculation involves numbers which are

the approximate measures of magnitudes such as length, area, volume, etc., the accuracy with which the calculation can usefully be carried out is that of the least accurate of the numbers. Thus, in an experiment it is well to know which measurement can be made with the lowest degree of accuracy, and then the remainder of the necessary measurements need only be made to at least this accuracy. In an attempt to find the volume of a rectangular block, suppose that the accuracy with which the thickness can be measured is about 1 part in 250 parts, *e.g.*, 1·25 in. If the length of the block is about 500 in. and the breadth about 20 in., what accuracy is necessary in these measurements ? The greatest error allowable in the thickness is 1 part in 250 parts, and this is at least maintained if the length is measured to the nearest inch and the breadth to the nearest tenth of an inch. Suppose the volume of the block is given by

$$V \text{ cu. in.} = 511 \times 22{\cdot}5 \times 1{\cdot}25 \text{ cu. in.} = 15{,}200 \text{ cu. in.}$$

The volume is stated correct to the third significant figure, and the accuracy of the result is 50 cu. in. in a total of 15,200 cu. in., *i.e.*, about 1 part in 300. It is useless to leave more significant figures in the answer simply because they have turned up during the multiplication, indeed to leave these additional figures in the result can only mislead anyone who reads it.

With practice the average person may expect to estimate length correct to the nearest tenth of one-tenth of an inch, *i.e.*, 1/100th in., or to the nearest 1/4 mm.

EXAMPLES I

1. If the earth is assumed to be a sphere and the correct radius is taken as 3,960 miles, calculate the error in the circumference (3·1416 × diameter) due to an error of +5 miles in the radius. Calculate also the fractional error.

2. The length of a rectangle is measured as 10·2 in. and the breadth as 14·6 in. If there is a possible error of ±0·05 in. in each of the measurements, calculate (correct to one decimal place) the greatest and least values which the area may have, and from these values find the greatest possible fractional error which may be made in calculating the area of the rectangle.

3. In measuring the diameter of a wire of uniform cross-section the error was ±0·005 cm. and the diameter was read as 0·12 cm. With what accuracy should the length of the wire (about 24 cm.) be measured ?

9. The Vernier Scale.

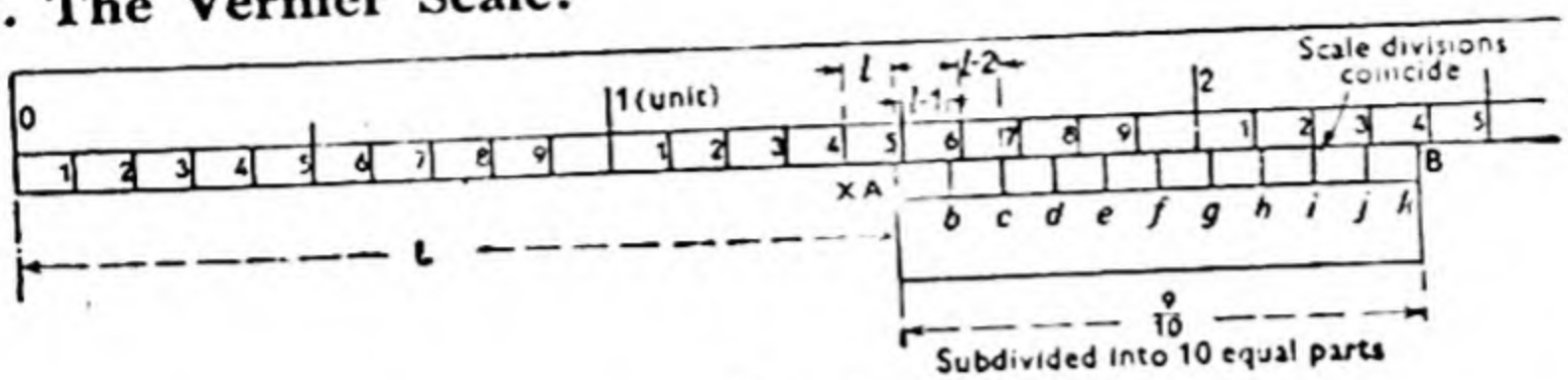

FIG. 3.

The vernier scale was invented by Pierre Vernier (1580-1637), a Spaniard. An example showing the use of the scale is illustrated above. The distance L is to be measured. It is evident that the required length lies between 1·4 and 1·5 units, and the fraction XA of one of the small divisions remains to be accurately determined. This is achieved by means of a subsidiary scale AB of length 9/10ths of 1 unit on the scale. AB is subdivided into 10 equal parts, each of which is therefore 9/100ths of 1 unit. The difference in length between a small division on the scale and a small division on the vernier AB is

$\left(\frac{1}{10} - \frac{9}{100}\right)$ units, *i.e.*, 1/100th unit, or 0·01 unit.

Suppose that A is l hundredths of a scale unit from X, the 1·4 scale division, then

Division b is $(l-1)$ hundredths of a scale unit from the 1·5 scale division.

Division c is $(l-2)$ hundredths of a scale unit from the 1·6 scale division.

Division i is $(l-8)$ hundredths of a scale unit from the 2·2 scale division.

But the division i coincides exactly with the 2·2 division, and therefore l is 8/100th units, and the length L is 1·48 units.

Reading the vernier therefore consists in first finding the pair of division lines on the two scales which coincide most nearly, then the number of vernier divisions which have been passed over to reach this point is the number of hundredths of a scale unit which should be added to the length of the line up to the scale mark X which precedes the zero on the vernier.

Vernier scales are used on a great variety of instruments—calipers, theodolites, barometers, etc. For any forward reading vernier suppose that n divisions of a vernier equal $(n-1)$ scale divisions, the difference between n divisions of the vernier and n scale divisions is 1 scale division, and the difference between 1 division of the vernier and 1 scale division is $1/n$th scale division.

In measuring the length of a line if m divisions of the vernier must be passed over before a vernier mark coincides with a scale mark, m/nths of a scale division is the fraction which should be added to the length of the part of the line up to the last scale mark which comes before the zero on the vernier scale.

10. Backward Reading Vernier Scales.

Sometimes vernier scales are backward reading. In this case the vernier division is greater than the scale division. Suppose that n vernier divisions are equal to $n+1$ scale divisions, then n vernier divisions exceed n scale divisions by 1 scale division, and 1 vernier division exceeds 1 scale division by $1/n$th scale division. With this type of vernier, the vernier scale marks get further and further ahead of the scale marks.

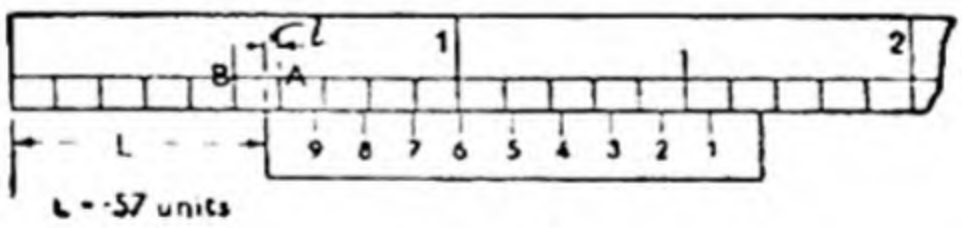

FIG. 4.

Thus, in the above figure, if the zero of the vernier is l/nths of a scale behind the scale mark A which follows it, the number of vernier divisions passed over before a vernier mark coincides with a scale mark is equal to l. This quantity must be subtracted from the reading of the scale up to the first mark A after the zero of the vernier, or, if the vernier is numbered backwards, the reading on the vernier should be added to the reading up to the scale mark B, which precedes the zero on the vernier.

EXAMPLES II

1. Find the length of the distance marked X on the figure in inches and fractions of an inch. Explain how you obtain your result.

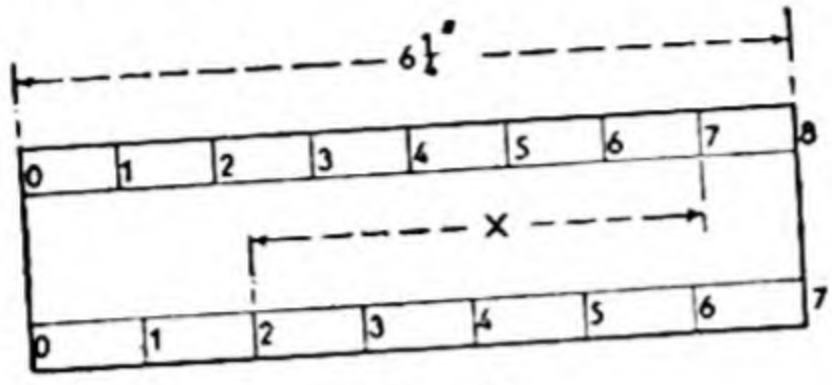

FIG. 5.

2. The following are two examples of vernier scales. What is the value of the length L in each case ?

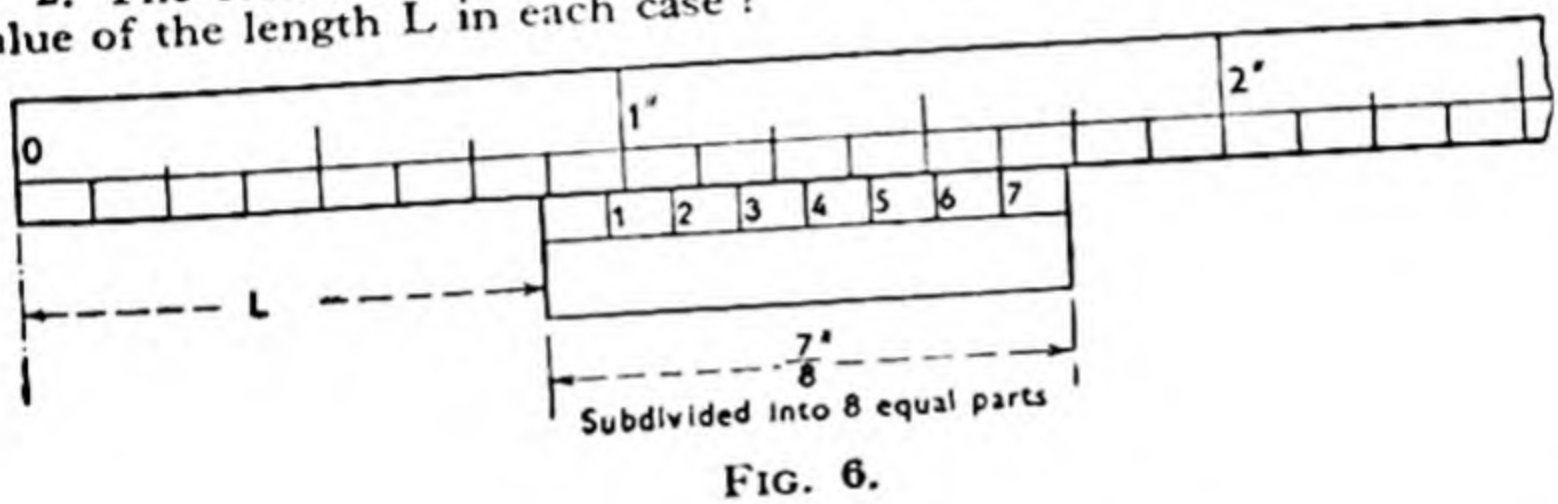

FIG. 6.

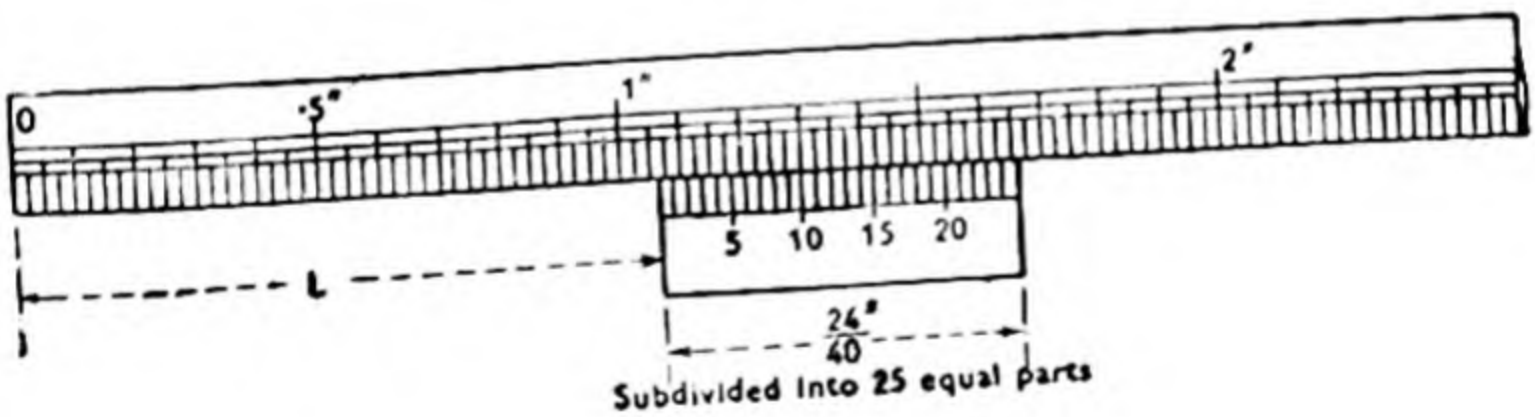

FIG. 7.

3. Obtain the length of the distance marked X correct to 1/100 in.

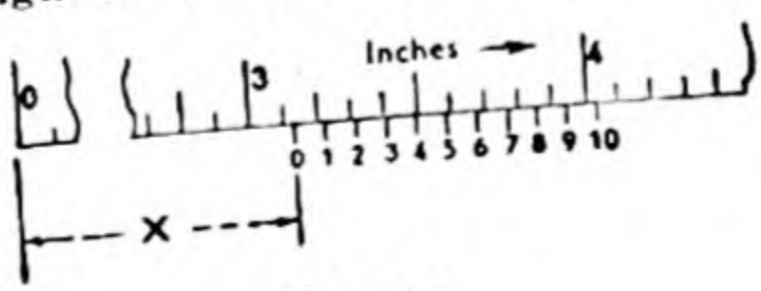

FIG. 7A.

4. The figure shows a vernier used to enable angles to be measured correct to the nearest minute. Describe how the vernier accomplishes this accuracy, and hence show that the reading shown is 20° 33′.

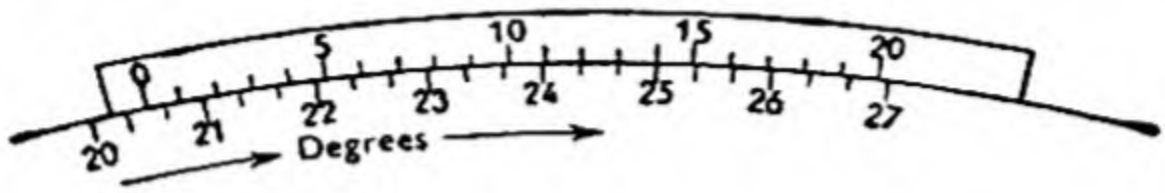

FIG. 8.

5. Construct a vernier scale suitable for use with a scale divided into inches and eighths of an inch to read to the nearest sixty-fourth of an inch.

6. Construct a vernier scale suitable for use with a circular scale on a theodolite on which the degrees are subdivided into 3 parts, so that an angle may be measured correct to the nearest minute.

7. Construct a backward reading vernier for use with a scale in inches and tenths of an inch, so that a length may be measured correct to one-hundredth of an inch.

11. Vernier Calipers.

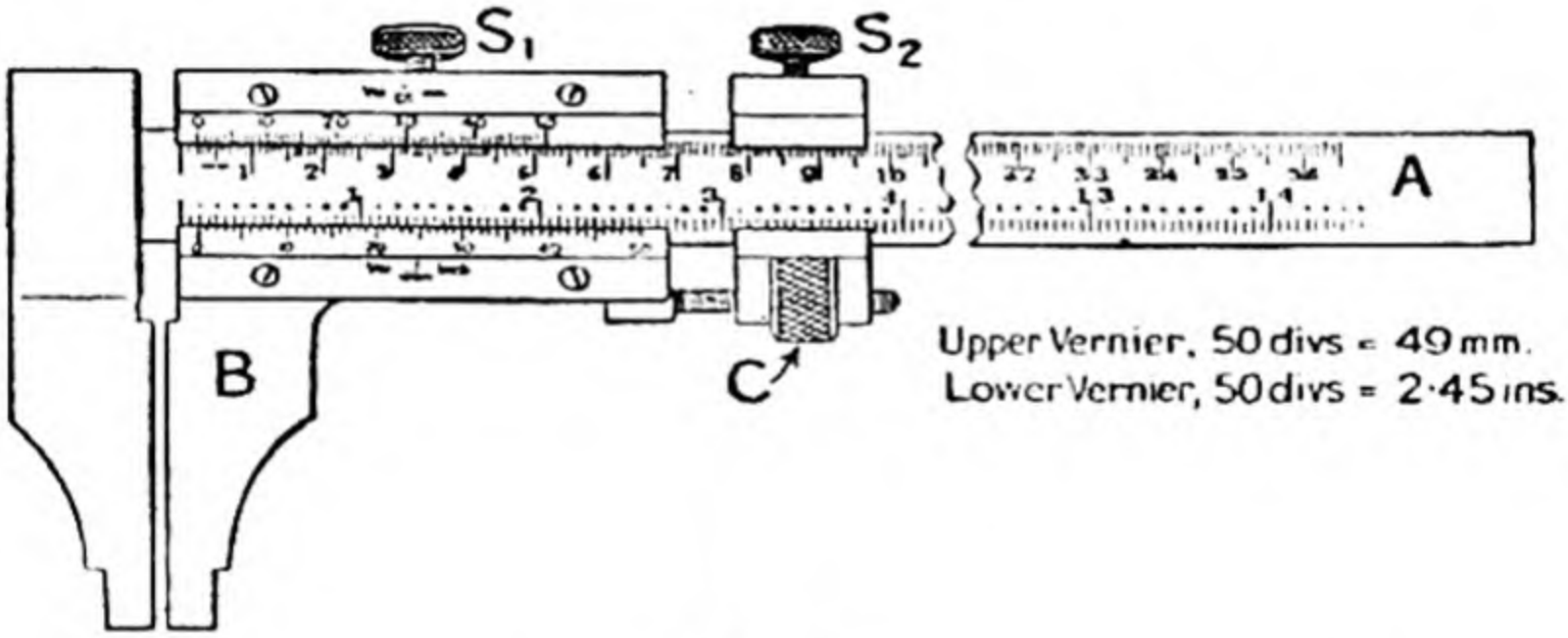

FIG. 9.

A pair of vernier calipers is shown in the above figure. By releasing the screws S_1 and S_2 the vernier scale may be slid along the stem A, which carries the ordinary scale. If S_2 is locked, fine adjustment of the position of the portion B, which carries the vernier, may be made by means of the screw C. The zero position of the calipers when closed should be tested from time to time, and if any error is found a correction should be made when using the calipers. To determine, say, the diameter of a cylinder,

several readings should be made at different places, and the average of these taken as the required diameter.

12. The Micrometer Screw Gauge.

For many purposes the micrometer screw gauge is replacing the vernier scale. Moreover, it is capable of giving a higher degree of accuracy, the readings requiring less strain on the eyesight than in the case of the vernier.

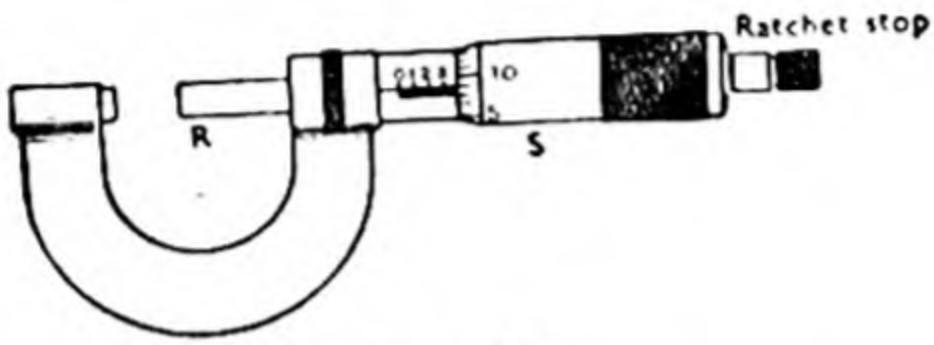

FIG. 10.

The instrument consists of a sieeve S, which is attached to a shaft with a very accurate thread cut on it. When the sleeve S is rotated the rod R at the end of the shaft is moved forward with the sleeve. One complete rotation of the sleeve moves the rod R forward (or backward) a distance equal to the pitch of the the screw thread. If the pitch of the screw is 0·25 mm. and the circumference of the edge of the sleeve is divided into 25 equal parts, 1/25th of a turn moves the rod forward 0·01 mm., and such a micrometer could therefore read easily to an accuracy of 0·001 cm. In order to prevent rough usage from wearing the threads the sleeve can usually be turned by a ratchet device attached at its extremity. When the rod presses with a certain force against the object to be measured, held in the jaws of the instrument, the ratchet handle turns without moving the screw. It is necessary to test the zero of the instrument from time to time when the jaws are together, and allowance can then be made for any error introduced if this reading is not exactly zero. The figure below shows a micrometer screw with the additional refinement of a vernier scale. The edge of the sleeve is divided into 25 parts, and one turn of the sleeve moves it forward 0·25 mm. The

vernier allows the instrument to be read correct to 1/10th of a sleeve division, *i.e.*, to 1/10,000 cm.

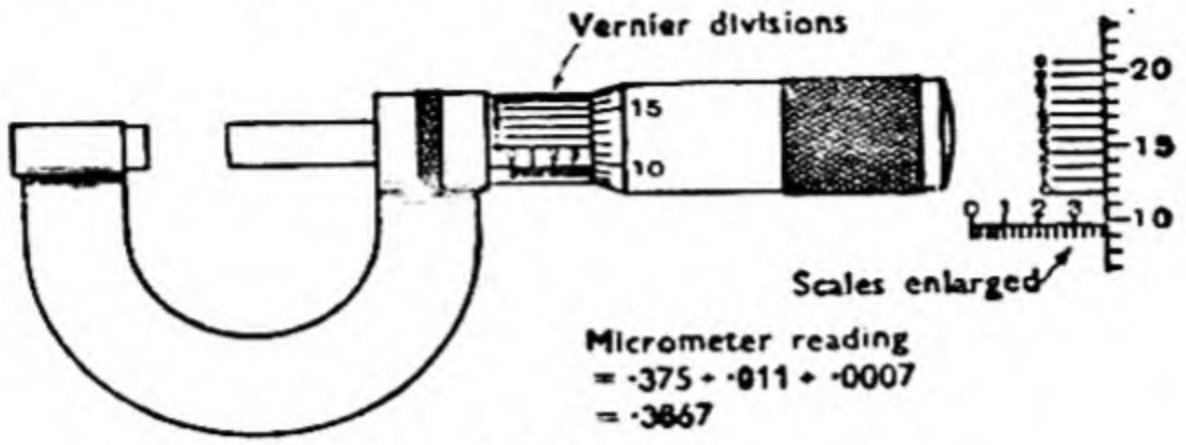

FIG. 11.

EXAMPLES III

1. Find the reading of the micrometer screw gauge which is shown in the figure. One division = 1 mm., pitch of thread = ½ mm., and the sleeve is divided into 25 parts.

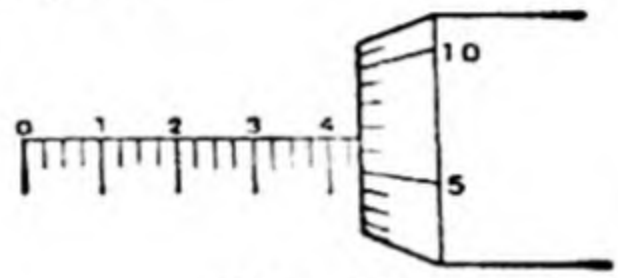

FIG. 12.

2. Obtain the reading of the micrometer shown below. One division = 1 mm. The sleeve is divided into 25 equal parts. Pitch of thread, ½ mm.

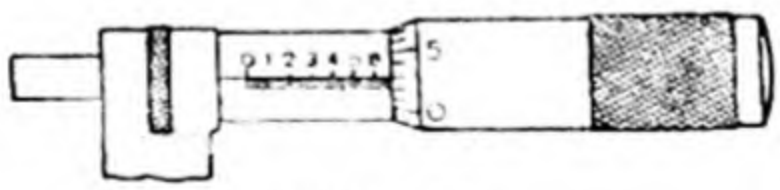

FIG. 13.

3. Obtain the reading of the micrometer correct to 0·0001 cm. The sleeve is divided into 25 equal parts, and the pitch of the thread is ½ mm. One scale division = 1 mm.

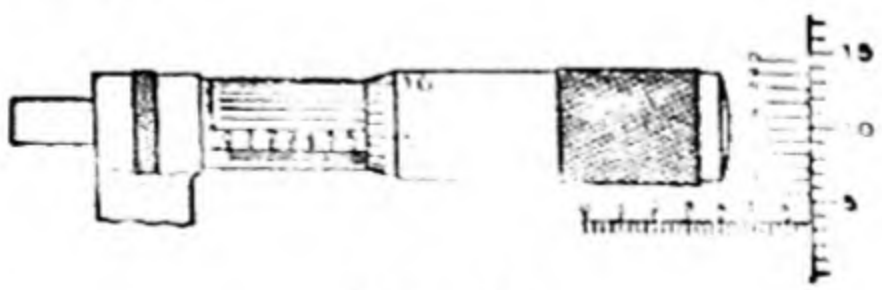

FIG. 14.

CHAPTER II

FIRST EXPERIMENTS WITH FORCES

13. Force.

Force and the effects of force at once claim our attention whichever of the branches of mechanics we proceed to study. A force with which we are all acquainted is that which is required to balance the weight of an object. When we carry a bag we experience a force which we call the weight of the bag. This force acts vertically downwards. Again, suppose that a book rests on a desk and we attempt to push it along the desk with a finger. A gentle push may not move the book, but we are aware of the force which is being exerted by our sense of touch. A harder push will move the book, and the harder we push the more quickly will it move along the desk. A dog straining at his lead is exerting a force, and if the lead breaks the force which he exerts causes him to move forward in the direction indicated by the previously taut lead. If a flat stone is set sliding along a level stone floor, it is quite soon brought to rest by the friction, or rubbing, between the stone and the floor. If the same stone is set sliding along ice, it moves a considerably greater distance than in the first case. In both cases the stone will travel in a straight line unless it be deviated from this path by a force such as might be produced by an unevenness in the floor or ice, or, again, by the stone striking some other object. If it were possible to have a perfectly smooth horizontal surface, and if there were no air resistance, the stone would continue to move uniformly along the surface, and would only cease to do so if a force were exerted upon it in some way to slow it down or cause it to deviate from its straight path. We have, therefore, sufficient information to enable us to state that FORCE *changes or tends to change a body's state of rest or of uniform motion in a straight line.*

14. The Measurement of Weight.

The weight of a body has been given as an example of a force. This weight is the force with which the earth attracts the body. The direction of this force is found at any place by fastening one end of a thread to a small leaden bob and the other to a fixed support. This constitutes a plumb-line, which you have probably seen used by builders to ensure that walls are accurately vertical. The direction of the plumb-line is that of the vertical at the place and the direction

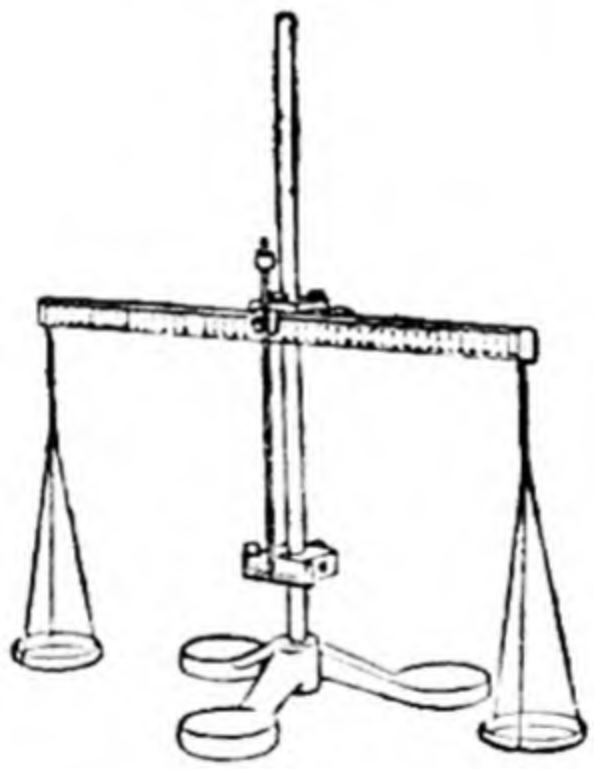

FIG. 15.

in which the attraction of the earth acts. The force of attraction of the earth on a body is different at different parts of the earth's surface, and is, in fact, greatest at the poles and least at the equator. This, however, does not prevent us from being able to decide when two bodies have equal weights. *If the attraction of the earth on two bodies is the same at the same place on the earth's surface, we say that they have equal weights.* Suppose that we take a simple pair of scales and in one scale pan place a body A. In the other pan place enough brass, shall we say, to more than balance A. Carefully file the brass until it just balances A. Repeat this with a second piece of brass. We have thus made

two brass *weights* which have the same weight.[1] It may now be seen that a system of weights which are multiples or fractions of a standard weight are not difficult to construct. The British standard of weight is that of a piece of platinum which is preserved in the House of Commons, and is *one pound weight*. The gram weight is another standard weight, which is commonly used in most branches of science where accurate weighing is carried out, and is one of the system of weights used in France and other continental countries.

1 lb. wt. = 453·6 gm. wt.
2·205 lb. wt. = 1,000 gm. wt. = 1 kgm. wt.

15. Method of Measuring Force.

A convenient instrument for measuring force in whichever direction it acts has long been used for household weighing; the common *Spring Balance*. In order to examine how this instrument works perform the following experiment.

EXPERIMENT.—Fasten one end of a spiral spring to a fixed support

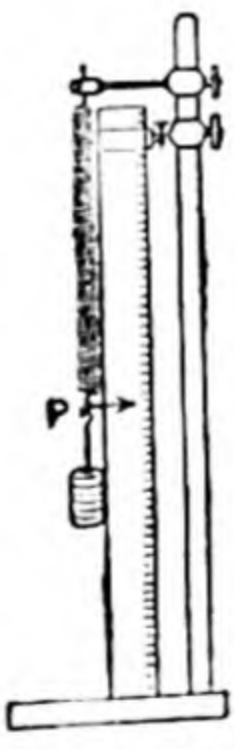

FIG. 16.

[1] Note the two different meanings given to the word "weight"; in one case it describes an object used for the purpose of weighing and in the second case it is used to describe a force, viz., the force of attraction of the earth.

so that the spring hangs vertically. Examine the coils of the spring and be sure that no coil is touching either of the adjacent coils above or below it. At its lower end attach a pointer P, in a horizontal position, and a pan to carry weights. Behind the pointer erect a suitable graduated rule in a vertical position. A metre rule should be used if your weights are gram weights, and a rule graduated in inches and tenths of an inch if your weights are in fractions of a pound. Note the position of the pointer and add equal weights in succession to the pan, noting the readings of the positions of the pointer corresponding to each new weight. The readings should be taken when the scale pan is at rest, and at least ten are necessary. Remove the weights one at a time, and again take the readings of the positions of the pointer. Care should be taken to see that the spring is not overloaded (see Example IV., No. 1, on p. 17).

Load (lb.) .	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
Extension (in.) .	0·0	0·29	0·60	0·90	1·21	1·50	1·80	2·11	2·40	2·69	3·00
Extension (in.) .	0·01	0·30	0·60	0·91	1·20	1·50	1·81	2·10	2·40	2·70	..

Make a table of your results, as shown above, and draw a graph on which the vertical scale represents the extension of the spring (calling the position of the pointer zero when the pan carries no weights), and the horizontal scale represents the load upon the scale pan. Compare your results with the graphs shown below.

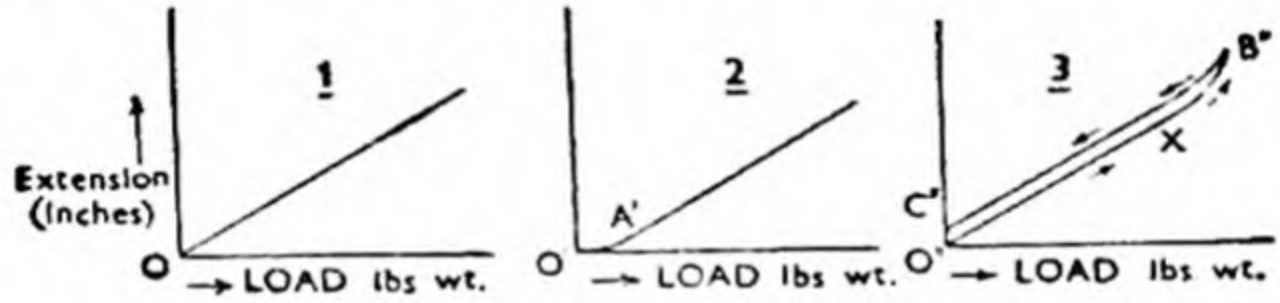

FIG. 17.

Notice that in graph (1) all of the points plotted lie on one straight line, and equal increases or decreases in the weight produce equal increases or decreases respectively in the extension of the spring. **In fact the weight in the pan is directly proportional to the extension it produces.** This is known as HOOKE'S LAW. A spring which has this property and keeps it, however much it is used, supplies us at once with a force measurer. Whatever the load on the spring in the experiment for which the results are given, the extension per 0·1 lb. wt. is 0·3 in. If we suppose that your spring behaves in this way (and it is essential that it should do so), stick a piece of paper on the vertical scale and mark the positions corresponding to the various weights with the appropriate numbers. The spring may now be used to weigh other objects, *i.e.*, to measure the force of attraction of the earth upon them.

The instruments used in practice are robustly made so that the spring, which is usually encased, cannot be overstrained when too great a force is applied to it. Another name for this instrument is the DYNAMOMETER. If the spring is fairly strong the effect on its readings caused by neglecting the weight of the spring itself and the permanent attachments is very small. Error is only introduced when the spring is used for measuring

FIG. 18.

FIG. 19.

forces which are not vertically downwards. Little difficulty is found in allowing for these effects, for we have only to take the difference of readings of the spring caused by the force to be measured.

EXAMPLES IV

1. (*a*) In the second graph above explain the shape of the part O′A′. (*b*) In the third graph O″B″ corresponds to weight increasing and B″C″ to weight decreasing. Explain the shape of the graph.

2. What do you understand by "force"? Give examples to illustrate your answer. Sketch a piece of apparatus which measures force, and explain why it does so.

3. Describe briefly with sketches an experiment to determine the relation between the load and the extension of a close-coiled spiral spring. In such an experiment the following values of the load W lb. and the extension *e* inches were obtained :—

W	2	4	6	8	10	12	14	16	18	20
e	0·24	0·47	0·70	0·94	1·17	1·40	1·63	1·87	2·10	2·34

Draw the graph of W against *e*, and find the load per inch of extension.

4. A spring is subjected to gradually increasing loads with the following results :—

Load W.	Extension *e*.	Load W.	Extension *e*.
Lb.	In.	Lb.	In.
0·0	0·0	0·5	0·74
0·1	0·14	0·6	0·84
0·2	0·28	0·7	1·06
0·3	0·42	0·8	1·30
0·4	0·56	0·9	1·61

Plot a graph showing the relation between the extension and the load, and explain the significance of any peculiarity in the shape of the graph.

16. First Experiments with Forces.

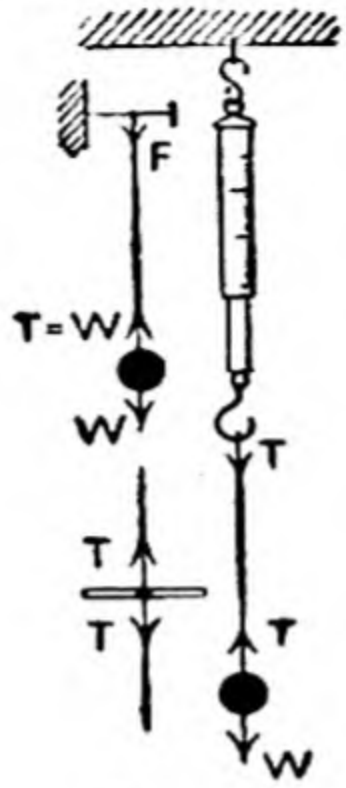

FIG. 20.

EXPERIMENT (1).—Fasten a piece of lead to a light string and secure the string by means of a loop to a fixed nail. What is the force on the nail and what is the pull or TENSION in the string? Lift the string from the nail and pass the loop over the hook of a spring balance. The spring is stretched by the weight of the lead (and, of course, the string). The force registered by the spring is the force which was previously experienced by the nail and is equal to the combined weights of the lead and the string. The latter weight is probably too small to affect the reading of the spring. The string immediately above the piece of lead supports the lead, therefore it pulls the lead vertically upwards with a force equal to the weight of the lead. Thus the tension in the string acts vertically downwards on the spring and vertically upwards on the lead with equal forces, which act in the same vertical straight line. (Here we neglect the weight of the string.)

Example.—*A chain consists of nine links each of weight* 1 *lb. It is suspended by its top link from a fixed horizontal peg.*

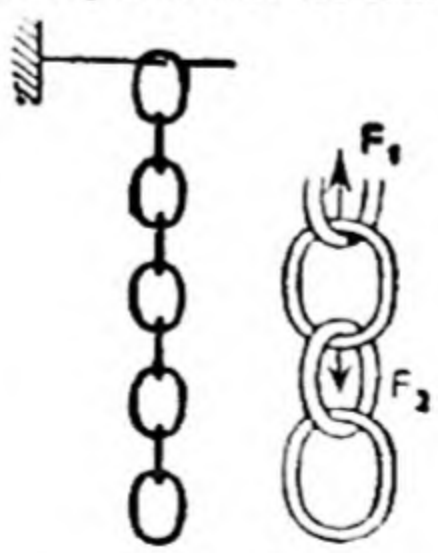

FIG. 21.

(*a*) What is the force exerted on the peg by the top link? What is the force exerted on the top link by the peg?
(*b*) What is the force F_1 exerted by the third link on the fourth link?
(*c*) What is the force F_2 exerted by the fifth link on the fourth link?
(*d*) Account for the difference between the forces in (*c*) and (*b*).

Carry out the following experiment and modify it if necessary to show that your answers to the above questions are correct.

EXPERIMENT (2).—Support a weight by a spring balance which is itself supported by a second spring balance secured to a fixed support. Note that the tension in the upper spring is greater than that in the lower. Why is this?

Example.—*A plank* AB *is supported on the banks of a ditch, the supports being near its ends. A boy proceeds to walk across the plank from one side of the ditch to the other, starting at the end* A *of the plank. How does the force exerted by the end* A *of the plank on the ground change as the boy walks across the plank? When are the forces exerted by the ends of the plank on the opposite sides of the ditch equal? Verify your reasoning by the following experiment.*

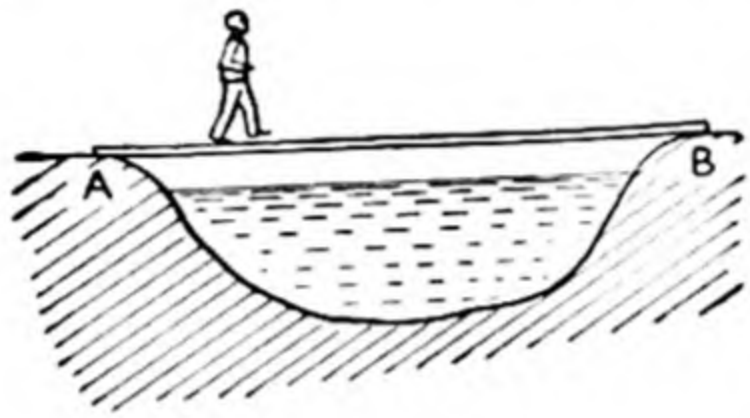

FIG. 22.

EXPERIMENT (3).—Support a light rigid rod horizontally by the two

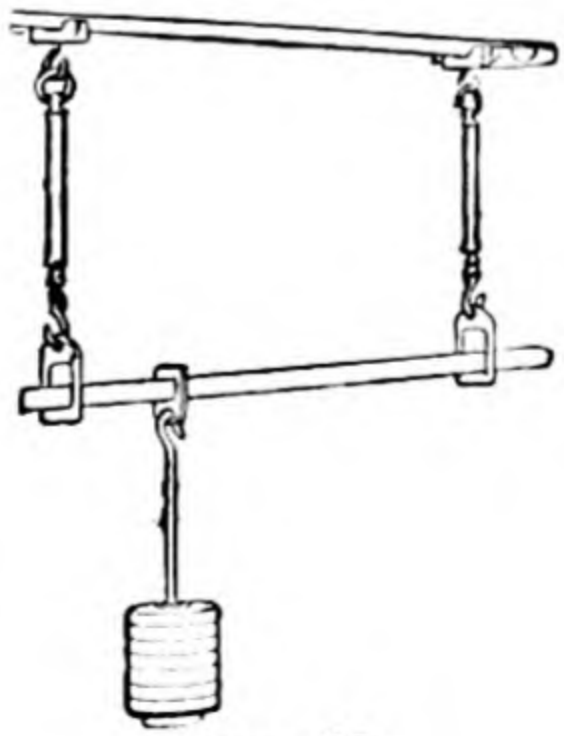

FIG. 23.

hooks of two vertical spring balances. Suspend a weight of 2 lb. from the rod so that each spring registers the same tension. Note that each tension is a little more than 1 lb. wt. Why is this? Arrange the weight so that the springs do not register equal tensions. Notice that the sum of the tensions registered by the springs is a little greater than 2 lb. wt. If T_1 and T_2 represent the tensions and W the weight,

$$T_1 + T_2 = W + \text{weight of the rod, etc.} = \text{a constant force.}$$

As an illustration of these last two experiments, suppose that a man carries balanced on his shoulder a pole of weight 2 lb. on which hang weights of 15, 10 and 17 lb. The total upward force exerted by his shoulder in order to support the pole and the weights is 44 lb. wt. If the man now stands upon a weighing machine then the weight registered is his own weight together with the load he carries.

EXPERIMENT (4).—Support a weight by two spring balances which

FIG. 24.

are not vertical (see diagram). Notice that the sum of the tensions is greater than the weight supported, and that this sum is greater the more closely this angle approaches two right angles. When are the tensions equal?

Two boys carry a cricket bag, each holding one of the two handles From the above experiment we see that the total vertically upward force which their arms would have to apply if these were vertical would be the weight of the bag. The sum of the forces which they exert is least in this case, but we know from experience that the boys would tend to topple towards each other if they attempted to carry the bag in this position. In practice each boy pulls the bag in directions a little out of the vertical and away from each other. In this physically more comfortable position they therefore pull with forces the sum of which is greater than the weight of the bag; moreover, the advantage gained in carrying the bag in the same manner is lost very quickly if the boys attempt to carry it by pulling away from each other too much.

EXPERIMENT (5).—Support a 2 lb. wt. by a string in a vertical plane

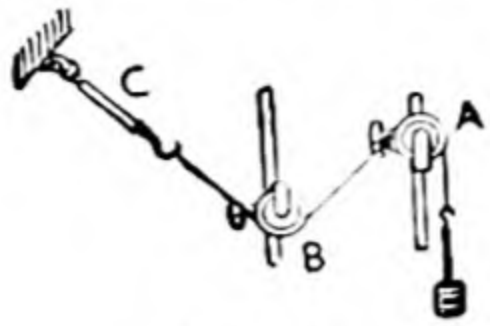

FIG. 25.

so that the string passes over a pulley A and under a pulley B and is then attached to a spring balance C, as in the above figure. The bearings of the pulley, the grooves of the pulley and the string should be as smooth as possible. The spring registers a tension of very nearly 2 lb. wt. Remove the string from pulley B and allow the spring to take the tension in the string in the new direction, and note that the tension registered remains practically the same as before. This shows that when a light string passes over a number of smooth pulleys and is pulled taut the tension throughout the string is constant.

EXPERIMENT (6).—Pass a string which supports two equal weights, one at each end, over two smooth pulleys so that the string between the pulleys is horizontal. What is the tension in the horizontal part of the

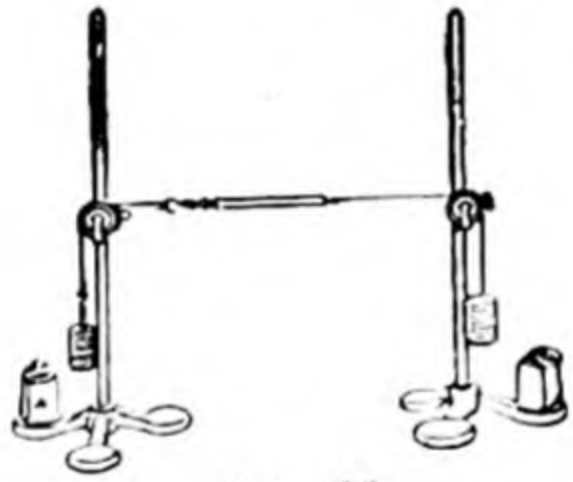

FIG. 26.

string? Cut the string near one pulley and secure the ends to a spring balance, which is now supported in such a position that the string between the pulleys and the spring are horizontal and in one straight line. The tension in the spring is seen to be equal to one of the weights, and is precisely what is required to support either weight *alone*. This confirms the conclusion of Experiment (5).

EXPERIMENT (7).—Fasten two light strings to two points on the edge of a piece of three-ply wood about 1 ft. square (a piece of strong cardboard will do) and secure each string to a spring balance. Pull the springs apart parallel to the table so that the strings are horizontal and the wood remains at rest upon the table. Place the edge of a metre rule along one of the strings and notice that it is in the same straight line as the other string and the two springs. It will also be seen that the springs register equal tensions. Thus the two equal and opposite tensions acting horizontally upon the wood keep it at rest. Two equal and opposite

forces which act in the same straight line upon a body are said to be in EQUILIBRIUM. The weight of the wood which is balanced by the upward force exerted on it by the table on which it rests has not been taken

FIG. 27.

account of in this experiment. The wood remains at rest on the table when we detach the strings, so that the vertical forces which now act on it are in equilibrium and so do not affect the conclusion arrived at by the experiment with regard to the horizontal tensions.

Pull the springs apart again so that the wood is at rest, as described above. Mark the line of the strings along the top of the wood and attach one of the strings at different points along this line instead of at the rim. The wood is observed to remain at rest. We may therefore deduce that **if a force acts on a body at any point on its own line of action it produces the same effect upon the body.** This is known as the Principle of the Transmissibility of Force.

Two equal and opposite forces acting upon a body in the same straight line are in equilibrium.

17. Action and Reaction.

Support a spring balance in a vertical position. Press the hook downwards with a finger. The force which the finger exerts on the hook is balanced by the equal and opposite force exerted by the hook on the finger. The magnitude of this force is registered by the spring balance. If we wish to know the tension in a string, we have seen that we actually measure the force exerted by one part of the string on an adjacent part, and when the string is at rest this force is balanced by the equal and opposite force exerted by the " adjacent " part of the string upon the first part. A pair of forces such as these are known as an **action** and a **reaction.** We have had ample confirmation in our experiments that the ACTION of a force upon a body is balanced by an equal and opposite REACTION. It is very important when we describe a force that we should state :

(*a*) The body upon which it acts.
(*b*) The direction in which it acts upon the body.
(*c*) The magnitude of the force.

EXAMPLES V

1. In a tug-of-war one team pulls on the rope with a force of 300 lb. wt. With what force does the other team pull on the rope and what is the tension in the rope ? What is the tension in the rope between the

second and third men from one end? (Assume that each man pulls with the same force.)

2. A load of 2 tons wt. is supported by two vertical parallel ropes at their ends. What is the tension in each rope?

3. A 10-stone man stands on the ground and grips an overhead bar. If he pulls vertically downwards on the bar with a force of 28 lb. wt., what is the vertically upward force exerted by the floor on his feet?

4. The ten links of a chain hanging vertically from a nail, which supports the first link, are each of weight 1 lb. Find (i) the force exerted by the chain on the nail, (ii) the force exerted by the sixth link on the fifth link, (iii) the force exerted on the fourth link by the third link. In each case state the direction of the force.

5. A truck weighs 1 ton and carries a 15-ton load which is evenly spread over the bottom of the truck. If the truck is at rest, what is the force exerted by the rails on each of the four wheels which may be considered as situated at the corners of a rectangle.

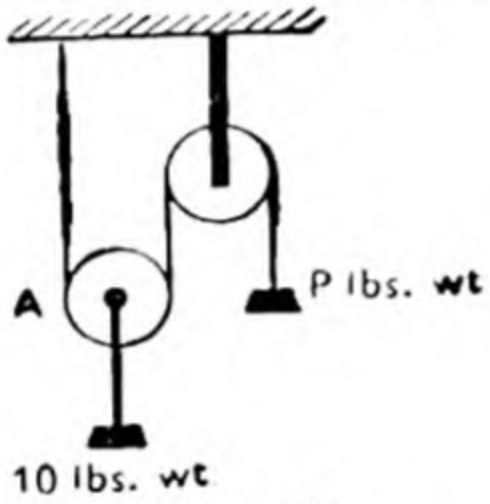

FIG. 28.

6. If the pulleys in Fig. 28 are supposed smooth, what must be the value of P, (i) so that the load of 10 lb. wt. may be at rest, (ii) so that the load of 10 lb. wt. may move vertically downwards or vertically upwards with uniform speed?

7. In Fig. 28, if the pulley A weighs 5 lb., what is the value of P required to keep the pulleys at rest. Suppose the pulleys are now separated horizontally so that the string parts of the system, except that supporting P, are no longer vertical, what will happen? Explain why.

8. If the parts of the string not in contact with the pulleys in Fig. 29

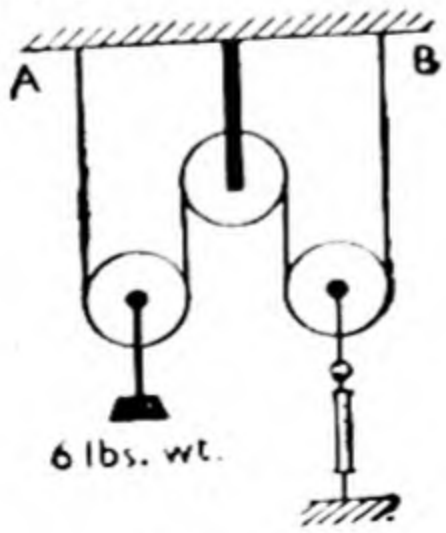

FIG. 29.

are vertical, what is the tension in the string and the reading of the spring balance ? What is the total force on the beam AB ?

9. In Fig. 30 the parts of the strings not in contact with the pulleys may be assumed to be vertical.

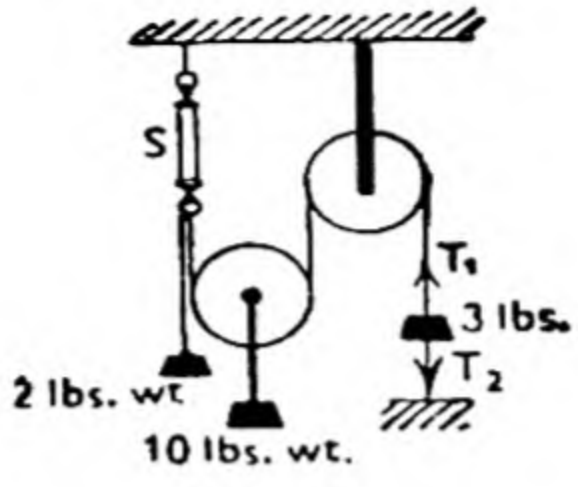

FIG. 30.

Find the reading of the spring balance S and the tensions T_1 and T_2.

CHAPTER III

PARALLEL FORCES, CENTRE OF GRAVITY, MOMENTS

18. Parallel Forces.

Let us suppose that we are in a mechanics laboratory. Consider the forces which are exerted upon the floor of the room; the weights of stools, benches, apparatus and our own weights are some of these forces which are all vertical and therefore parallel to each other. The floor of the room is usually supported by beams, which in their turn are carried by the walls of the room. Before constructing the floor of a room it is necessary to decide how strong the beams are to be and how thick the walls of the room to support them. It is not, of course, possible to try different strengths of beams and thicknesses of walls in practice, but we can estimate the forces which each must bear, and so choose beams of adequate strength and walls of sufficient thickness to support them. On a small scale we can arrange to study the effects of parallel forces acting upon a beam by carrying out the following experiments :—

EXPERIMENT (1).—Make a light scratch through the middle point of

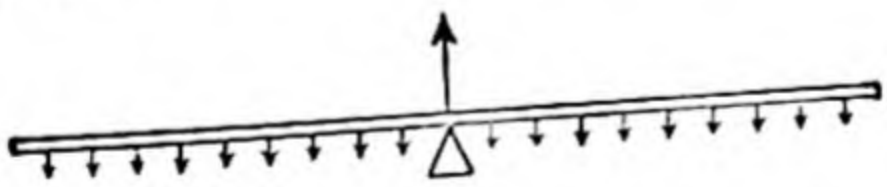

FIG. 31.

the ungraduated surface of a metre rule perpendicular to its length. Balance the rule on a knife-edge so that it rests on this mark. The rule will probably not remain perfectly balanced in a horizontal position. To make it balance accurately drop small splashes of sealing-wax on to the ruler at appropriate points, or scrape the ungraduated side. Now fasten a piece of string round the rule so that the part of the string on the

2944

under side of the rule lies along the scratch. Fasten the string to a spring balance. Notice that the rule is supported in a horizontal position by the tension in the string, which is vertical. The reading of the spring is the weight of the rule. This experiment shows that the attraction of the earth on all the particles of the rule can be balanced by a single upward vertical force acting through a certain point of the rule. Thus the weight of the rule must be equivalent to a single force which acts vertically downwards through this point. This result is extremely important.

EXPERIMENT (2).—Since the metre rule used in the last experiment is likely to bend too much when it has to carry additional weights, if it is supported in the manner indicated above, it is usual to make a number of small holes through it and support it with its graduated surface vertical. In this position the amount of bending is negligible for our purposes

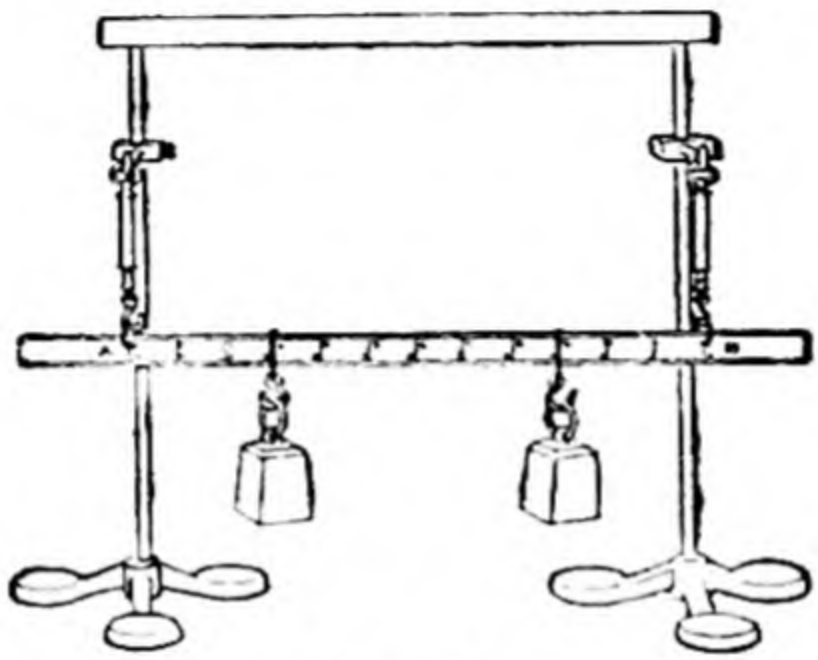

FIG. 32.

Balance the rule accurately, supported by a pivot through the central hole, as already described. Now suspend the rule in a horizontal position from a spring balance. On either side of the support add additional weights fastened by looped strings to the rule. Adjust the distances of these strings from the point of support until the rule is again horizontal. Notice that the spring registers the sum of the weights of the rule and the two additional weights. Repeat this experiment with three different pairs of weights, and carefully check the above result in each case. Support the rule at a point which is not its centre. Add weights at a number of points on the rule and arrange them until they balance and the rule is horizontal. Verify as before that the reading of the spring is equal to the sum of the weights added and the weight of the rule.

In this experiment a single vertically upward force has been shown to balance the effect of a number of vertically downward forces. The single vertically upward force is called the **Equilibrant** of the vertically downward forces, and since all the forces acting upon the rule keep it at rest, they are said to be in **Equilibrium.** Since the vertically upward force would be in equilibrium with an equal but vertically downward force having the same line of action, this latter force would have the same effect as the number of vertically downward forces acting on the rod. A

single force which produces the same effect upon a body as a number of other forces is called the **Resultant** of these forces.

Support the rule by TWO or more vertical spring balances and load it with various weights. The sum of the tensions registered by the springs is equal to the sum of the added weights and that of the rule.

Repeat the above experiments with the rule in a position inclined to the vertical.

If the forces which act vertically downwards on the rule are called positive forces and those which act vertically upwards negative forces, the conclusion drawn from the above experiments may be stated as follows :—

When a number of parallel forces act upon a body and keep it at rest the algebraic sum of these forces is zero.

19. Centre of Gravity.

We have already stated that the force of attraction of the earth upon a body is known as the WEIGHT of the body, and that this force is different at different places on the earth's surface. A body at the surface of the moon or one of the planets is acted upon by a similar force of attraction ; thus the same body would have different weights on the earth, the moon or the planets. Moreover, the earth and the planets are themselves attracted towards the sun by this same force, which enables them to travel in their orbits around the sun. From these facts we deduce (as you may have already anticipated) that **all bodies attract one another.** This force of attraction between bodies is called **the force of gravity.** When a body is suspended by a string or rope and hangs in equilibrium under the actions of the force of gravity and the tension in the string or rope, this tension acts vertically upwards upon the body. The attraction of the earth upon all the particles of the body constitute a very large number of vertical parallel forces. The tension, which is a single vertically upward force, is, however, in equilibrium with an equal vertically downward force, which must therefore be exactly equivalent to the whole attraction of the earth upon the body. If the string is attached in turn to different points of the body, and in each case the body is allowed to hang freely in equilibrium, it is found that the lines of action of the tensions required to support it all pass through one point of the body. This point is called **the centre of gravity of the body.** It is not easy to demonstrate this in the case of an irregular opaque body, but the

following experiments are sufficiently convincing for our present purposes.

Note (1).—We have tacitly assumed that the vertical lines through different points of a body are parallel. This is not strictly true, but the assumption is justified for bodies which are small in comparison with the earth. If A and B are two points on the earth's surface 1 mile apart, and the verticals through A and B are imagined to be drawn upwards for 1 mile above these points, then the upper ends of these lines are only 7 ft. farther apart than A and B.

Note (2).—The weight of a body at different places of the earth's surface does not alter by more than 1 part in 300, and in fact is so little different at different places in the British Isles that British engineers use 1 lb. wt and 1 ton wt. as their units of force. These units of force are known as **gravitational units of force.**

20. EXPERIMENT (1).—Cut an irregular figure out of a thin sheet of tin

FIG. 33.

or strong cardboard and suspend it by means of a fine string attached to a point in its rim. When in a position of rest, draw on the sheet a line in the direction of the string and in the same straight line. Repeat the procedure when the sheet is suspended from a number of other points on its rim. Verify that the straight lines drawn on the sheet all pass through one point. This point is the **centre of gravity** of the sheet. It is clear, of course, that the centre of gravity is inside the material of the body. Make a small hole in the sheet at this point, and support it by passing the point of a pair of compasses through this hole. Notice that the figure stays in equilibrium whatever angle its plane makes with the vertical.

Find by the above method the positions of the centres of gravity of sheets of tin or cardboard in the shapes of some of the plane figures, such as the rectangle, square, parallelogram, triangle and quadrilateral.

EXPERIMENT (2).—Cut an irregular solid, as shown in the diagram,

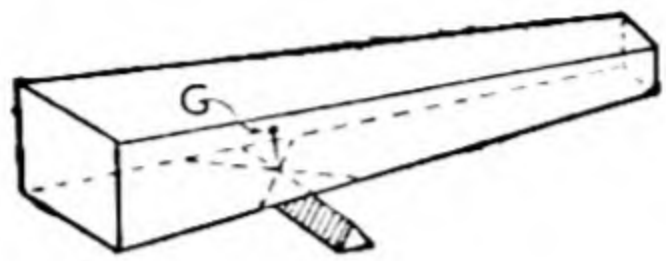

FIG. 34.

which is capable of being balanced about a long knife-edge in contact with its various faces. Balance this solid upon the knife-edge and mark its line of contact with the solid. Twist the solid horizontally and balance it again. Mark the new line of contact. Repeat a number of times, and notice that all the lines of contact pass through one point of the surface. The weight of the solid therefore acts through this point. Repeat the experiment, using the opposite face of the solid, when it will again be found that the lines of contact of the knife-edge on this face corresponding to balanced positions of the solid pass through one point. The centre of gravity of the solid lies on the line joining these two points only if the corresponding faces of the solid are parallel; if they are not parallel, perpendiculars to these two faces at these points meet at the centre of gravity.

21. The Turning Effect of a Force.

If we ask why the handle of a door is placed as far from its hinges as is practicable, we have no difficulty in answering that the door is easier to open with the handle in this position than if it were placed nearer to the hinges. The force required to crack a Brazil nut is much greater than most people can apply with fingers alone, but by using a pair of nutcrackers the task becomes an easy one. Standing with one arm outstretched is comparatively easy for a short time, but why does it so soon become tiring? Moreover, if a weight of 2 lb. is carried in the hand in this latter instance, why is the task made so much more difficult, for a 2 lb. wt. is very easy to carry with the arm in a vertically downward position? The answers to the above questions involve the explanation of principles which are of the greatest importance in mechanics. It is therefore desirable that the

answers should be sought by carrying out experiments designed to make these principles as clear as possible.

EXPERIMENT (1).—For this experiment use a metre rule through which

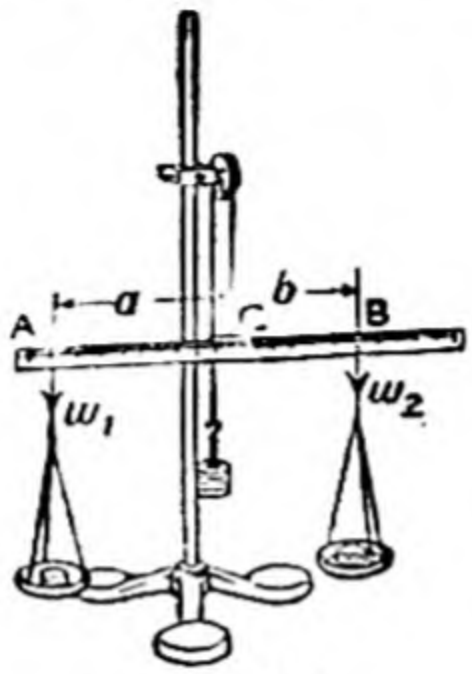

FIG. 35.

small holes have been carefully drilled at regular intervals along its entire length. Support the rule through the centre hole and balance it in a horizontal position, as previously described. Suspend weights at points A and B on either side of the centre C, and arrange the distances from C to the points of suspension, so that the rule rests in a horizontal position. Keeping one of the weights in its same position double the second weight and arrange its point of suspension so that the rule is again horizontal. Do you notice any connection between the weight and the distance from its point of support to the pivot? Try various multiples and fractions of the same weight, and verify that the product of the weight and the distance from the point of its support to the pivot is constant. Now alter both weights, and having again balanced the rule, note the weights and the distances CA and CB. Suppose the weights are W_1 and W_2 and their respective distances from the pivot a and b, then it should be found that

$$W_1 \times a = W_2 \times b.$$

From the above experiment we see that if W_1 is kept unaltered in weight and position, if W_2 is multiplied by any number, then in order that it may still balance W_1 the distance from its point of support to the pivot must be divided by the same number. Again, if W_2 is increased m times and b is increased n times, then if a is unaltered, W_1 has to be increased mn times to maintain the balance; if W_1 is unaltered, b must be increased mn times. The tendency of a weight to rotate the rule about the pivot is called the TURNING EFFECT of the weight, or otherwise the MOMENT of the weight about the pivot. **Thus the turning effect of a weight is not only proportional to the weight itself but also to the distance from the pivot to the point of support of the weight.**

Experiment (2).—We may apply the result of the last experiment

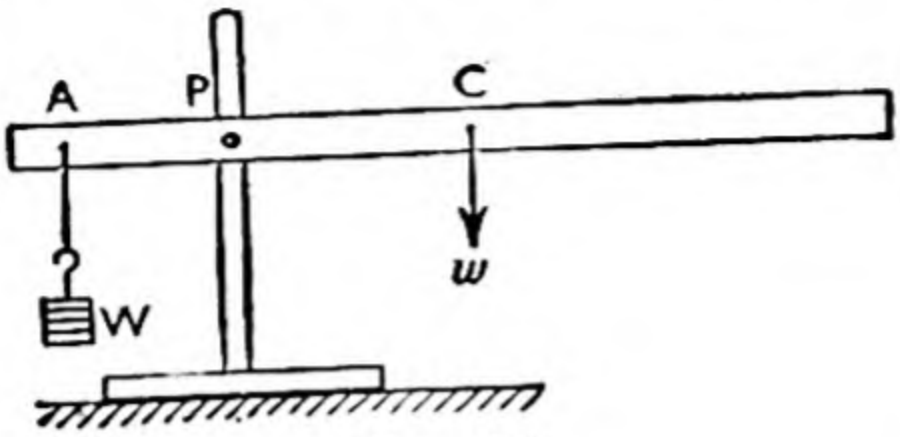

Fig. 36.

to find the weight of the metre rule. Support the rule by pivoting it at a point P away from its centre. If w is the weight of the rule and W that of the balancing weight,

$$w \times \mathrm{CP} = \mathrm{W} \times \mathrm{AP},$$

and therefore

$$w = \frac{\mathrm{W} \times \mathrm{AP}}{\mathrm{CP}},$$

where C is the centre of the rule and A is the point of support of the weight. Repeat the experiment and take the average of three values. Use this average value in the succeeding experiments when the weight of the rule is required.

Example.—*A uniform beam* AB, 10 *ft. long and of weight* 20 *lb., is pivoted at its centre and rests in a horizontal position. A weight of* 100 *lb. is placed on the beam at* C, 1 *ft. from* A. (*a*) *At what point on the beam should a weight of* 120 *lb. be placed so that the beam may be balanced?* (*b*) *What weight placed on the beam at* B *will balance it, and what is the force on the pivot in this case?*

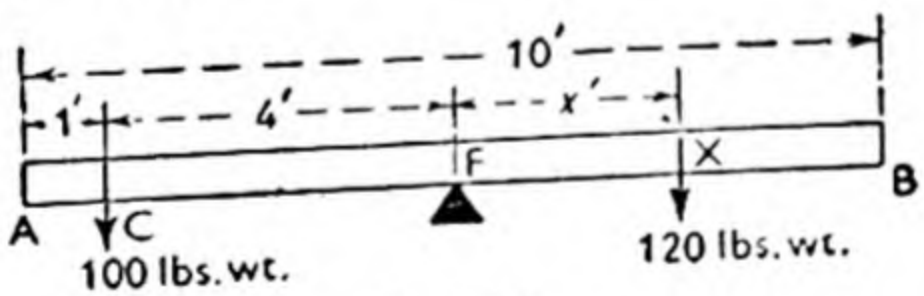

Fig. 37.

(*a*) F is the position of the pivot. X is the position of the 120 lb. wt.

Let

$$\mathrm{FX} = x \text{ ft.}$$

Since the beam is balanced the anti-clockwise moment of the 100 lb. wt. about F is equal to the clockwise moment of the 120 lb. wt. about F; therefore,

$$120x = 4 \times 100,$$

$$x = \frac{400}{120} = 3\tfrac{1}{3}.$$

The weight must therefore be applied 1 ft. 8 in. from B.

(*b*) Let W lb. wt. = the weight which must be placed on the beam at B to balance it.

Since the beam is balanced the clockwise moment of W lb. wt. about F is equal to the anti-clockwise moment of the 100 lb. wt. about F; therefore,

$$5W = 4 \times 100,$$

$$W = \frac{400}{5} = 80.$$

Hence 80 lb. wt. placed at B will balance the beam.

The sum of the vertically upward forces acting on the beam is equal to the sum of the vertically downward forces.

Let F lb. wt. be the vertically upward force of the pivot exerted on the beam.

$$F = 100 + 120 + 20 = 240.$$

The downward force of the beam on the pivot is 240 lb. wt.

EXAMPLES VI

Beam with Single Pivot

1. A uniform beam AB is balanced at its midpoint C. A weight of 20 lb. is placed 3 ft. from C. Where should a weight of 30 lb. be placed to balance the beam?

2. A uniform rod is pivoted at its centre, and a weight of 3 lb. is suspended from it 9 in. from the centre. If the rod is balanced by a weight of 5 lb., where must it be applied? What is the total force on the pivot and in which direction does it act?

3. A 50 lb. wt. is placed at one end of a uniform plank 18 ft. long, which is supported at a point 4 ft. from the weight. If the plank is balanced, find its weight.

4. Two boys whose weights are 100 and 112 lb. sit on either side of a see-saw, which is balanced before they sit on it. If the lighter boy sits 5 ft. from the pivot, how far from the pivot should the other boy sit to maintain the balance? What force does the pivot exert on the see-saw if its weight alone is 60 lb.?

5. The centre of gravity of a billiard cue of weight 2 lb., 5 ft. in length, is 1 ft. 3 in. from the heavy end. If the cue is supported at its mid-point, what force (*a*) applied at the heavy end, (*b*) applied at the light end, is required to maintain it in a horizontal position?

6. A non-uniform beam AB of length 6 ft. balances about a point 2 ft. from the end A when a weight of 10 lb. is hung at that end. When the pivot is at the midpoint of the beam a weight of $6\frac{2}{3}$ lb. must be hung at B to balance it. What is the weight of the beam and where is its centre of gravity?

22. Principle of Moments.

Experiment.—It will be assumed in this experiment that the weight of the rule is known. By means of three weights P, Q and R, three parallel forces are arranged to act upon the rule, and the rule is pivoted at a point away from its centre. Arrange the positions of the weights until the rule is horizontal. Call F the position of the pivot and A, B, C

the respective points of support of P, Q and R. Each weight exerts a turning effect upon the rule, and, in addition, we have to take into account

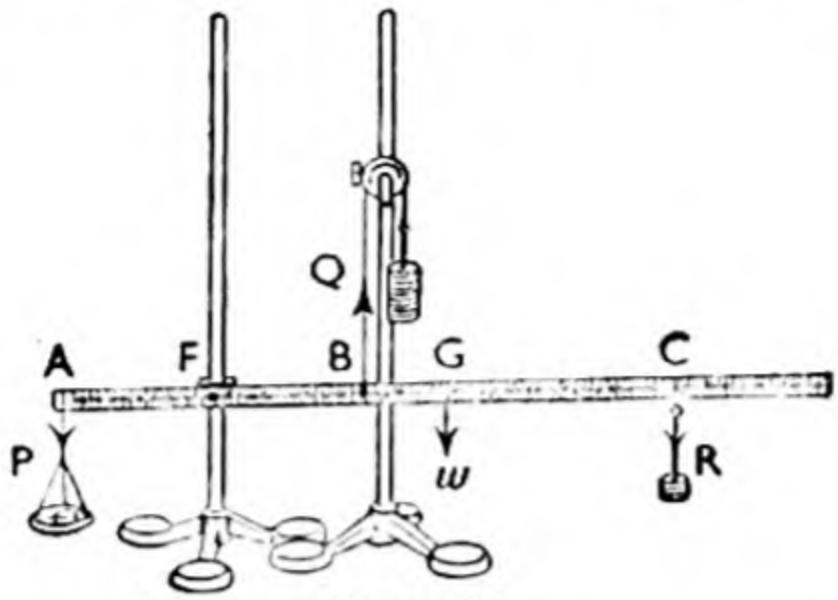

FIG. 38.

the turning effect of the weight of the rule. Since the rule is balanced in a horizontal position the turning effects of the various weights, some tending to turn the rule in a clockwise direction and the others in an anti-clockwise direction, must exactly balance each other.

Call G the middle point of the rule. Applying the result of the last experiment and referring to the diagram below, the sum of the clockwise moments is

$$w \times GF + R \times CF.$$

Also the sum of the anti-clockwise moments is

$$P \times AF + Q \times BF.$$

Calculate each of these sums and verify that they are equal. Now support the rule at F by means of a spring balance. Note the reading of the spring balance and the distances from one end of the rule to the points of application of all the forces. Calculate the sum of the moments of all the forces acting on the rule about this end, which are in a clockwise direction. The tension alone registered by the spring balance or the weights alone make up this moment according to whichever end of the rule we take. Now, calculate the sum of the anti-clockwise moments about the same end of the rule. Verify that the sum of the clockwise moments is equal to the sum of the anti-clockwise moments about the same point. Thus the sum of the anti-clockwise moments minus the sum of the clockwise moments about the same point of the rule is zero. It is usual to call the moments of the forces which tend to turn the rule in an anti-clockwise direction about a point POSITIVE MOMENTS, and those which tend to make it turn in a clockwise direction NEGATIVE MOMENTS. **We may thus state that the algebraic sum of the moments of the forces acting on the rule in this experiment about the point F in the first place and about one end in the second place is zero.**

Finally, support the rule by two or more spring balances, and load it as already described. Show that the algebraic sum of the moments of the forces acting on the rule about two or more points along the rule chosen at random is zero.

23. Example (1).—AB *represents a straight rod whose weight* W *acts at* G. *Forces* P *and* Q *act vertically upwards at the points* C *and* D *respectively on the rod. A force* W_1 *acts vertically downwards on the rod at* E.

It is required to find the forces P *and* Q, *given the distances of the points of application of* P, Q, W *and* W_1 *from the end* A, *and the magnitudes of* W *and* W_1 *when the rod is in equilibrium.*

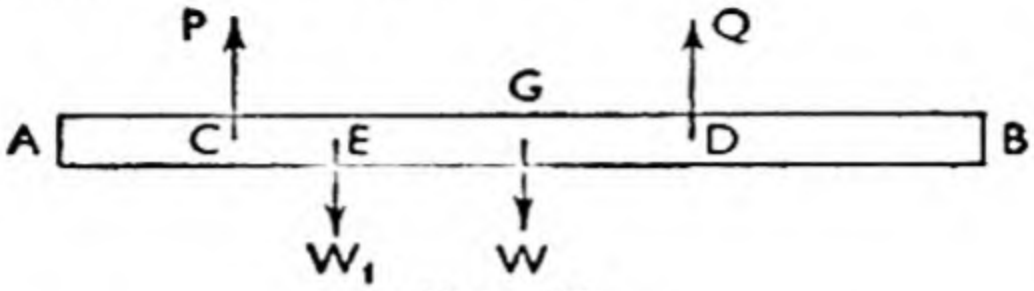

FIG. 39.

Since the rod is in equilibrium the sum of the upward forces acting on it is equal to the sum of the vertically downward forces,

$$P + Q = W + W_1.$$

Again, considering the moments of the forces acting on the rod about C,

$$(P \times O) - W_1 \times CE - W \times CG + Q \times CD = 0,$$

i.e.,

$$Q \,.\, CD = W_1 \,.\, CE + W \,.\, CG.$$

This equation at once enables us to calculate Q. Put this value for Q in the first equation and so calculate P. As an alternative to this last procedure P may be found by taking moments about D. **Notice that by taking moments about a point on the line of action of one of the forces, this force does not appear in the resulting equation.**

Example (2).—*A beam* 10 *ft. long carries weights of* 25, 40 *and* 50 *lb. at distances of* 2, 4 *and* 6 *ft. respectively from one end* A. *The weight of the beam,* 80 *lb., acts at its midpoint. Where should the pivot be placed in order that the beam may be balanced?*

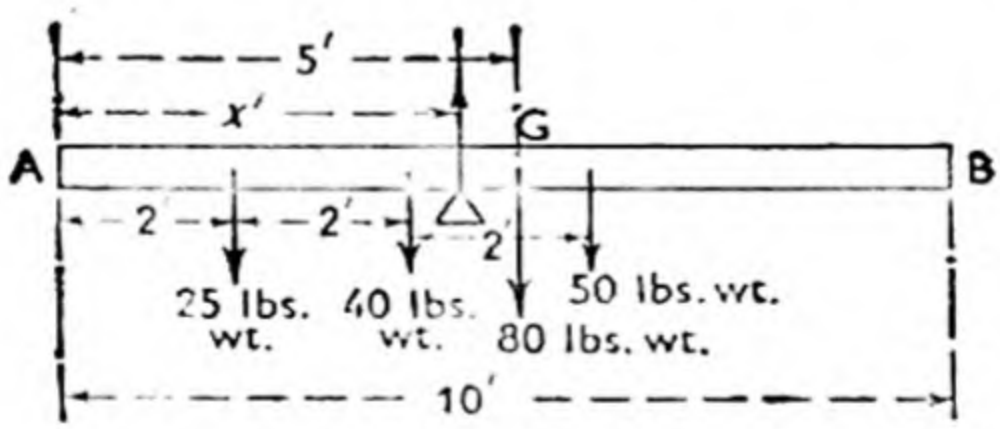

FIG. 40.

Let the pivot be placed x ft. from A between the 40 and 80 lb. wt. forces.

The 25 lb. wt. is $(x - 2)$ ft. from the pivot.
" 40 " " $(x - 4)$ " " "
" 80 " " $(5 - x)$ " " "
" 50 " " $(6 - x)$ " " "

Take moments about the pivot :

$$25(x-2)+40(x-4)-80(5-x)-50(6-x)=0,$$

$$195x=910,$$

$$x=4\tfrac{8}{13}.$$

The pivot should be placed 4 ft. 8 in. from A.

Example (3).—*A uniform beam* AB, 6 *ft. long and weighing* 80 *lb., rests on two supports, one at each end. A load of* 40 *lb. wt. is carried by the beam at a point* C, 2 *ft. from the end* A. *Calculate the pressures at the supports. If the support at* B *is moved to a point* 1 *ft. from* B, *calculate the new pressures on the supports.*

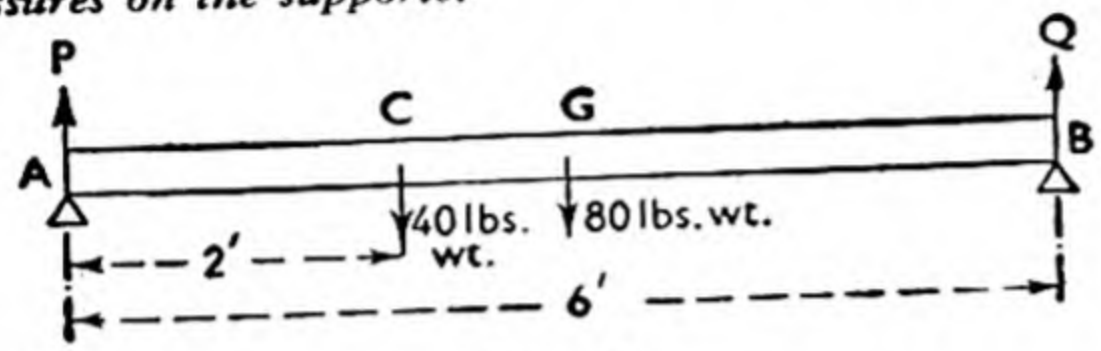

FIG. 41.

(i) Since the beam is uniform its weight acts vertically dowards at its middle point.

The forces acting on the beam may now be represented as shown on the diagram below :—

The sum of the pressures on the supports is equal to the sum of the weights of the beam and the load.

Thus if P lb. wt. =the vertically upward pressure at A and Q lb. wt. =the vertically upwards pressure at B,

$$P+Q=80+40,$$

$$P+Q=120.$$

Taking moments about A,

$$Q\times AB-80\times AG-40\times AC=0.$$

Therefore,

$$Q=\frac{80\times 3+40\times 2}{6}=53\tfrac{1}{3}.$$

Putting this value for Q in the first equation

$$P=120-53\tfrac{1}{3}=66\tfrac{2}{3}.$$

The pressure on the support at A is $66\frac{2}{3}$ lb. wt., and the pressure or. the support at B is $53\frac{1}{3}$ lb. wt.

(ii) Consider now the case in which the beam is supported at A and at a point 1 ft. from B.

Using the same notation as before to represent the pressures

$$P+Q=120.$$

Taking moments about A,

$$Q\times 5-80\times 3-40\times 2=0.$$

Therefore

$$5\,.\,Q=320,$$

$$Q=64.$$

Put 64 for Q in the first equation and we see that P = 56. The pressures on the supports are therefore,

56 lb. wt. at A vertically downwards
and 64 lb. wt. 1 ft. from B vertically downwards.

EXAMPLES VII

Moments of Parallel Forces

1. A uniform beam AB, of weight 20 lb. and length 6 ft., is supported at A and B. A weight of 10 lb. is placed 1 ft. from A. Find the pressures on the supports.

2. In question 1 the support at B is removed. Where should a new support be placed so that the pressures on the two supports may be equal ?

3. A uniform plank AB, 12 ft. long and of weight 40 lb., rests on a pivot 4 ft. from A and is supported at B by a vertical rope. Find the tension in the rope and the pressure on the pivot.

4. A uniform beam, of length 12 ft. and weight 48 lb., rests on supports at A and B. It carries weights of 16 and 20 lb. 2 and 4 ft. from A respectively. Find the pressures on the supports. If the supports are removed, at what point of the beam would it balance on one support ?

5. A uniform bar 2 ft. long weighs 2 lb. and is supported at its two ends. A 7-lb. weight is hung from the bar 6 in. from one end and a 4-lb. weight 9 in. from the other end. Find the pressures on the two supports. (*O.C.*)

6. The top of a trestle table is 10 ft. long, is uniform, and weighs 60 lb. The trestles are placed 2 ft. from each end. Weights of 10 and 20 lb. are placed on the table at distances of 1 and 3 ft. from one end. Find the load supported by each trestle and the position of the centre of gravity of the table top and its load. (*O.C.*)

7. Two men carry a weight of 60 lb. suspended from a pole 6 ft. long, which is held by one man at each end. Where must the weight be suspended that one of the men may have to exert a force of 25 lb. ?

8. A golf club balances about a point 26 in. from the end of the handle. When a 2-lb. weight is hung from that end, the balancing point has to be shifted through 17 in. What is the weight of the club ? (*O.C.*)

9. A heavy uniform beam AB, weighing $\frac{1}{2}$ ton, is supported at one end A and at a point C one-quarter its length from the other end B. A load of $1\frac{1}{2}$ tons rests on the end B. What force acts on the support A ? (*O.C.*)

10. A horizontal girder, weighing 21 lb. per foot length, rests at its ends on supports 20 ft. apart. Vertical loads of 1 and 2 tons are supported by the girder at points 5 and 14 ft. respectively from the left end. Find the reactions of the supports.

11. One end of a pole AB, 10 ft. long, lying on the ground can just be lifted off the ground by a force of 20 lb. applied 6 in. from A. A force of 30 lb. must be applied 1 ft. from B so as just to lift the end B. Find the weight of the pole and the distance of its centre of gravity from A. (*L.M.*)

CHAPTER IV

THE PRINCIPLE OF MOMENTS

24. The Turning Effects of Forces which are not Parallel.

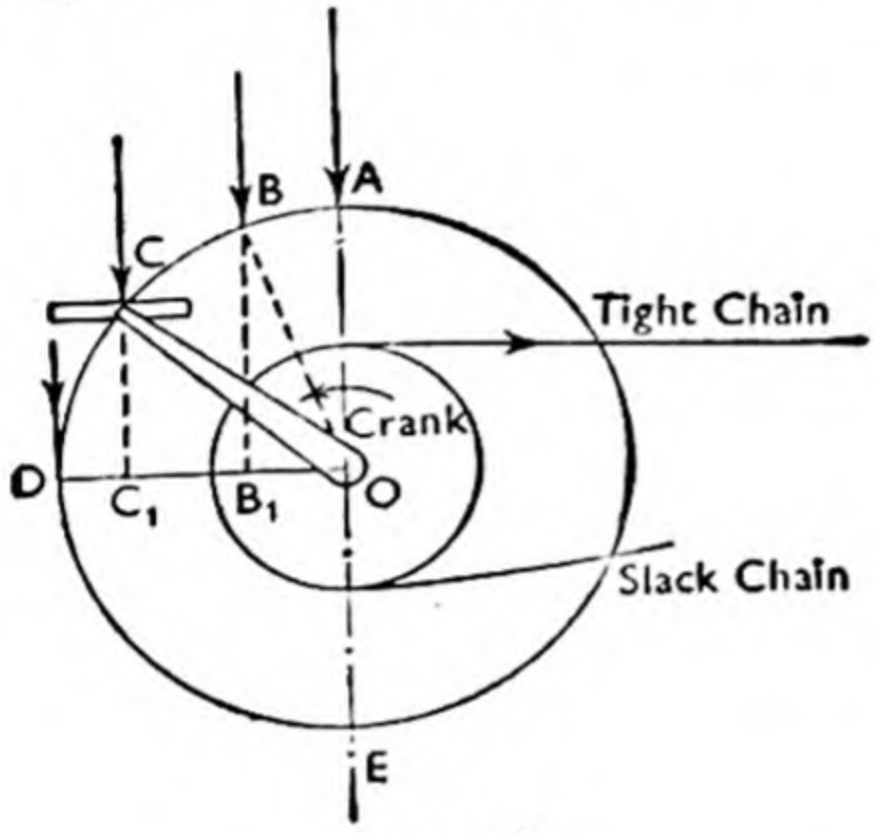

FIG. 42.

In the above figure OC is the crank of a bicycle, and the pedal is at C. Suppose that the force exerted on the pedal is always vertically downwards. The non-parallel forces acting are the vertically downward force on the pedal and the tension in the upper part of the chain on the teeth of the sprocket wheel. We know by experience that the force which is exerted upon the pedal by a cyclist is practically zero in the position OA, and gradually increases as the crank passes through the positions OB and OC until at OD the force is greatest. Afterwards the force decreases until in the position OE it is again least. This is especially noticeable if the going is at all difficult. Evidently a cyclist is aware that the force exerted on the pedal in the position D is more effective for turning the crank than in any other position. In the position OA a vertical force exerted on the

crank simply pushes the crank against the axle bearing at O, and has no turning effect about O If the same force be exerted vertically downwards on the pedal in all positions the turning effect of this force about O increases as the perpendicular distance from O to the vertical line through the pedal increases (compare OB_1, OC_1 and OD). If the friction at the axle bearing is neglected, the turning effect of the force exerted on the pedal is balanced by that of the tension in the upper part of the chain of the bicycle, and this tension is applied to the hub of the back wheel in the opposite direction. In the following experiment we shall examine more fully the turning effects of non-parallel forces acting on a body.

25. Experiment.—

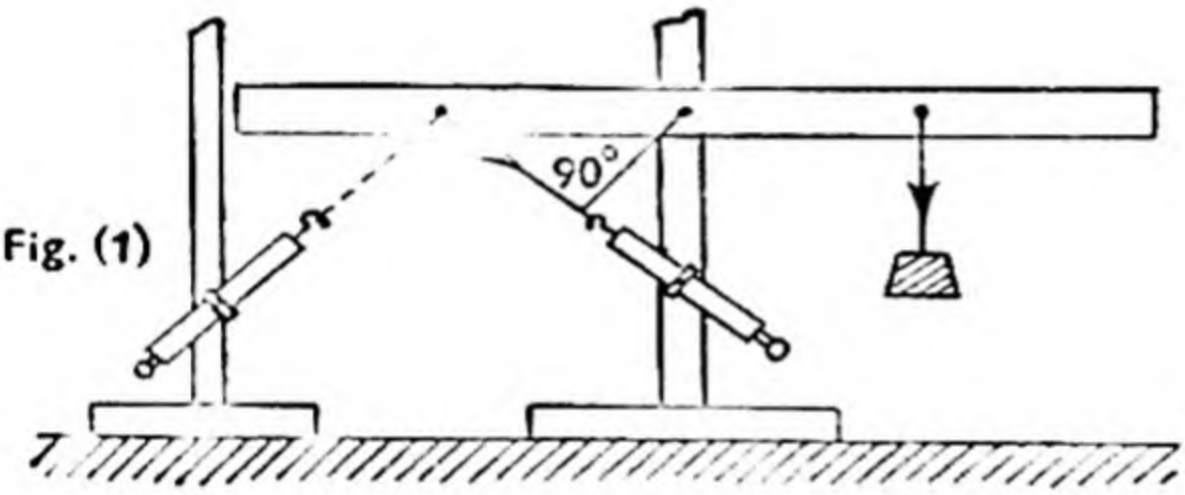

Fig. (1)

FIG. 43.

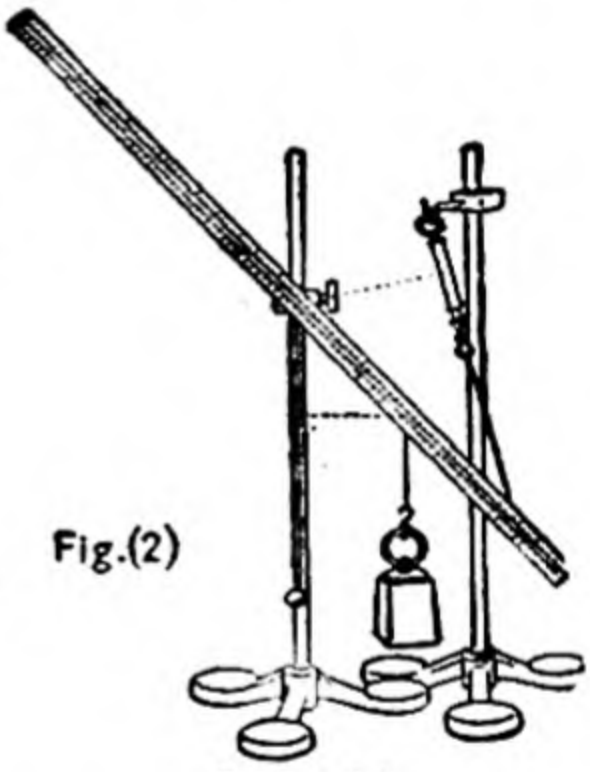

Fig.(2)

FIG. 44.

The object of the experiment is to find the turning effects on the rule of forces which are not parallel. Support the rule at its centre and apply a weight of 0·5 lb. at a distance of 25 cm. (say) from the pivot. Fasten a string through a hole in the rule near one end on the opposite side of the pivot. Fasten the other end of the string to a spring balance and the latter to a support in such a way that the string and spring are in one straight line inclined at an angle to the rule and in the same vertical plane. Take the reading of the spring balance. Alter the angle between the string and the rule, and when it is again balanced in a horizontal position take the reading of the spring (Fig. 1). Repeat for various different angles. Notice how the reading of the spring depends on the angle. Repeat the experiment, and for each reading of the spring measure the perpendicular distance from the pivot to the straight line in which the spring and string lie. Multiply each reading of the spring by the corresponding perpendicular distance, and observe that the products so obtained are substantially equal. Let us suppose

that the angles used have been less than a right angle. Repeat the experiment for angles greater than a right angle. In this case in order to measure the perpendiculars it is necessary to produce the line of the string above the rule. In any of the balanced positions in the above experiment, apply two or more times the weight which has been used up to the present. Keeping the string in the same direction, note the new tension in the spring required to maintain the rule in a horizontal position. We deduce from the above experiment that the turning effect of a force is not proportional to the distance from the pivot to the point of action of the force on the rule but to the PERPENDICULAR DISTANCE from the pivot to the straight line in which the force acts. As in the two previous experiments, the above experiment confirms that the turning effect of a force is proportional to the magnitude of the force. **Thus a measure of the moment of a force about a point may be taken as the product of the force and the perpendicular distance from the point to the line of action of the force.** In comparing the moments of various forces about a point it is essential that the same units of force and length should be used. Repeat the experiment with the rule inclined to the horizontal, and verify that the above result is still true (Fig. 2).

26. EXPERIMENT.—

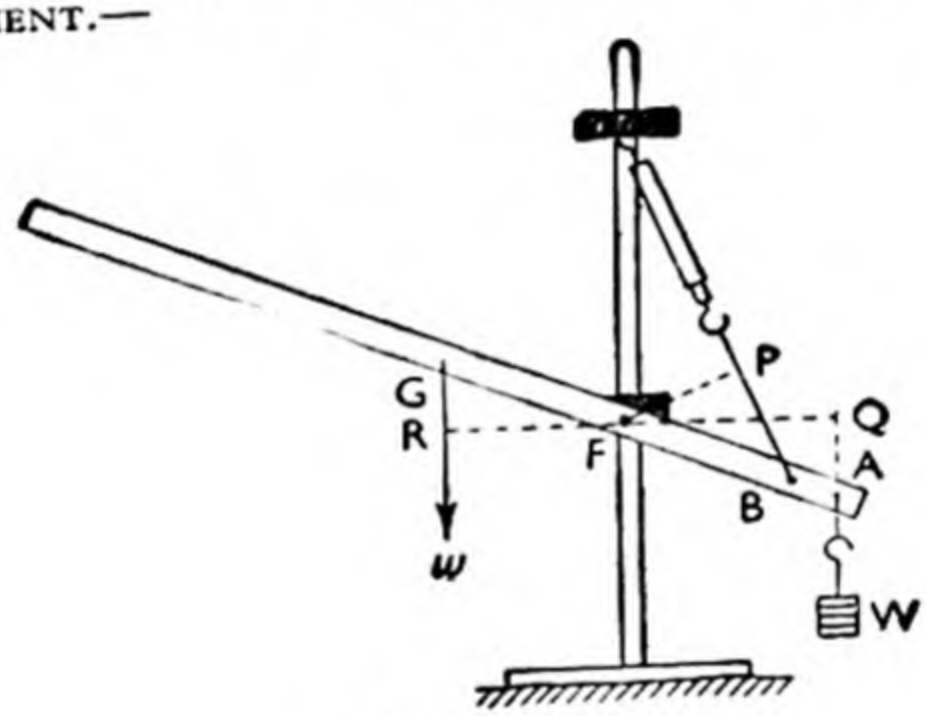

FIG. 45.

Pivot the rule at a point F and suspend a weight W at a point A to the right of F. Balance the rule by a tension T applied at a point B on the rule in a direction making an acute angle with the vertical. The tension may be applied by a spring balance or by a weight suspended from the string after it has been passed over a smooth pulley. Measure the perpendicular distances from the point F to the lines of action of all the forces acting on the rule, and verify that the algebraic sum of the moments of the forces acting on the rule, about F, is zero. Thus, if FP is the perpendicular distance from F on to the line of action of T, referring to the diagram,

$$w \cdot FR + T \cdot FP - W \cdot FQ = 0.$$

If $\angle$ PBF, QFA and GFR are measured, this may be written

$$w \cdot FG \cdot \cos \angle GFR + T \cdot FB \cdot \sin \angle PBF - W \cdot FA \cos \angle AFQ = 0$$

EXPERIMENT.—The apparatus consists of a horizontal table on which a sheet of tin is free to roll on steel balls. A number of horizontal forces are arranged to act upon the tin by means of weights attached to

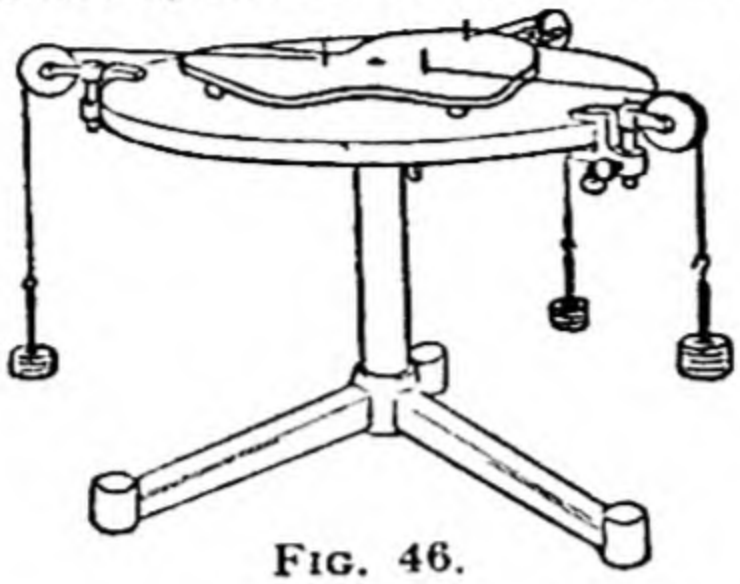

FIG. 46.

strings which pass over pulleys secured to the table. When the tin is at rest, mark the lines of action of the tensions acting upon it, together with their magnitudes and directions *as they act upon the tin*. Remove the strings from the tin and take any point O in it. From this point draw perpendiculars to the lines of action of the tensions. Calculate the sum of the clockwise moments about O and the sum of the anti-clockwise moments. Hence show that the algebraic sum of the moments of the tensions which kept the tin at rest about O is zero. Take two other points, O_1 and O_2, such that O, O_1 and O_2 are not in the same straight line, and verify the PRINCIPLE OF MOMENTS as for the point O. (Three forces only are shown in the drawing.)

EXPERIMENT.—Determine the position of the centre of gravity of a sheet of tin and arrange it so that its plane is vertical and it is at rest under the action of a number of forces which act in its plane. Mark the positions of the lines of action of the forces, INCLUDING THE WEIGHT OF THE TIN on its surface, together with their magnitudes and directions. (This means that the weight of the tin should be found.) As in the previous experiment, verify the principle of moments.

27. The Principle of Moments.

We are now in a position to state the principle of moments in the following form :—

WHEN A NUMBER OF FORCES ACTING ON A BODY IN ONE PLANE KEEP THE BODY IN EQUILIBRIUM THE ALGEBRAIC SUM OF THEIR MOMENTS ABOUT ANY POINT IN THEIR PLANE IS ZERO.

Units.—When force is measured in lb. wt. and distance in feet, the moment of a force is in lb. wt. ft.

28. Rigid Bodies.

Forces acting upon a body cause a change in the shape of the body. We describe a body as being RIGID if the alteration in its

shape caused by forces which act upon it is so small that the effect of this change is negligible. Consider the particular case of the turning effect of a force which acts upon a body. The change in shape caused by the force will result in a corresponding change in the perpendicular from the point about which the turning effect is considered to the line of action of the force. If this change in the perpendicular is less than, say, one-thousandth of its original length, then we may agree that this effect is too small to take into account for practical purposes. It may therefore be assumed that if a body is stated to be rigid it retains its original geometrical shape to a degree of accuracy sufficient for practical purposes.

Example (1).—*A weight of* 10 *lb. is supported in a vertical position by a string of length* 3 *ft. It is then pulled aside by a horizontal force until the string is inclined at* 30° *to the vertical. It is required to find the horizontal force.*

A is the point of suspension and B the position of the weight. BN is the perpendicular from B on to the vertical through A. Let F lb. wt. be the horizontal force which has a turning effect about A just sufficient to balance that of the weight of 10 lb.

Take moments about A.

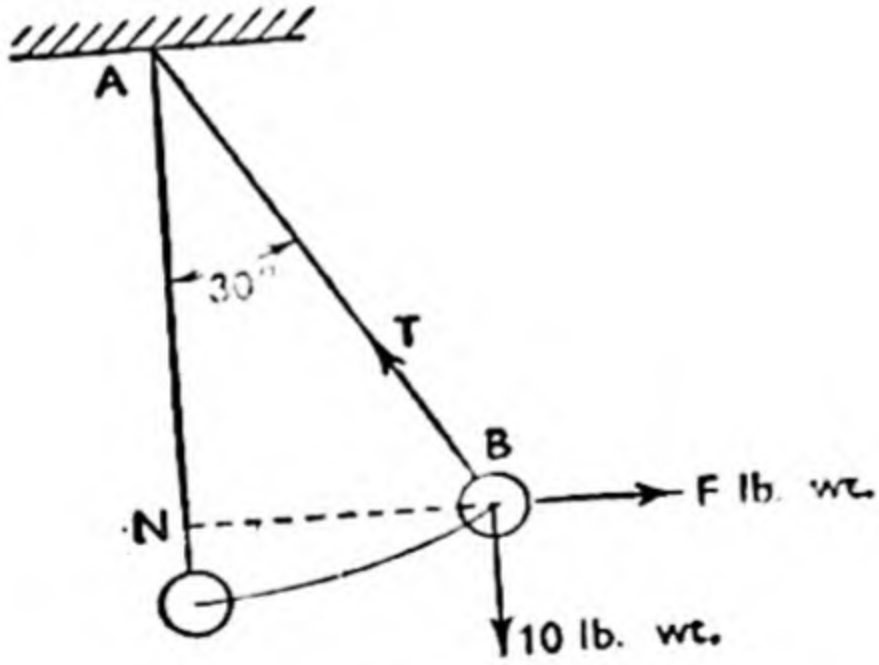

FIG. 47.

Since the tension T in the string has no turning effect about A,

$$F \times AN = 10 \times BN,$$

$$F = \frac{10 \times BN}{AN} = 10 \times \frac{BN}{AN} = 10 \times \tan 30° = 10 \times 0{\cdot}5774 = 5{\cdot}774.$$

Example (2).—ABC *is a jib-and-tie arrangement.* AB *is the jib which is uniform and of length* 10 *ft. The jib is hinged at* A *and the tie is secured to a point* C, 10 *ft. vertically above* A. *The jib is of weight* 1 *cwt. and carries at* B *a load of* 1 *ton wt.* ∠BAC *is* 45°. *It is required to find the tension in the tie.*

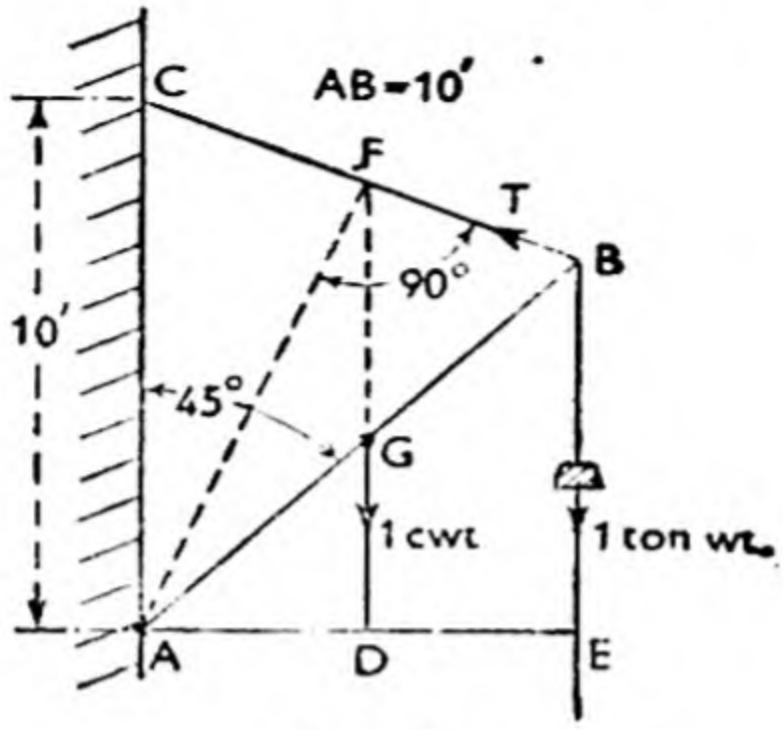

FIG. 48.

The forces which act on the jib are in equilibrium. They are :

The tension in the tie along BC.
The load of 1 ton wt. vertically downwards at B.
The weight of the beam vertically downwards at G its centre of gravity.
The reaction of the hinge at A.

Draw an accurate diagram to a suitable scale as large as possible, say, $\frac{1}{2}$ in. = 1 ft.

Draw ADE perpendicular to the lines of action of the load and weight of the beam and meeting them at E and D respectively.

Draw AF perpendicular to BC.

Let T cwt. be the required tension in BC.

Take moments about A, then

$$20 \,.\, EA + 1 \,.\, DA - T \,.\, FA = O.$$

If EA, DA and FA are measured, T can at once be calculated. Alternatively EA, DA and FA may be calculated as follows :—

$$EA^2 + EB^2 = AB^2,$$

but $$EA = EB,$$

therefore $$2 \,.\, EA^2 = 100,$$

$$EA = \sqrt{50},$$

$$EA = 7{\cdot}071,$$

$$\frac{AF}{AB} = \cos \angle BAF,$$

$$AF = 100 \times 0{\cdot}9239 = 9{\cdot}239,$$

$$20 \times 7{\cdot}071 + 1 \times 3{\cdot}536 - T \times 9{\cdot}239 = O.$$

$$T = \frac{144 \cdot 96}{9 \cdot 239} = 15 \cdot 7.$$

The tension in the tie is 15·7 cwt.

Example (3).—*A garden roller has a diameter 2 ft. 6 in. and weight 2 cwt. It is required to pull it over a step 3 in. high, as shown in the diagram. The handle is pulled in a direction making 60° with the vertical. What force is required just to lift the roller from the ground? If this force is maintained can the roller be drawn over the step? In what direction should the handle be pulled so that the force applied to pull the roller over the step may be the least possible? Find this smallest force.*

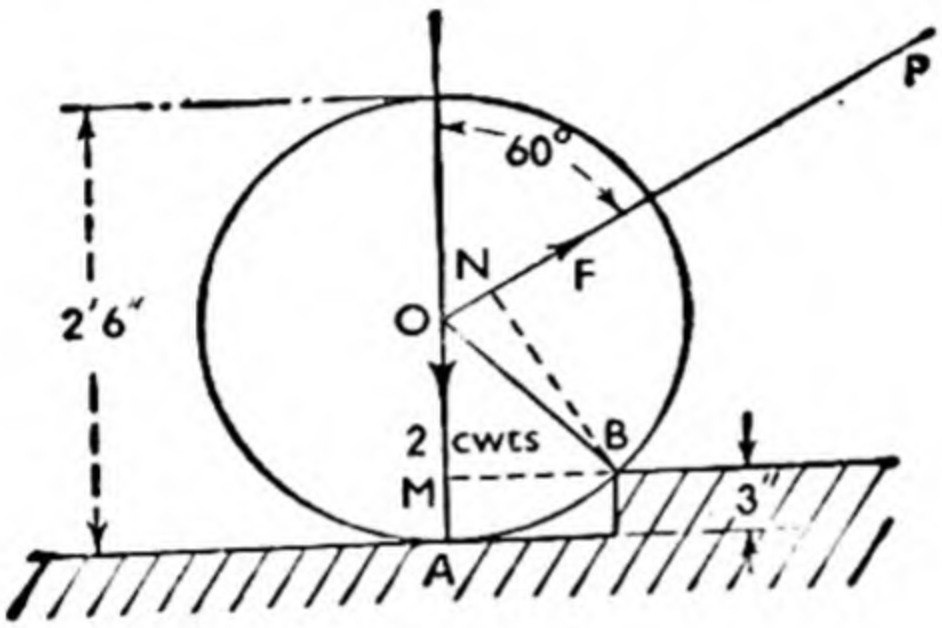

FIG. 49.

The roller is in contact with the ground at A, and B is the edge of the step. OP is the direction of the handle. Draw an accurate scale diagram. Draw the perpendiculars from B to OA and OP, BM and BN respectively.

Let F cwt. be the force along OP, which has a moment about the pivot B just sufficient to balance the moment of the weight of the roller about the same point. Then

$$F \cdot BN = 2 \cdot BM,$$

$$F = \frac{2 \cdot BM}{BN}.$$

Measure BM and BN and thus calculate F.

Otherwise,
$$BM^2 = OB^2 - OM^2 = (1\tfrac{1}{4})^2 - 1^2 = \tfrac{9}{16},$$
$$BM = \tfrac{3}{4}.$$

Now,
$$\sin \angle MOB = MB/OB = \tfrac{3}{4}/\tfrac{5}{4} = 0 \cdot 6,$$
$$\angle MOB = 36° 52'.$$

Therefore $\angle BON = 83° 8'$,

$$\frac{NB}{OB} = \sin \angle BON = 0{\cdot}9929,$$

$$NB = 1{\cdot}25 \times 0{\cdot}9929 = 1{\cdot}241.$$

Thus $$F = 2 \times 0{\cdot}75/1{\cdot}241 = 1{\cdot}21.$$

The force which just lifts the roller is 1·21 cwt.

If this force is maintained the roller will be pulled over the step, for as OB turns about B, the turning moment of F about B increases, for BN increases.

When BN is greatest the force required to move the roller is least. Thus F should be applied perpendicular to BO in the first place. The smallest force is therefore given by

$$F = 1{\cdot}5/1{\cdot}25 = 1{\cdot}2.$$

The least force required is 1·2 cwt.

EXAMPLES VIII

1. The figure illustrates a yarn strength tester which, when used for doubled yarns, has a 1-lb. weight attached to the end of the indicator arm. If the pulley to which the indicator arm is fastened is 3½ in. diameter,

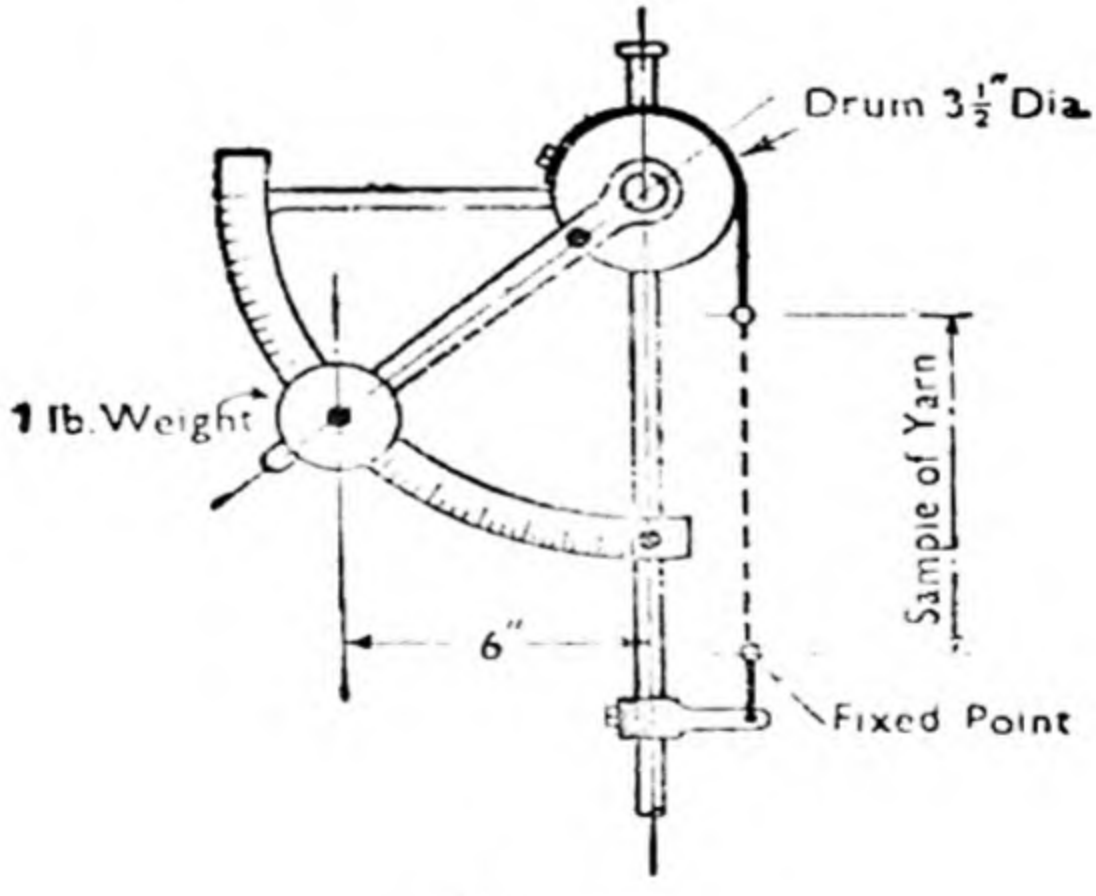

FIG. 50.

calculate the pull on the yarn when the weight is 6 in. from the vertical centre line of the pulley.

2. The arms of a bell-crank lever are at right angles to each other, and

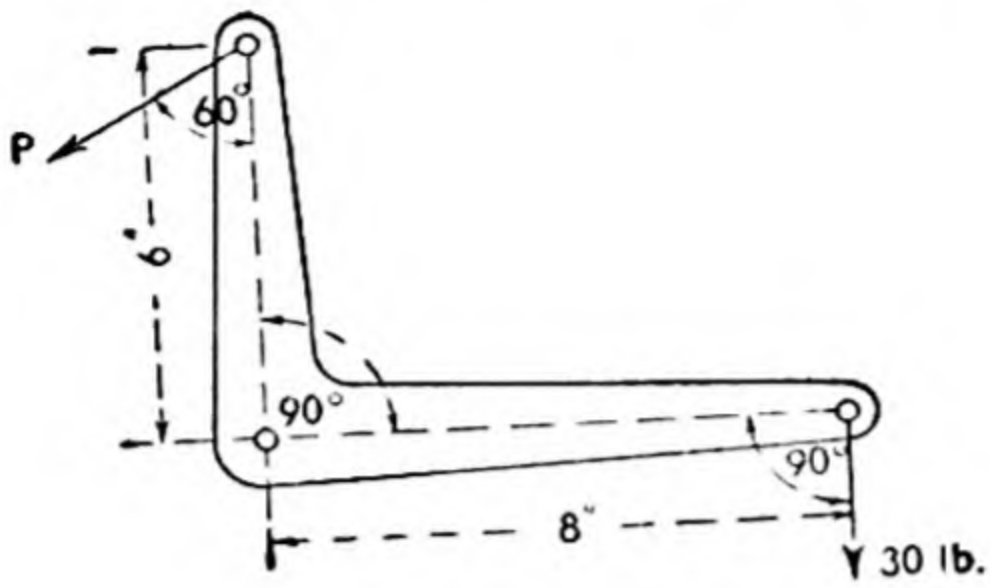

FIG. 51.

are 8 and 6 in. long respectively. A force of 30 lb. wt. acts at the end of the 8-in. arm, as shown in the figure. What should be the magnitude of force P acting at the end of the 6-in. arm in order to keep the lever in equilibrium? (Neglect the weight of the lever.)

3. DE is a flagstaff of length 50 ft. and weight 150 lb. lying horizontally and to be raised by a rope AD. The flagstaff will turn about E; what is the tension in the rope when the flagstaff is just clear of the ground (except at E)? $\angle ADE$ is 30°, and A, D and E are in the same vertical plane, and the centre of gravity of the flagstaff is 20 ft. from E.

4. A uniform bar AB is hinged at B and carries a load of 100 lb. at A. The bar weighs 50 lb. and is 6 ft. long. It is supported in a horizontal position by a chain attached to A and to a point C vertically above B. The perpendicular BD from B on to AC is 3 ft. in length. Calculate the tension in the chain AC.

5. A square sheet of metal ABCD of uniform thickness and weight 100 lb. is hinged along CD, which is horizontal. AB is 3 ft. long, and its midpoint is fastened by a chain to a point 3 ft. vertically above the midpoint of CD. Calculate the tension in the chain when:

(*a*) ABCD is horizontal.
(*b*) ABCD is inclined at 60° to the vertical.

6. A rectangular block of stone, 1 ft. × 2 ft. × 3 ft. of weight 800 lb., rests on a horizontal plane on one of its largest faces. What is the least force which, applied to one of its upper edges in a horizontal direction, will raise the stone

(*a*) so that it may rest on a 1 ft. × 2 ft. face;
(*b*) so that it may rest on a 1 ft. × 3 ft. face?

(Assume that there is no slipping.)

7. In a jib-and-tie arrangement the jib AB is hinged at A, and is inclined at 45° to the vertical. The tie BC is horizontal, and a load of 1 ton wt. is carried at B. What is the tension in the tie? If the tie is perpendicular to AB, calculate the corresponding tension. (Neglect the weight of the jib.)

8. A leaden weight of 4 lb. is suspended by a string 3 ft. long. A horizontal force applied to the lead maintains it at rest in a position such that the string is inclined at 60° to the vertical. Find this horizontal force.

9. A uniform rod AB, of length 2 ft., is hinged at A. Initially it hangs in a vertical position. It is drawn aside by a horizontal force of 10 lb. applied at B, and takes up a position of rest inclined at 30° to the vertical. Calculate the weight of the rod.

10. A vertical wireless pole 24 ft. high is pulled horizontally at its top point by a force of 4 lb. In the same vertical plane as the pole and this force, a wire is attached to the pole 12 ft. from its base and secured to a point in the ground. If the wire makes an angle of 45° with the pole and balances the turning effect of the horizontal force, find the tension it must carry.

11. A garden roller of weight 3 cwt. is to be pulled over a step of height 6 in. The diameter of the roller is 2 ft. 6 in., and the force is to be applied along the handle, which is to remain inclined at 60° to the vertical. State where the force exerted is least, and calculate this least value.

12. From the particulars given in the figure, find:

(*a*) The force exerted by the brake on the brake drum.
(*b*) The force that must be applied by the operative's foot (as represented by the arrow A) in order to relieve entirely the pressure of the brake.

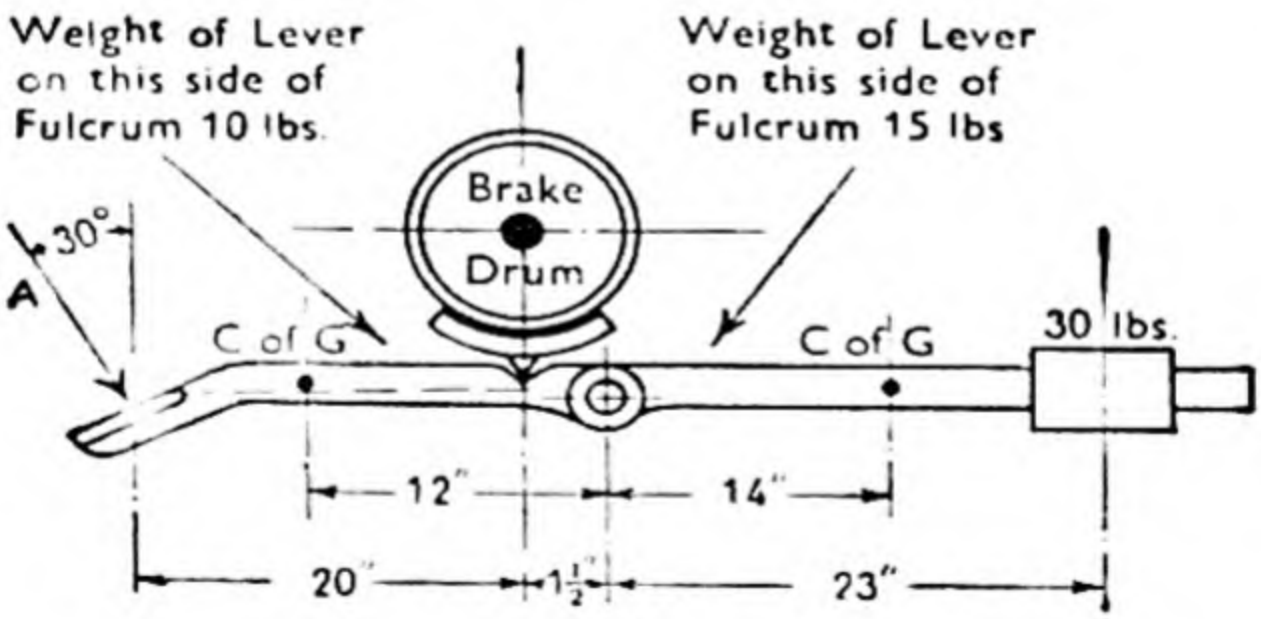

FIG. 52.

13. Explain what is meant by the moment of a force about a point. What is the least tangential force necessary to roll a uniform cylinder, 2 ft. in diameter and weighing 250 lb., up a 4-in. step? (*G.S.*)

14. Define the moment of a force about a point.

A uniform cylinder of diameter 2 ft. and height 2 ft. and weighing 180 lb. stands on level ground with its axis vertical. What is the least force applied at the upper edge which will cause tilting? (*G.S.*)

15. The tail board of a lorry is 6 ft. wide and 2 ft. tall, and it is hinged to the floor. It is allowed to drop so that it is in a horizontal plane and is supported in this position by two strings, which are at 45° to the board in parallel planes at right angles to it, *i.e.*, they are fastened to the lorry 2 ft. vertically above the hinge of the board. If the board weighs 30 lb., find the tension in the strings.

Find what this tension becomes when a boy weighing 60 lb. sits on the board so that his centre of gravity is above the central line of the board, *i.e.*, 3 ft. from either side, and 6 in. from the outer edge of the board. (*L.M.*)

16. What is meant by the moment of a force about a point? Explain how you would carry out experiments to show the application of the theory of moments to a lever, with the forces on the same side of the fulcrum.

A fisherman's rod, 8 ft. long, firmly fixed at one end and at an angle of 45°, is pulled with a force of 2 lb. at right angles to the rod and at a distance of 1 ft. from the end. The line runs to a fish from the top of the rod at an angle of 30° to the horizon. Find graphically or otherwise the tension the fish exerts on the line. (*O.*)

17. A non-uniform beam, 5 ft. long and weighing 25 lb., rests with one end on rough ground, and a horizontal rope attached to the upper end maintains the beam at 45° to the vertical by means of a pull of 10 lb. By applying the principle of moments, find the point in the beam at which its weight acts. (*J.*)

18. BC and BD are a pair of shear legs used for lifting heavy weights out of a ship, and AB is the back leg. A, C and D are in a horizontal

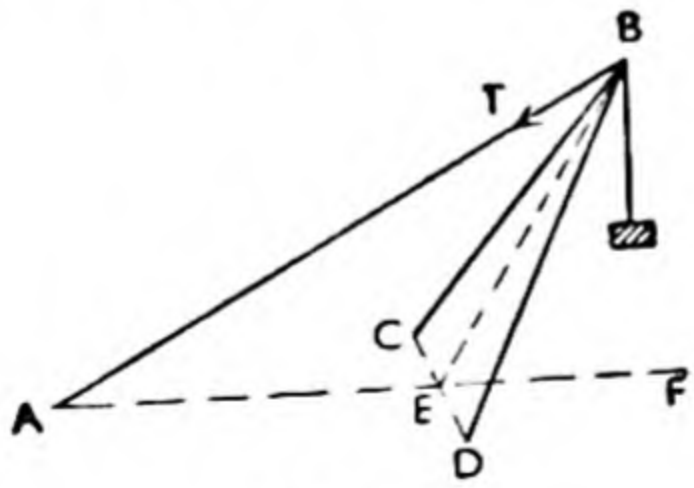

FIG. 53

plane. AE is perpendicular to CD, and E is the midpoint of CD. CD = 10 ft., BD = CB = 13 ft., BE = EA and AB = 19·2 ft. If the load carried at B is 1 ton, calculate the tension in AB.

19. A steel plate is tilted about one end from a horizontal to a vertical position by a force applied at the other end perpendicular to the length.

If the plate weighs 490 lb., what is the force necessary when the plate is (*a*) horizontal, (*b*) at an angle of 30°, (*c*) at an angle of 60° ? (*L.M.*)

20. A uniform rigid bar CD, of weight 100 lb., carries at D a weight of 500 lb. Calculate the thrust in the jib AB.

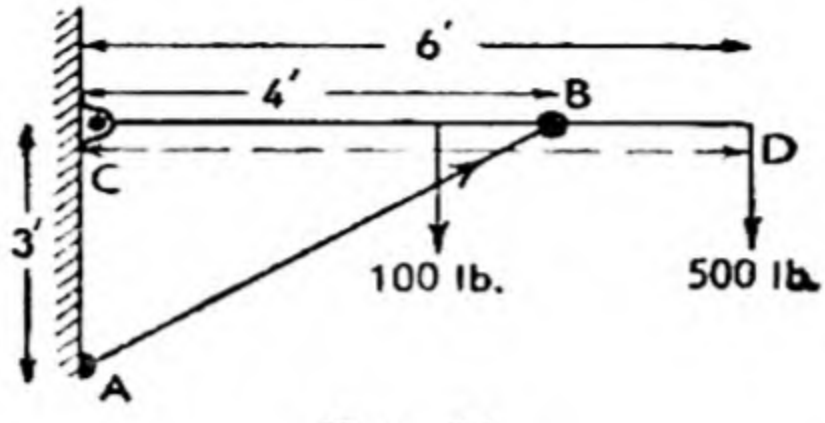

FIG. 54.

CHAPTER V

MACHINES

29. Distributed Force or Pressure.

It is convenient in practice to describe a force as acting at a point in a body when in fact its effect is distributed over a small area. All that we require in such a case is that the position of the point of action of the force may be taken as the position at which the whole effect of the force may be considered to act for practical purposes.

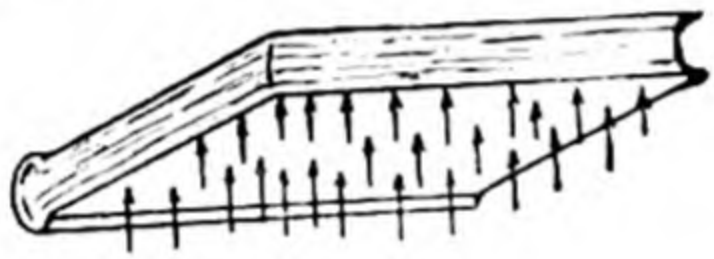

FIG. 55.

In some cases the effect of the force is distributed over a comparatively large area. Thus if a book rests upon a table the upward force of the table on the book is distributed over the surface of the book which is in contact with the table. If the book is supported on the unsharpened end of a pencil, the force exerted on the book is now distributed over a much smaller area, and, in fact, we may state that the book is supported at a point which can be located with an accuracy of, say, $\frac{1}{8}$ in. In this way a distributed force may be considered to act as if its whole effect were concentrated at a point in the body.

Consider the usual method of supporting brick walls which have a tendency to sag outwards. The force distributed over the metal surface of the plate C is balanced by the tension in the metal rod. In such a case this tension may be stated to act at a point in the wall located to the nearest inch. Thus a statement that a force acts at a point implies that the whole effect of the

force may be considered to act at that point, and the accuracy

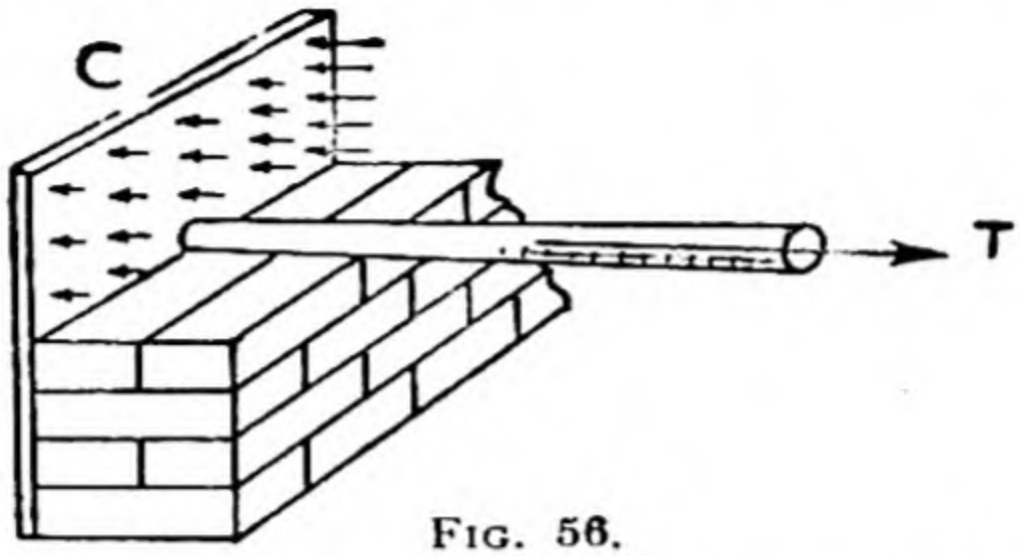

FIG. 56.

of its location depends upon circumstances such as have been described in the two above examples.

Distributed force is sometimes called PRESSURE, and it is usual to describe pressure as force per unit area. Thus the total force exerted by steam in the cylinder of an engine upon the piston may be 20,000 lb. wt., and if the area of the piston is 100 sq. in. the pressure of the steam may be described as 200 lb. wt. per sq. in.

30. Machines.

We usually associate machines with movement and think of them as performing work of one kind or another. We shall return to this idea of machines later, but for the time being let us see how the study of the turning effect of a force enables us to explain the action of a number of the simpler machines.

We have learned that the moment of a force about a point may be measured by the product of the force, and the perpendicular distance from the point to the line of action of the force. Thus the turning effect of a force about a point may be made larger by increasing the force alone or increasing the perpendicular from this point to its line of action, or, lastly, by increasing both of these quantities. It is the first two cases which find important practical applications in machines, for by this means we are able to balance the turning effects of forces of widely different magnitudes.

31. Levers.

A lever consists of a rigid bar which is hinged or pivoted at a

point of the bar called the FULCRUM. The Greeks at the time of Archimedes (287-212 B.C.) divided levers into three classes, which are considered below.

32. Levers having the Fulcrum between the Effort and the Load.

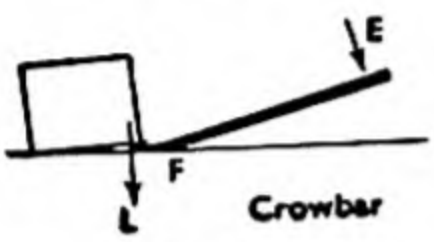

FIG. 57.

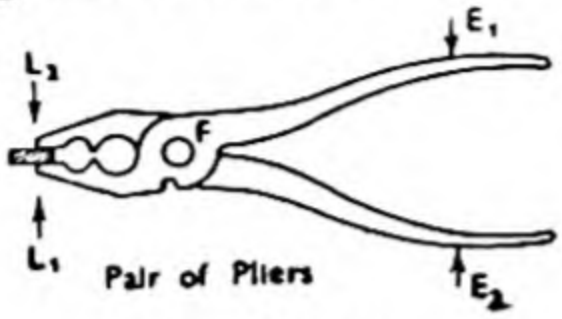

FIG. 58.

FIG. 59.

Consider the lever AB shown above, and suppose that the load and the effort are applied at A and B respectively in directions perpendicular to the lever. The moment of the effort E about the fulcrum F is balanced by the moment of the load L about the fulcrum, and therefore

$$L \cdot AF = E \cdot BF.$$

$$\frac{E}{L} = \frac{AF}{BF}.$$

By making the distance AF from the load to the fulcrum small in comparison with BF, the distance from the effort to the fulcrum, the effort required to balance a load is small in comparison with that load. Examples of this type of lever are :

(*a*) A crowbar.
(*b*) A pair of pliers. This is an example of a double lever.

33. Levers having the Load between the Fulcrum and the Effort.

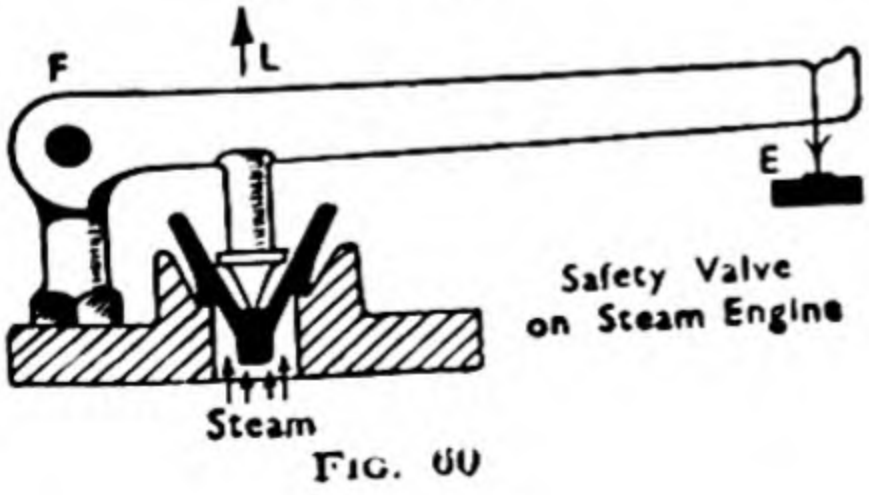

FIG. 60

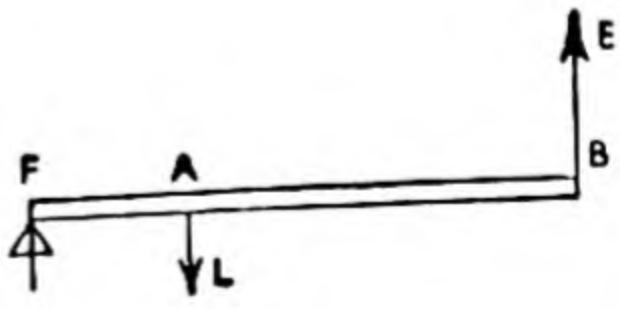

FIG. 61.

Levers of this type are sometimes referred to as being of the nutcracker type, since a pair of nutcrackers is an example of a double lever of this kind. The fulcrum is at one end of the lever and the load between the fulcrum and the effort. Referring to Fig. 61 on previous page and taking moments about the fulcrum F :

$$E \,.\, BF = L \,.\, AF.$$

As in the case of levers of the first kind, we may again balance the moment of the load about the fulcrum by an effort small in comparison with the load by making AF/BF small.

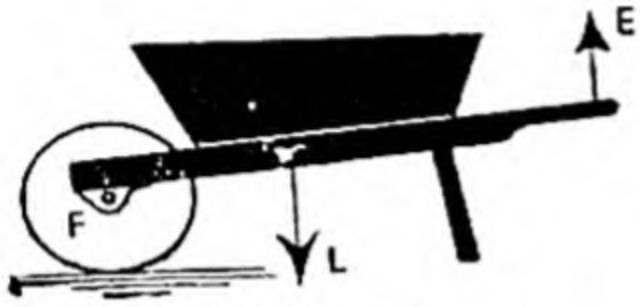

Fig. 62.

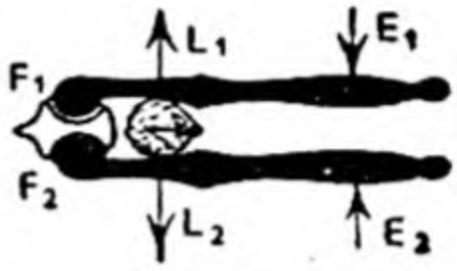

Fig. 63.

Examples of this type of lever are :

(*a*) A wheelbarrow.

(*b*) A pair of nutcrackers. (Double lever.)

34. Levers having the Effort between the Fulcrum and the Load.

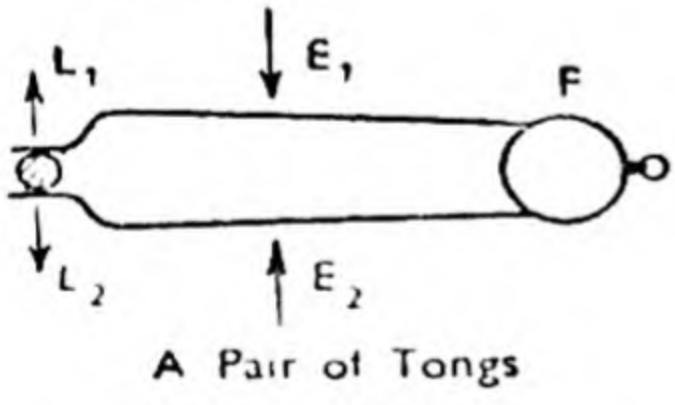

Fig. 64.

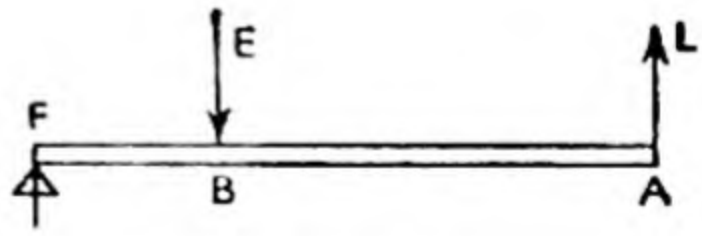

Fig. 65.

In this class of lever the effort is applied between the fulcrum, which is at one end of the lever, and the load, which is at the other. It thus follows that the effort applied is greater than the load in this case. The effort applied is " watered down," and an examination of levers of this kind shows that it is usually essential in their

application for a constant load to be maintained. Any variation in the effort automatically causes a *much reduced* variation in the load. Examples of this kind of lever are :

(*a*) A pair of tongs. (Double lever.)
(*b*) The treadle of a sewing machine.

Note.—Levers of the above three kinds are to be found in abundance on mechanical instruments such as pianos, sewing machines and bicycles, as well as the machines to be found in engineering workshops. It is left as an exercise for the reader to draw such levers and show carefully by arrowheads the directions in which the various forces act upon them.

35. Bent Levers.

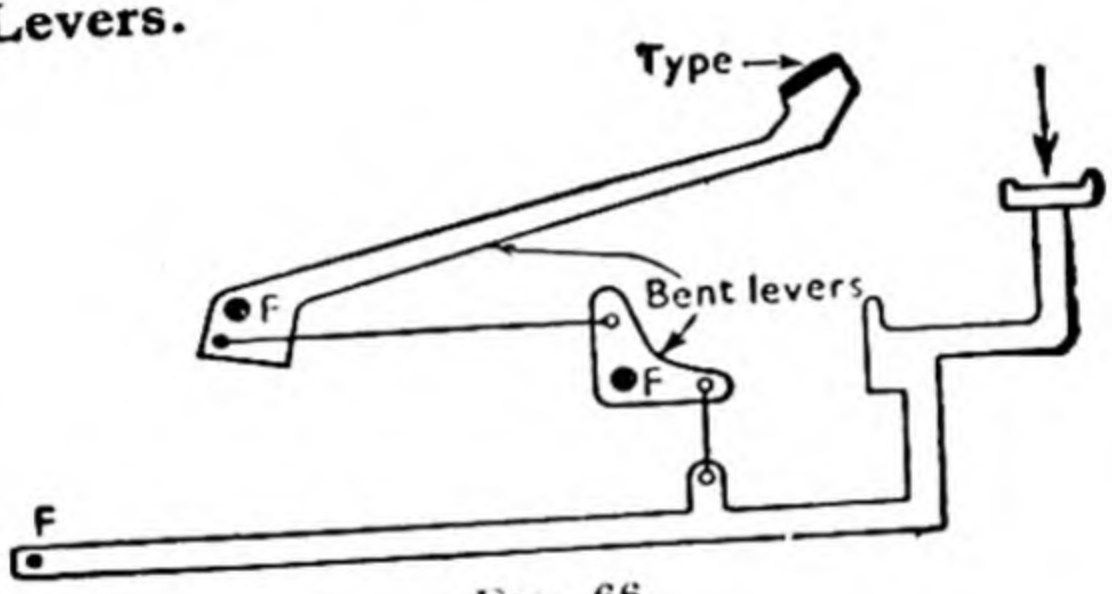

FIG. 66.

When the load upon a lever is in a direction which is not parallel to the direction in which the effort is applied, the lever usually consists of a bar which is bent at a point near the fulcrum, and so arranged that effort and load are in directions perpendicular to the inclined parts of the lever. What is the reason for this ? The above diagram shows a typical bent lever which is to be found on such a machine as the typewriter in which the type is propelled in quite a different direction from that in which the keys are struck.

36. The Wheel and Axle.

The wheel and axle are permanently attached. A rope which is wound round the axle has one of its ends fastened to the axle

and the other supports a load L. In the same way a rope which passes round a groove in the wheel is attached to a point in the

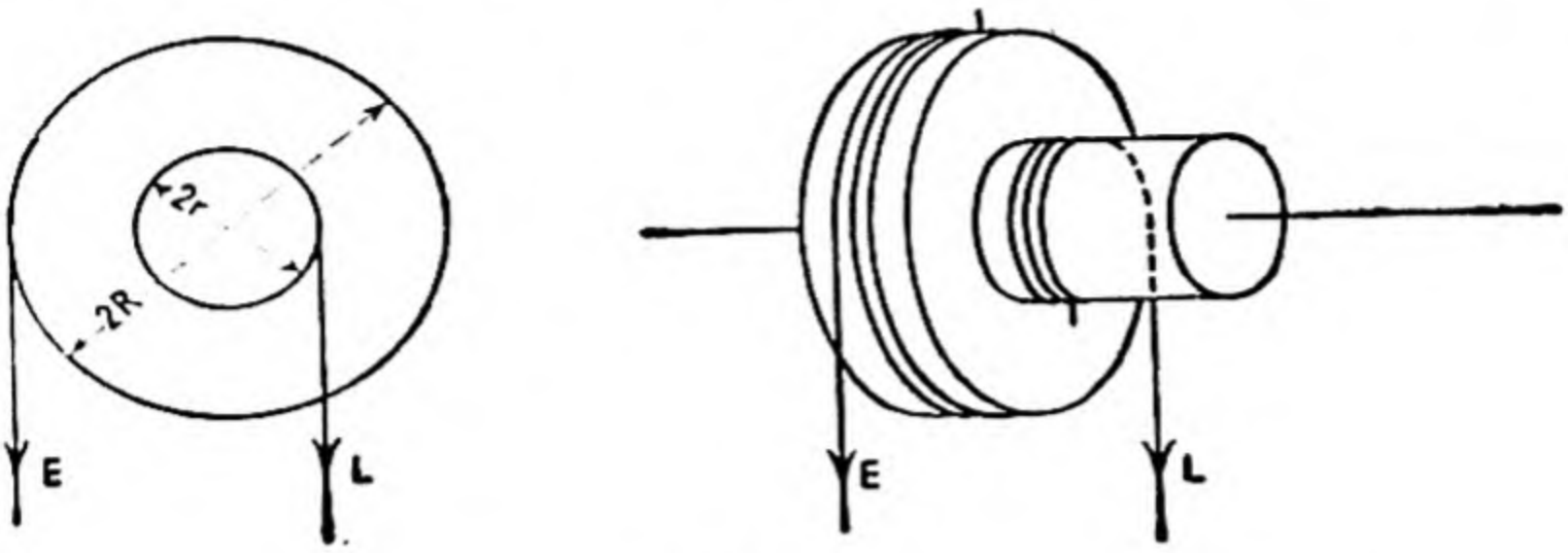

FIG. 67.

wheel and supports a weight at its other end. Let the weight E be such that it is on the point of descending and so raising the load. Remove E and replace it by a spring balance, which is at first held vertically. The reading of the balance is seen to be equal to E. Now rotate the balance in a vertical plane so that the load is still on the point of being raised. The reading of the balance remains the same. Thus so long as the pull of the rope round the axle is the same, the direction of the pull does not affect the load which can be supported. Let us now try different loads and determine the corresponding weights as above. We find that the ratio E/L is almost constant and equal to the ratio of the radius of the axle to that of the wheel. If R, r are the radii of the wheel and the axle

$$\frac{E}{L} = \frac{r}{R}.$$

The diameters are, of course, more easily measured than the radii.

If we consider the above diagram in the light of the previous experiments on the turning effect of a force, the above result may be arrived at by noticing that the turning effects of the load and the force applied to the wheel balance each other, and therefore

L × radius of the axle = E × radius of the wheel.

Moreover, provided that the tension in the rope round the wheel is the same, it is readily seen that its turning effect is the same whatever the inclination of the rope. By increasing the radius of the wheel in comparison with that of the axle, we may use this machine for raising large loads with comparatively small efforts.

37. The Windlass and the Capstan.

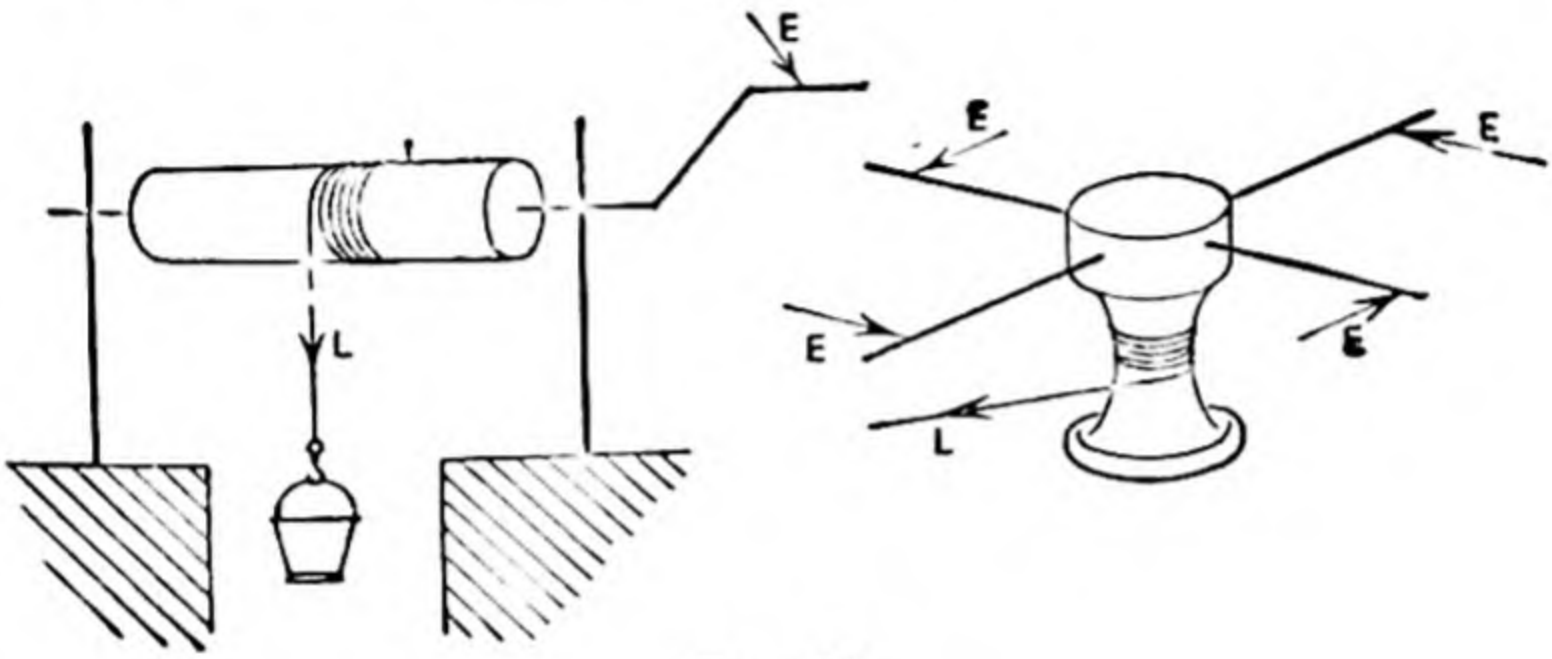

FIG. 68.

The windlass and the capstan are machines which work on the same principle as the wheel and axle. The barrel of a windlass is turned by a handle fixed centrally at one end of the drum. (In certain cases handles may be attached at both ends of the drum.) It is used for raising water from a well or similar hoisting purposes, and may also be used for haulage. The capstan consists of a vertical drum, and the effort is applied by means of horizontal bars secured to the top of the barrel. The bars act as levers, and several men pushing horizontally perpendicular to these bars turn the barrel about bearings at its lower end and thus wind cable on the drum; for example, the cable of a ship which is being secured in dock. The whole turning effect about the central axis of the drum exerted by the men is the sum of their individual turning efforts, and this sum balances the turning effect of the tension in the cable.

EXAMPLES IX

1. In the following arrangements of levers, determine the force P. All forces in lb. wt.

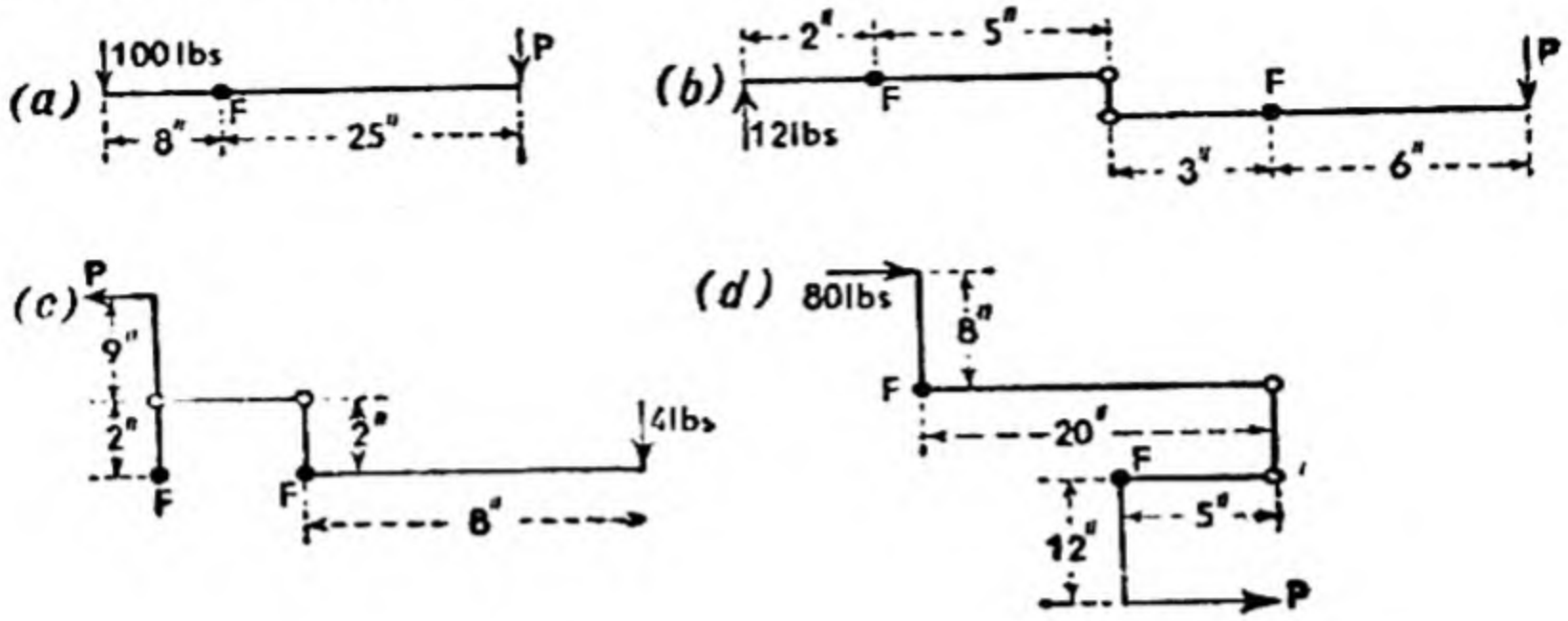

FIG. 69.

2. A wheel and axle is used to raise water from a well. The radius of the wheel (and that of the circle described by the handle attached to the wheel) is 1 ft. 6 in. and of the axle 4 in. What force must be applied to handle to raise a weight of 60 lb. ?

3. The axle of a wheel-and-axle arrangement has a diameter of 6 in. A force of 10 lb. wt. must be applied at the rim of the wheel to raise a weight of 45 lb. attached to a rope wound round the axle. What is the diameter of the wheel ?

4. Four sailors raise an anchor by means of a capstan, the radius of the drum being 5 in. Each man applies a force of 100 lb. to a spoke at a distance of 4 ft. 6 in. from the axis of the drum. Find the weight of the anchor.

5. A windlass consists of a barrel of diameter 6 in. to which is fastened a handle whose crank is 1 ft. 3 in. What force should be exerted perpendicular to the crank at its end to raise a pail of water weighing 24 lb. ?

6. Four men push the arms of a capstan horizontally, with equal forces perpendicular to the arms and at equal distances of 5 ft. from the axis of the barrel. A rope is coiled round the barrel, which has a diameter of 1 ft. at the point where the rope leaves the barrel. If the tension in the rope is 5 cwt., with what force should each man push ?

7. An irregular-shaped block of stone weighing 800 lb. is to be tilted about an edge A by means of a crowbar 4 ft. long, the fulcrum being placed 6 in. from the stone. The horizontal distances from A of

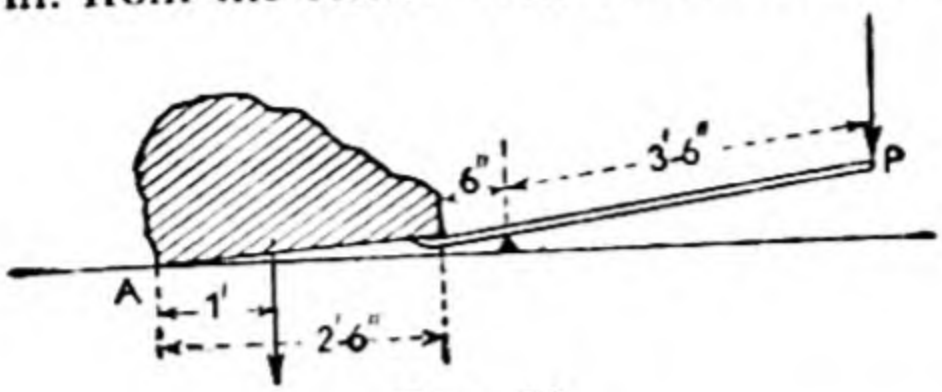

FIG. 70.

the point of contact of the crowbar and the centre of gravity of the block are respectively 2 ft. 6 in. and 1 ft. Find the effort which must be applied at the end of the crowbar.

38. The Steelyards.

FIG. 71.

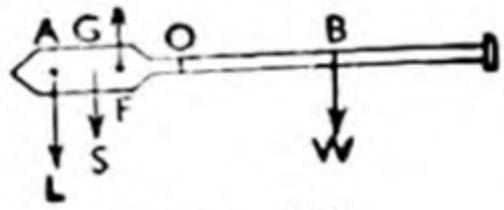

FIG. 72.

The Roman Steelyard.—This machine is used to determine the weight of a body, and is still in use to-day. It consists of a steel bar pivoted at a point F, the centre of gravity of the bar being at G to one side of the pivot. On the opposite side of the pivot to G is a long, straight, graduated bar along which a movable weight W slides. When W is moved to the zero graduation of the bar, the steelyard is balanced. Suppose that at a fixed point A, situated near the end of the bar opposite to the graduated portion, a load L is supported. In order to balance this load, W is moved along the graduated bar. The load L is then read directly from the graduated scale. In order to understand how

the graduations may be made, suppose that O is the position of the zero graduation and B the graduation corresponding to the load L. If S is the weight of the machine, then for no load

$$S \cdot GF = W \cdot OF.$$

Corresponding to the load L:

$$L \cdot AF + S \cdot GF = W \cdot BF = W \cdot (OF + BO).$$

Taking the first equation from the second, we see that

$$L \cdot AF = W \cdot BO.$$

Now AF is constant and W is constant, therefore the length BO is proportional to the load. Thus, equal increases in the load result in equal displacements of W along the graduated bar. If the total length of the graduated portion is $14 \times AF$, then 1 lb. wt. on the "14" graduation corresponds to a load of 1 st. wt. The intermediate graduations therefore measure the load in lb. wt.

Now near the end of the graduated bar, if we arrange a support for weights which balance loads of 1 st., 2 st., etc., with the riding weight at zero, the steelyard can be used to weigh in stones and lb. wt. In this form the machine is often referred to as the COMMON STEELYARD.

The Danish Steelyard.—In this machine the weight of the

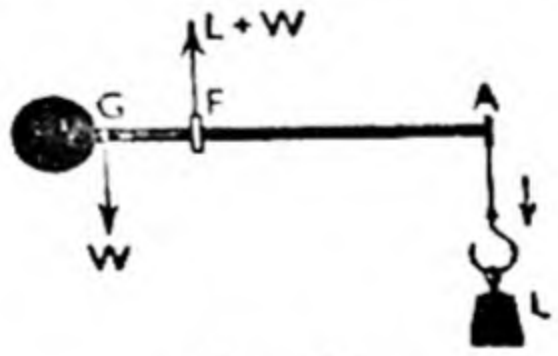

FIG. 73.

bar acts near one end at G. The load is applied at the opposite end in a fixed position, whereas the pivot is movable. If there is no load the pivot is at G. Suppose the load is applied at A and the pivot has then to be moved to F to balance the machine.

Let L be the load and W the weight of the bar. Since the bar is balanced the sum of the vertically upward forces is equal

to the sum of the vertically downward forces; therefore the upwards force of the pivot on the bar is $(L + W)$.

Taking moments about A,

$$(L + W) \cdot AF = W \cdot GA,$$

$$AF = \frac{W \cdot GA}{(L + W)}.$$

If $L = W, 2W, 3W \ldots$

$$AF = \tfrac{1}{2}GA, \tfrac{1}{3}GA, \tfrac{1}{4}GA \ldots$$

The graduation of the bar is thus readily carried out.

39. The Laboratory Balance.

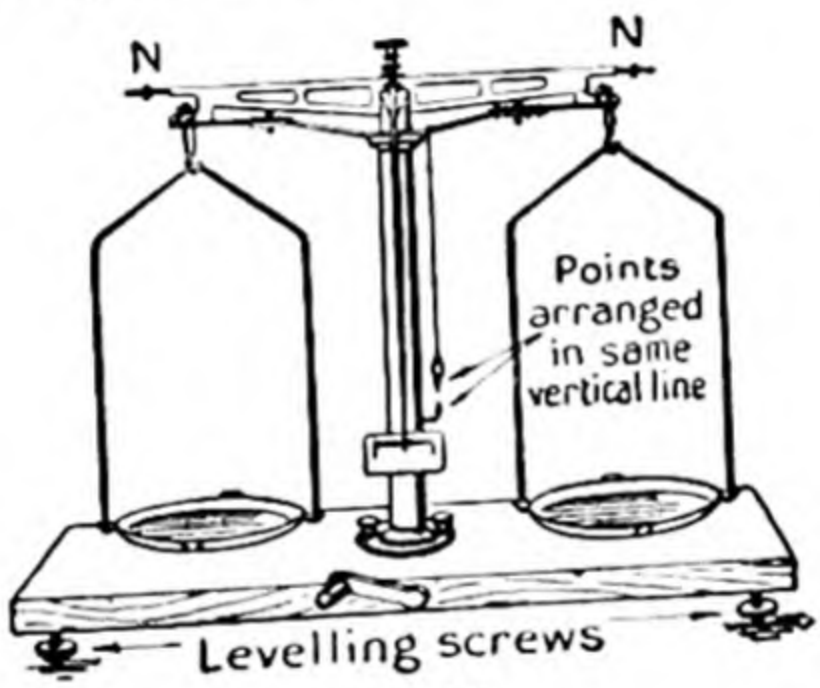

FIG. 74.

A laboratory balance consists of a beam which carries two scale pans suspended from points near its ends. A pointer is fixed to the beam, and the centre of gravity of the beam and pointer is just below the pivot about which the beam turns. The beam is only allowed to swing gently, and the pointer then oscillates in front of a fixed scale. When the scale pans are empty the beam is made to swing in such a way that the pointer passes over an equal number of scale divisions on either side of the central graduation. If the beam does not swing symmetrically the positions of the small nuts N at its extremities may be adjusted until it does so. If the weights of equal masses placed in two pans are to balance each other, the points of the beam from which

If a and b are the lengths of the arms of the balance, taking moments about the pivot,

$$Wa = W_1 b.$$

(*c*) Place the body in the right-hand pan and add weights W_2 in the other pan to balance it.

Taking moments about the pivot,

$$W_2 a = Wb.$$

Dividing the two equations,

$$\frac{W}{W_2} = \frac{W_1}{W},$$

$$W^2 = W_1 W_2,$$

$$W = \sqrt{W_1 W_2}.$$

Thus the correct weight is the square root of the product of the two apparent weights with the body first in one pan and then in the other.

When the arms of the balance are very nearly of equal length, the average of W_1 and W_2, viz., $\frac{W_1 + W_2}{2}$ is a closer approximation to the correct weight W of the body than either W_1 or W_2. If W_a is the average value, $W_1 = W_a - w$, and $W_2 = W_a + w$ where w is a small quantity in comparison with W_1 or W_2.

Then
$$W_1^2 - W^2 = W_1^2 - W_1 W_2 = W_1(W_1 - W_2) = -2wW_1,$$
$$W_2^2 - W^2 = W_2^2 - W_1 W_2 = W_2(W_2 - W_1) = 2wW_2,$$
$$W^2 = W_1 W_2 = (W_a + w)(W_a - w) = W_a^2 - w^2.$$
$$\therefore \quad W_a^2 - W^2 = w^2.$$

Because w is small in comparison with W, w^2 is small in comparison with $2wW_1$ or $2wW_2$.

The value W_a is therefore a more accurate approximation to the true weight of the body than either W_1 or W_2.

41. The Sensitivity of a Balance.

In the figure above AB represents the beam in an unsymmetrical position which it has taken up when nearly equal

loads W_1 and W_2 are suspended from the knife-edges at A and B respectively. The angle θ is the small angle turned through

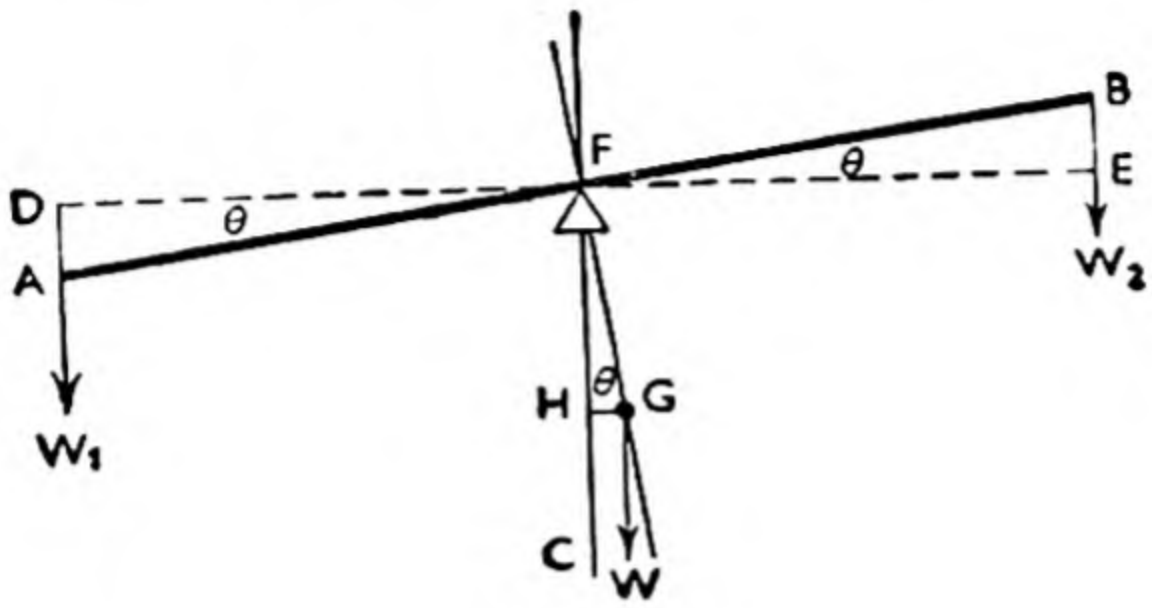

FIG. 77.

by the beam. G is the position of the centre of gravity of the beam and the point through which the weight W of the beam and pointer may be considered to act. $FG = h$.

If the moments of the forces W_1, W_2 and W acting on the beam are considered about the fulcrum F, and it is assumed that $AF = FB = l$, we have

$$W_1 \times DF - W_2 \times FE - W \times HG = 0,$$

$$W_1 l \cos\theta - W_2 l \cos\theta = W \,.\, FG \sin\theta,$$

$$(W_1 - W_2) l \cos\theta = W \,.\, h \sin\theta,$$

$$\frac{(W_1 - W_2) l}{Wh} = \frac{\sin\theta}{\cos\theta},$$

i.e.,

$$\tan\theta = \frac{(W_1 - W_2) l}{Wh}.$$

The sensitivity of a balance may be judged from the angle through which the beam turns before taking up its equilibrium position when weights which differ by a definite amount are carried in the scale pans. The greater the angle through which the beam turns under conditions the more sensitive it is judged to be.

The tangent ratio of an angle between 0° and 90° increases

as the angle increases. Thus the angle is large when tan θ is large. Referring to the formula for tan θ we see that

(*a*) a long beam l,
(*b*) a beam of small weight W,
(*c*) the centre of gravity of the beam a small distance h below the pivot,

are all factors which increase the sensitivity of a balance. In practice, however, a highly sensitive balance is found to require a long time for its swings, and is an extremely delicate instrument to manipulate. The common laboratory balance is built to withstand considerable wear and tear. It is well to remember, however, that weights should never be added to the pans when the beam and the pans are supported on their knife-edges, and the process of raising the beam and pans on to their knife-edge supports must be carried out gently.

EXAMPLES X

The Steelyards; the Balance

1. The bar of a common steelyard is 3 ft. long and weighs 4 lb. It is suspended at a point 3 in. from one end, and its centre of gravity is 4 in. from this end. If the movable weight is 5 lb., what is the greatest weight which can be weighed, assuming it to be suspended from a point 2 in. from the fulcrum?

2. A steelyard of weight 10 lb. (with the scale pan) is 4 ft. long, and its centre of gravity is 4 in. from the fulcrum on the side of it at which the scale pan is suspended. If the scale pan is suspended at a point 6 in. from the fulcrum, and the movable weight is 4 lb., find the distance from the fulcrum of the 20-lb. graduation mark.

3. A uniform rod is 2 ft. in length and weighs 1 lb. It is to be used as a steelyard, the fulcrum being 2 in. from the end to which the body to be weighed is attached. The sliding weight is $\frac{1}{2}$ lb. Find the greatest and least weights which can be weighed by the machine and the distance between successive 2-oz. graduation marks.

4. A steelyard is 18 in. long, and with the scale pan weighs 2 lb. The centre of gravity of the steelyard is 2 in. from the end at which the scale pan is attached. If the greatest weight which can be ascertained is 7 lb. and the movable weight is 1 lb., determine the position of the fulcrum and the distance between successive $\frac{1}{2}$-lb. marks on the graduated part of the steelyard.

5. A Danish steelyard with scale pan weighs 5 lb., and its centre of gravity is 2 ft. from the end at which the scale pan is suspended. At what point should the bar be supported when weighing 20 lb.?

6. When a Danish steelyard is used to weigh a 20-lb. weight the point of support is 3 in. from the end at which the scale pan is attached, and when used to weigh an 11-lb. weight the point of support is 5 in. from the

scale pan end. Find the weight of the steelyard (including the pan) and the distance of its centre of gravity from the scale pan end.

7. How is a steelyard used to find the weight of an object? Explain clearly the principles upon which it acts. What adjustment can be made so that still heavier objects may be weighed?

8. Describe a Danish steelyard and explain how it is graduated.

9. A Danish steelyard weighs 5 lb., and its centre of gravity is 2 ft. from the end at which the scale pan is supported. Find the distance apart of the graduations corresponding to 10 and 15 lb.

10. A rod AB is used as the beam of a simple balance, the pivot being at G, the centre of gravity of the rod, which is not uniform. When a 4-lb. weight is suspended at A, a $4\frac{1}{2}$-lb. weight at B is required to balance it. If AG is 9 in., find the length of AB.

11. The two arms of a balance are unequal, and its centre of gravity is at the fulcrum. A 12-lb. weight at the end A of the beam balances the weight of a body suspended at the end B, and a weight of 14·52 lb. at the end B balances the weight of the same body suspended at A. What is the weight of the body?

12. Describe a laboratory balance, and state the factors which influence its sensitivity.

13. Describe two methods of obtaining the true weight of a body when using a balance whose arms are unequal in length. Give numerical examples to illustrate your arguments.

14. The centre of gravity of a balance beam is not at the fulcrum. A 4-lb. weight in one scale pan is required to balance a 5-lb. weight in the other, and when the 5-lb. weight is transferred to the first scale pan a weight of 6·1 lb. in the second pan is required to balance it. Find the ratio in which the fulcrum divides the beam.

CHAPTER VI

CENTRE OF GRAVITY AND STABILITY

42. Centre of Gravity.

We may consider a body as made up of a very large number of particles extremely small in size compared with the body. The attraction of the earth on these particles forms a large number of equal parallel forces. Since the particles are very small the weight w of each particle may be considered to act at the position occupied by the particle.

The weights of any two such particles P and Q may be replaced by twice the weight acting at the midpoint of PQ.

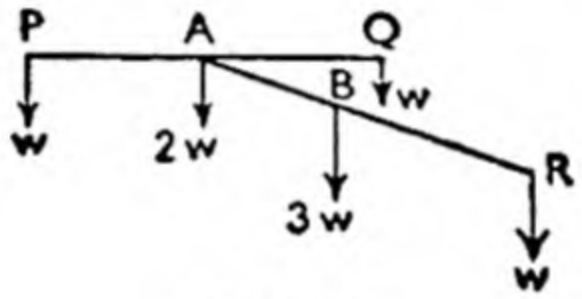

FIG. 78.

The force $2w$ may in like manner be combined with the weight of another particle at R. Thus the weight $2w$ at A and w at R may be replaced by their resultant $3w$ at B, such that

$$2w \,.\, AB = w \,.\, BR.$$

Proceeding in this manner we see that eventually we arrive at a point at which the whole weight of the body may be considered to act.

If the body is moved into any other position exactly the same procedure as has been described may be repeated and the points A and B, etc., obtained as before. *The same point as before is arrived at.* This point is called the centre of gravity of the body.

The centre of gravity of a body is a point at which the whole weight of the body may be considered to act, no matter in what position the body is placed.

The same definition of centre of gravity applies to any system of particles or bodies. For convenience in this case, however, we may imagine the particles or bodies to be rigidly connected by a light framework, so that we can still suppose the system to be turned into another position.

43. Centre of Gravity of a Thin Uniform Rod.

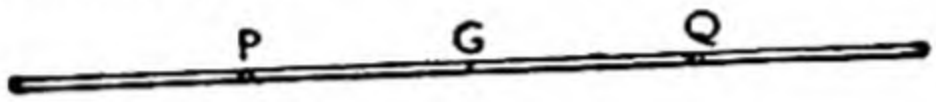

FIG. 79.

Consider any pair of equal particles P and Q of the rod situated at equal distance from the centre G. The weight of the pair of particles may be replaced by a particle at G of weight equal to twice the weight of each particle. The rod may be divided into pairs of particles similar to the above, which make up the whole rod, and it is therefore readily seen that the centre of gravity of the rod is at its centre G.

44. Centre of Gravity of a Lamina in the Form of a Parallelogram.

A plane lamina is a very thin sheet whose thickness is so small that it may be neglected in describing the position of the centre of gravity of the lamina.

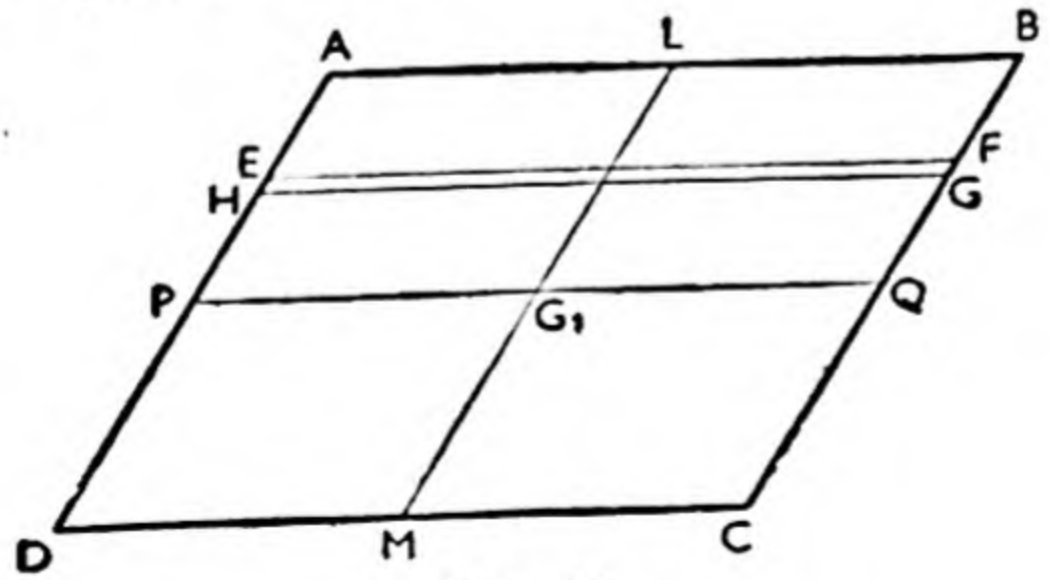

FIG. 80.

ABCD is a parallelogram. Divide the figure into thin strips, such as EFGH parallel to AB.

The line LM joining the midpoints of AB and CD bisects all straight lines, such as EF drawn parallel to AB.

When the strip EFGH is made extremely thin its centre of gravity lies on LM, and its weight may therefore be considered to act at this point.

Similarly, the centre of gravity of all the strips which make up the parallelogram lie along LM.

The centre of gravity of the parallelogram therefore lies along LM.

In a similar way it may be shown that the centre of gravity of the parallelogram lies on PQ, the line joining the midpoints of BC and DA.

The centre of gravity of the figure is therefore at the point of intersection of LM and QP, *i.e.*, at G_1.

45. Centre of Gravity of a Triangle.

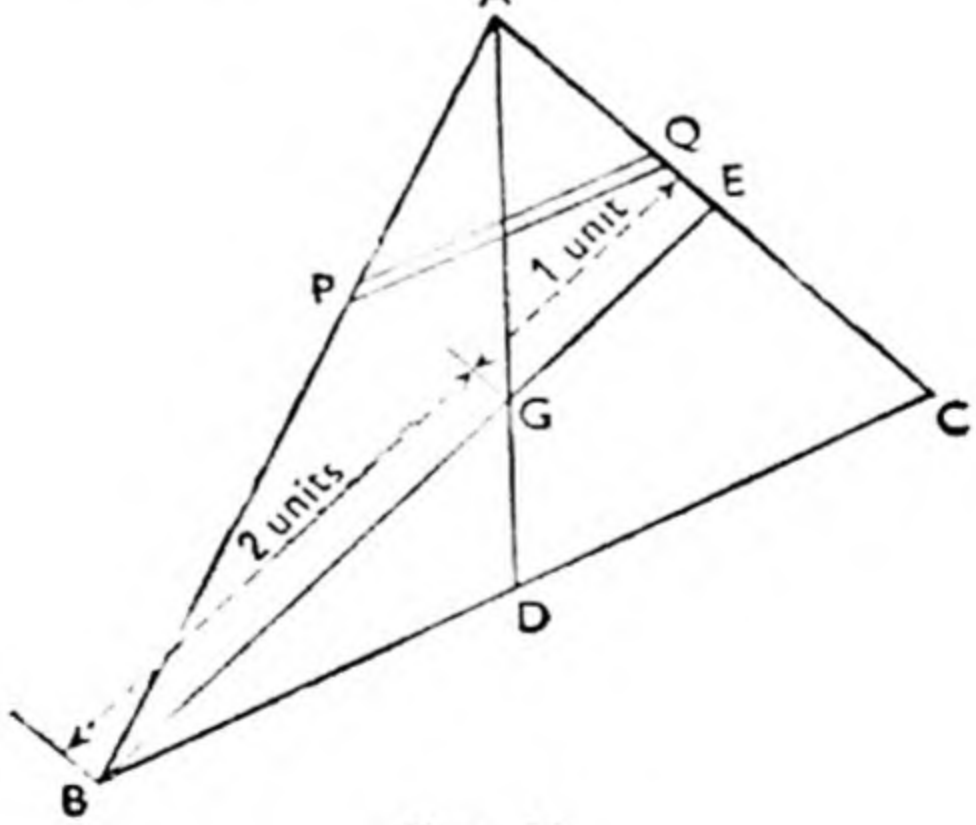

FIG. 81.

ABC is a triangular lamina. AD and BE are medians of the triangle which cut at G.

The median AD bisects all lines parallel to BC, which are terminated by the sides AB and AC.

Divide the triangle into thin strips parallel to BC.

The centre of gravity of a strip PQ lies on AD when the strip is made very thin.

Similarly, the centre of gravity of all strips which make up the triangle lie along AD, therefore the centre of gravity of the triangle lies in AD.

By dividing the triangle into strips parallel to CA, it may be shown that the centre of gravity of the triangle lies along the median BE. Hence the centre of gravity of the triangle is at the point of intersection of its medians, and divides each median therefore in the ratio 2 : 1.

46. The Centre of Gravity of a Number of Particles. (Geometrical Method.)

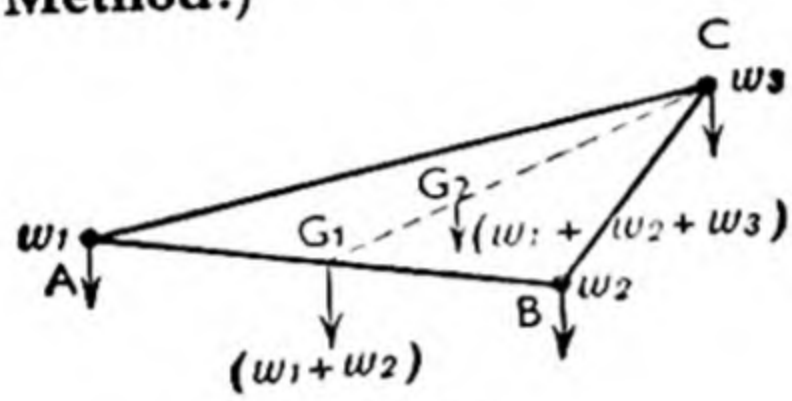

FIG. 82.

Let A, B, C be the positions of three particles of weights w_1, w_2, w_3 respectively.

The centre of gravity of w_1 and w_2 divides AB at G_1, such that

$$\frac{AG_1}{G_1B} = \frac{w_2}{w_1}.$$

Thus the whole weight $(w_1 + w_2)$ of the particles at A and B may be considered to act at G_1.

The centre of gravity of $(w_1 + w_2)$ at G_1 and w_3 at C is at G_2 such that

$$\frac{G_1G_2}{G_2C} = \frac{w_3}{(w_1 + w_2)}.$$

Thus the whole weight $(w_1 + w_2 + w_3)$ of the particles at A, B and C may be considered to act at G_2, which is therefore the centre of gravity of the particles.

It is readily appreciated that the above method may be continued and the centre of gravity of any number of particles found.

47. Centre of Gravity of Three Equal Particles at the Vertices of a Triangle.

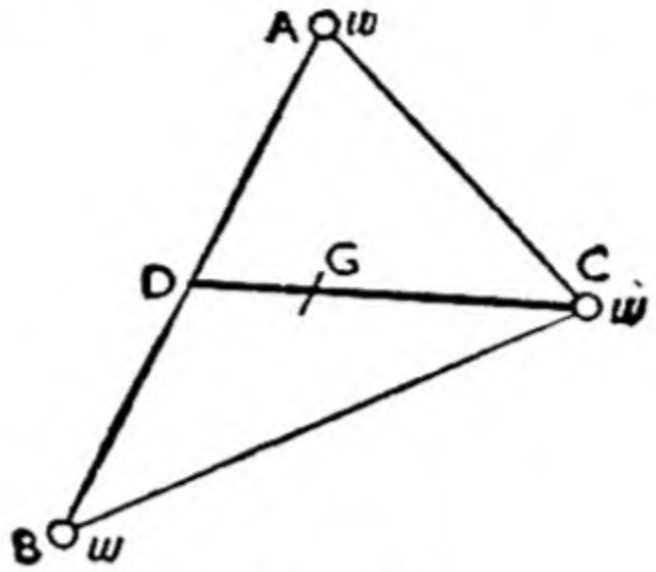

FIG. 83.

Let three equal weights w be situated at A, B and C the vertices of a triangle. w at A and w at B have a resultant $2w$ at the midpoint D of AB.

$2w$ at D and w at C have a resultant $3w$ at G, such that

$$\frac{DG}{GC} = \frac{w}{2w} = \frac{1}{2}.$$

Now CD is a median of the triangle ABC, and since it is divided at G in the ratio 2 : 1, it follows that G is the centre of gravity of the triangle ABC. Hence the centre of gravity of the three equal weights at A, B and C coincides with the centre of gravity of the triangular lamina ABC.

48. Centre of Gravity of a Quadrilateral.

Let ABCD be a lamina in the form of a quadrilateral.

Join AC and find G_1, the centre of gravity of the triangle ABC. Find G_2, the centre of gravity of the triangle ACD.

The centre of gravity of the whole quadrilateral lies on the line G_1G_2.

In the same manner if G_3 and G_4 are the centres of gravity of the triangles ABD and CBD respectively, the centre of gravity

of the quadrilateral lies on G_3G_4. Hence the centre of gravity of the quadrilateral is at G, the point of intersection of G_1G_2 and G_3G_4, and the only point common to both lines.

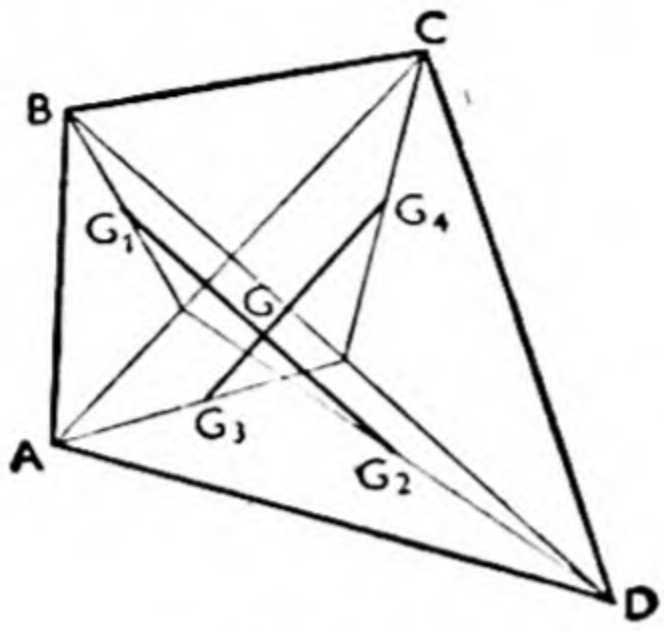

FIG. 84.

Imagine weights placed at G_1 and G_2 proportional to the areas of the triangles ABC and ACD respectively. The centre of gravity of these two weights coincides with that of the quadrilateral. Hence the centre of gravity of the quadrilateral is at G, a point on G_1G_2, such that

$$\frac{G_1G}{GG_2} = \frac{\Delta ACD}{\Delta ABC}.$$

The latter method may be used to find the centre of gravity of any plane lamina made up of triangles and parallelograms.

Suppose that the point B of the quadrilateral ABCD be raised vertically and that the lamina does not slide. Which of the edges AD or CD will remain in contact with the table? Where should the centre of gravity of the lamina be situated if the lamina can be raised at B with only the point D in contact with the table?

49. Centre of Gravity of a Prism of Uniform Density.

The centre of gravity of a prism whose plane ends are parallel may be found by imagining the prism to be divided into equal laminæ parallel to the plane ends. Let G_1 and G_2 be the centres of gravity of the plane ends of the prism shown in the figure

below. The laminæ may be replaced by equal weights placed along G_1G_2. The centre of gravity of the whole prism is therefore at the midpoint of G_1G_2.

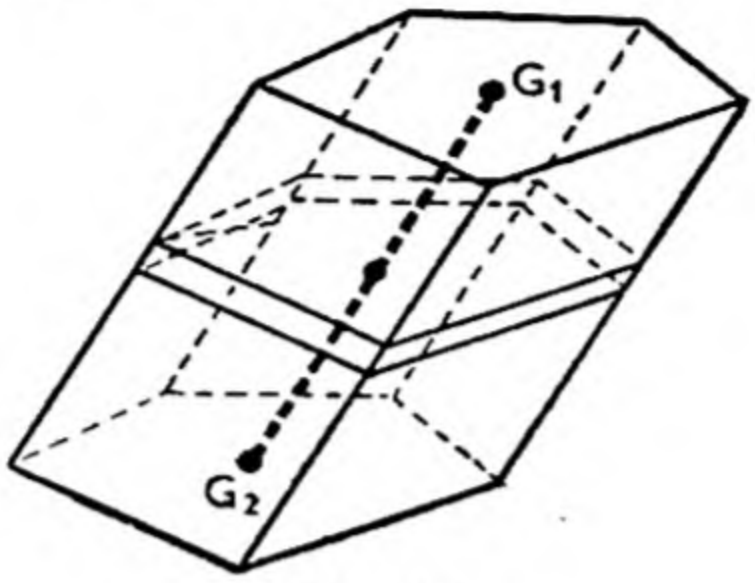

FIG. 85.

This argument is not restricted to cases in which the ends of the prism are perpendicular to the line G_1G_2.

50. Centre of Gravity of a System of Weights in One Straight Line.

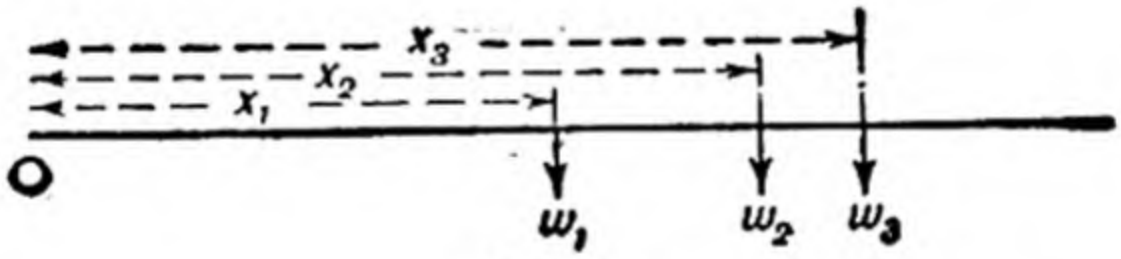

FIG. 86.

Consider a number of weights w_1, w_2 . . . situated at distances x_1, x_2 . . . from a given point O in the straight line AB.

Since the weights w_1, w_2 . . . form a system of like parallel forces, their resultant is equal in magnitude to their sum $(w_1 + w_2 + \ . \ . \ .)$.

The moment of the resultant $(w_1 + w_2 + \ . \ . \ .)$ about the point O is equal to the sum of the moments of the weights w_1, w_2 . . . about the same point.

Let R be the point in AB at which the resultant may be considered to act, and let $OR = \bar{x}$.

$$\bar{x}(w_1 + w_2 + \ . \ . \ .) = w_1x_1 + w_2x_2 + \ . \ . \ .$$

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ \ldots}{w_1 + w_2 + \ \ldots} = \frac{\Sigma wx}{\Sigma w},$$

where Σwx means the sum of all products of the kind wx.

Example.—*A uniform beam of weight 40 lb. and length 8 ft. is loaded with weights 20, 16, 24 lb. at points 3, 5, 7 ft. respectively from one end. At which point can the beam be balanced on a single support?*

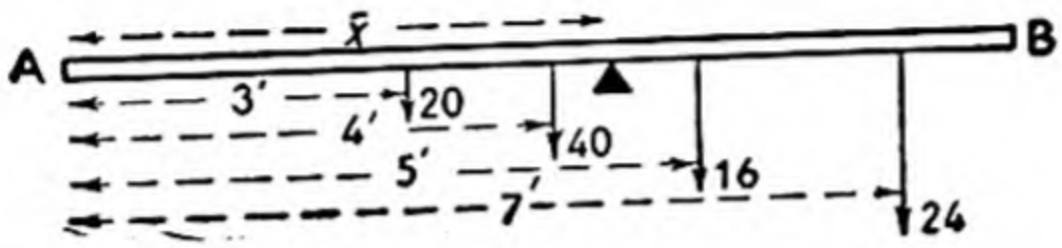

FIG. 87.

The support must be placed beneath the centre of gravity of the beam and the loads.

Sum of the vertical forces acting on the beam

$$= (20 + 40 + 16 + 24) = 100 \text{ lb. wt.}$$

Let $\bar{x}$ ft. be the distance of the support from A.

Taking moments about A,

$$\bar{x} = \frac{20 \times 3 + 40 \times 4 + 16 \times 5 + 24 \times 7}{100} = \frac{468}{100} = 4{\cdot}68.$$

The support must be placed 4·68 ft. from A.

51. Centre of Gravity of a Compound Body.

The method of finding the centre of gravity of a system of particles in one straight line may be applied to plane or solid bodies made up of parts whose centres of gravity lie in one straight line. By replacing the parts by particles of weights equal to those of the respective parts at their centres of gravity the problem becomes one of finding the centre of gravity of a system of particles.

It should be remembered that by equating the sum of the moments of the weights of the particles about their centre of gravity to that of their resultant weight about the same point, this sum is zero, since the resultant weight has

a zero moment about any point on its own line of action. Usually this is the most convenient way of finding centres of gravity, but occasionally it is convenient to take moments about a point away from the centre of gravity.

52. Example (1).—*Find the centre of gravity of the plane lamina composed of rectangles, shown in the figure.*

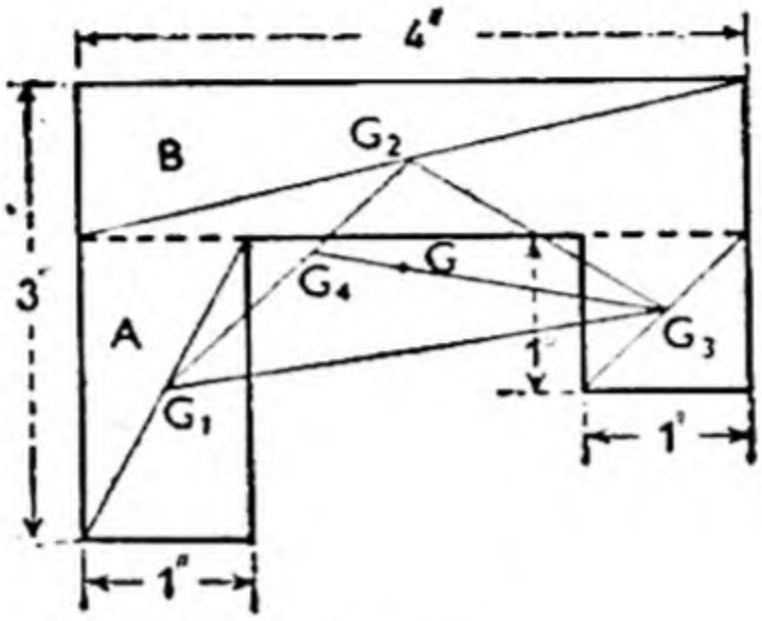

FIG. 88.

Area of rectangle A $= 2 \times 1$ sq. in. $= 2$.
" " B $= 4 \times 1$ " " $= 4$.
" " C $= 1 \times 1$ " " $= 1$.

The centre of gravity is at the midpoint of a diagonal.

Let G_1, G_2 and G_3 be the centres of gravity of the rectangles A, B and C respectively. Two units at G_1 and 4 units at G_2 have a resultant of 6 units at G_4, and

$$\frac{G_1G_4}{G_4G_2} = \frac{4}{2} = \frac{2}{1}.$$

Again, 6 units at G_4 and 1 unit at G_3 have a resultant 7 units at G, such that

$$\frac{G_4G}{GG_3} = \frac{1}{6}.$$

G is the required centre of gravity of the figure. If the point G is marked in position by the point of intersection of two threads, then the position of G may be checked by experiment.

Example (2).—*Find the position of the centre of gravity of a lamina consisting of a rectangle* ABCD *and an isosceles triangle* BCE, *as shown in the figure below.*

G_1 and G_2 are the centres of gravity of the figures ABCD and BCE. Consider the whole figure as made up of the figures ABCD and BCE.

G_1G_2 is the line of symmetry of the whole figure, and when produced it cuts BC at F and passes through the vertex E of the triangle BCE.

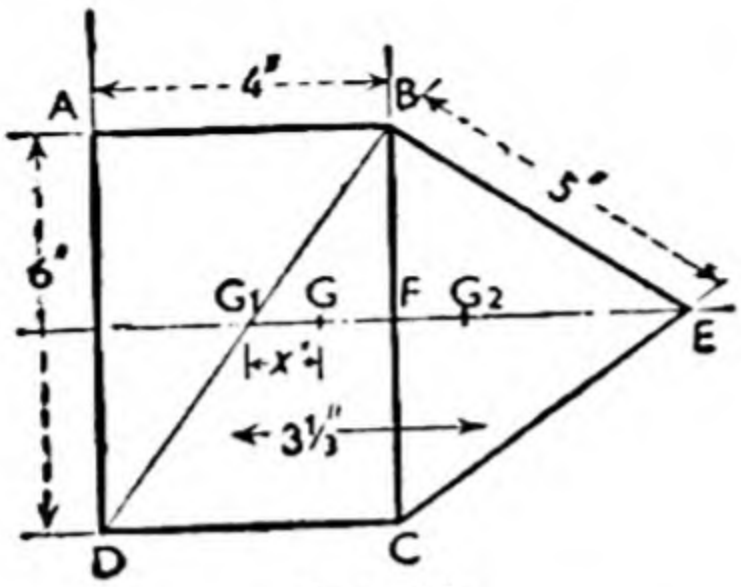

FIG. 89.

The centre of gravity of the whole figure lies along G_1G_2, let it be at G and let $G_1G = x$ in.

$$G_1G_2 = G_1F + FG_2 = (2 + \tfrac{1}{3}EF) \text{ in.}$$

$$FE^2 = BE^2 - BF^2 = 25 - 9 = 16.$$

$$FE = 4.$$

Therefore

$$G_1G_2 = 3\tfrac{1}{3} \text{ in.} \qquad GG_2 = (3\tfrac{1}{3} - x) \text{ in.}$$

Taking moments about G,

$$24x - 12(3\tfrac{1}{3} - x) = 0.$$

$$x = 1{\cdot}11.$$

The centre of gravity of the figure is 1·11 in. from G_2, or 4·89 in. from E along the line of symmetry of the figure.

Note.—The following will be of use in examples which follow :—

(*a*) If a line is divided in the ratio 3 : 8, the parts of the line are 3/11ths and 8/11ths respectively of the total length of the line.

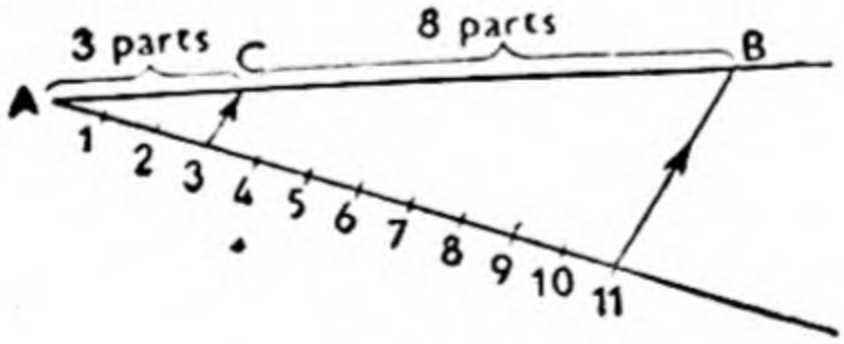

FIG. 90.

(*b*) To divide a line AB in the ratio 3 : 8, draw any line AX, and along it mark off 11 equal parts (of suitable size). Join the end of the last part to B, and draw a parallel to this line through the end of the third part to meet AB at C. Then AC : CB = 3 : 8.

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53. When the centre of gravity of what remains of a body is required after certain parts have been removed, it is usual to consider the whole body as made up of the part whose centre of gravity is required, together with the parts removed, exactly as in the last paragraph. Since the centre of gravity of the whole body is supposed known, moments are taken about a point whose position is known, whereas in the previous case the position of this point was required.

Example.—*Out of a metal disc of diameter* 4 *in. a hole of radius* $\frac{1}{2}$ *in. is punched. The centre of the hole is* 1 *in. from that of the disc. Find the position of the centre of gravity of the remainder of the disc.*

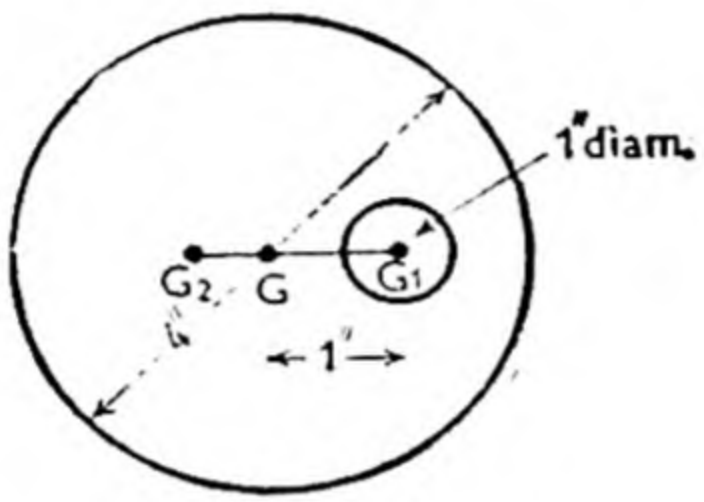


FIG. 91

Consider the whole disc as made up of the part whose centre of gravity is required and the part removed.

Let G_2 be the centre of gravity of the remainder.

G_1 that of the part removed.

G that of the whole disc.

The weight of the whole disc is proportional to the area $\pi 2^2 = 4\pi$ units.

The weight of the part removed is proportional to the area $\pi(\frac{1}{2})^2 = \frac{1}{4}\pi$ units.

The weight of the remainder is proportional to the area $\frac{15}{4}\pi$ units.

Take moments about G,

$$\frac{15}{4}\pi \,.\, G_2G - \tfrac{1}{4}\pi \,.\, GG_1 = 0,$$

$$15 \,.\, G_2G = 1 \qquad (GG_1 = 1 \text{ in.})$$

$$G_2G = \tfrac{1}{15} \text{ in.}$$

If a disc identical with that removed is stuck to the original disc with its centre along G_1G produced at G_3, such that $G_3G = GG_1$, is the centre of gravity of the new body at G_2? (Neglect the thickness of the discs.)

EXAMPLES XI

1. Find the position of the centre of gravity of the plane tee-piece shown in the figure and its distance from AB.

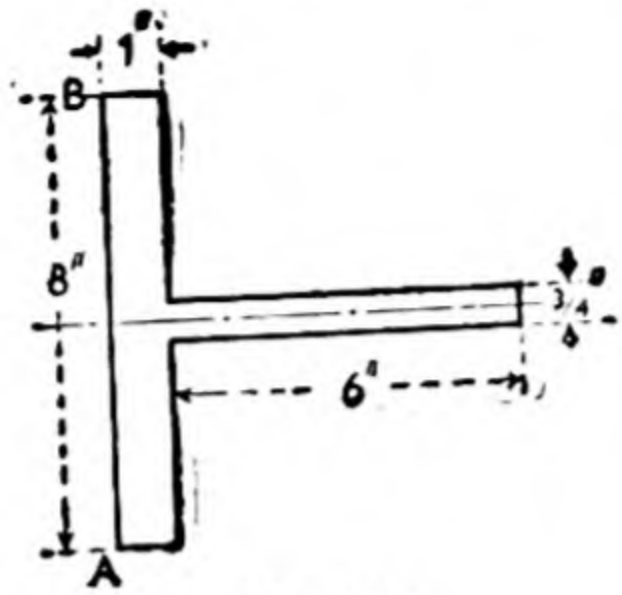


FIG. 92.

2. A piece of wood of uniform thickness consists of a plane square of side 8 in., surmounted by an isosceles triangle of height 6 in. Calculate

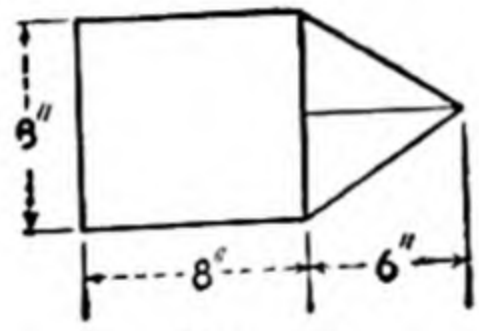


FIG. 93.

the distance of the centre of gravity of the piece of wood from the vertex of the triangle.

3. From a rectangular sheet of metal ABCD, 24 in. long and 16 in. wide, a rectangular piece AXYZ, 18 in. long and 12 in. wide, is cut,

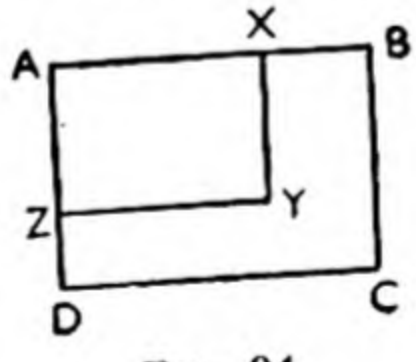


FIG. 94.

as shown in the figure. Find the position of the centre of gravity of the remainder, and determine its distances from BC and CD by drawing.

4. ABCD is a lamina in the shape of a parallelogram. From this parallelogram is cut the parallelogram XYZW, as shown in the figure.

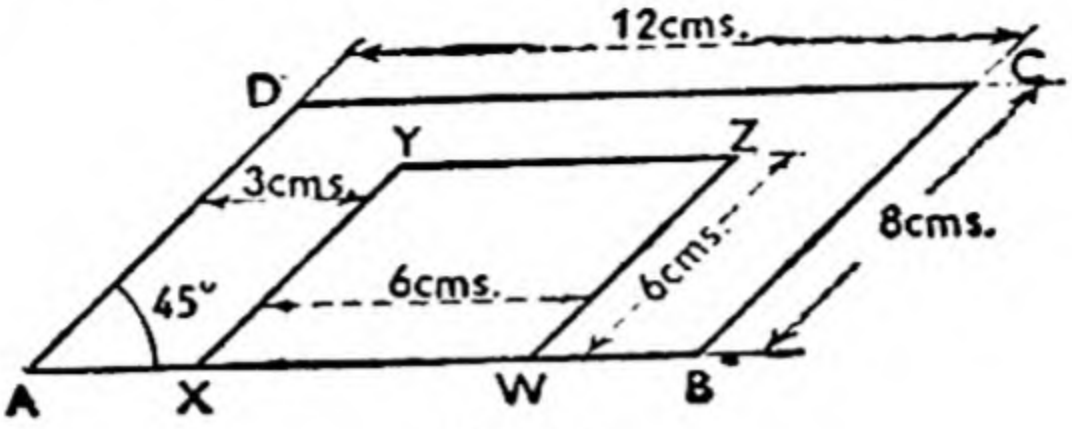


FIG. 95.

Find the position of the centre of gravity of the remainder of the lamina. If this remainder of the lamina is suspended from D, find the angle which the vertical through D will make with DC.

EXAMPLES XII

1. A uniform plank AB, of length 10 ft. and weight 50 lb., is loaded with weights 120 and 30 lb., 1 and 6 ft. respectively from the end A. Find the point at which the plank may be supported by a single prop.

2. A girder AB carries loads of 2, 4 and 10 tons at distances 4, 8 and 12 ft. from the end A. Find the distance of the centre of gravity of these loads from A.

3. A piece of metal consists of three cylindrical portions of different diameters whose lengths are 5, 8 and 11 in. and whose weights are 2, 5, and 18 lb. respectively. Calculate the distance of the centre of gravity of the metal from the end of the cylindrical part of length 5 in.

4. A sheet of metal of uniform thickness consists of a square of side 10 in., surmounted by an isosceles triangle whose equal sides are 13 in. long. Calculate the distance of the centre of gravity of the metal from the vertex of the isosceles triangle.

5. From a rectangular sheet of metal ABCD, in which AB = 16 in. and BC = 24 in., a triangular piece ABX is removed, such that AX = BX = 10 in. Calculate the distance of the centre of gravity of the remainder from the side AB.

6. Determine the position of the centre of gravity of a thin steel disc 8 in. diameter through which a circular hole $5\frac{1}{2}$ in. diameter, with its centre $\frac{1}{2}$ in. from the centre of the disc, had been bored. How would you prove experimentally that the position you had determined was correct ?

7. ABC is a triangle in which AB = 13 in., AC = 13 in. and BC = 10 in. The triangle is divided into two parts by a line parallel to BC and 3 in. from A. Calculate the distance from BC of the centre of gravity of the remainder of the triangle ABC when the part in the shape of a triangle is removed.

8. A cylindrical can of diameter 4 in. and length 3 in. is closed at one end. Assuming that the weight of any part of the can is proportional to its area, find the distance of the centre of gravity of the can from the centre of the closed end.

9. Find the centre of gravity of a solid made up of a cylinder of diameter 10 cm. and length 8 cm. which is surmounted by a right circular cone of height 6 cm. (The volume of a cone is one-third of that of a cylinder with the same base and perpendicular height, and the centre of gravity of a solid cone is one-quarter of its perpendicular height from its base.)

10. The figure shows a framework made up of six rods equal in

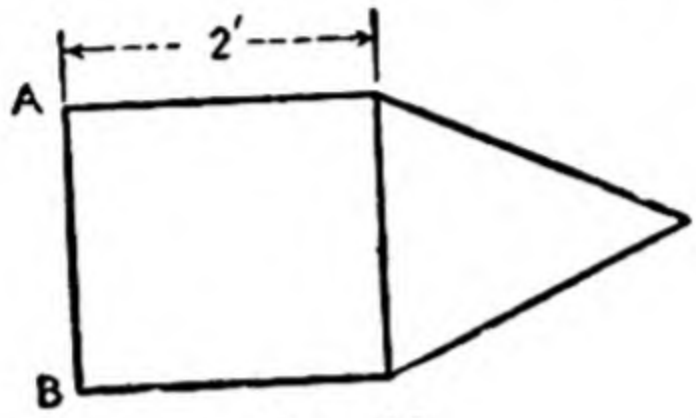


FIG. 96.

length and weight. Find the distance of the centre of gravity of the framework from the rod AB.

11. Part of the worm shaft of a motor car rear axle was 6 in. long and was made in three lengths as follows : 3, $2\frac{1}{2}$ and $\frac{1}{2}$ in. The corresponding diameters were $1\frac{1}{4}$, 2 and $2\frac{1}{2}$ in. Find the position of the centre of gravity.

12. The metal of a telescope consists of three hollow tubes each 6 in. long, the weights of the tubes are in the ratio 3 : 4 : 5, and the centre of gravity of each tube is midway between its ends. When the telescope is pulled out the tubes may be taken to lie end to end in a straight line. Find the distances from the eyepiece to the centre of gravity of this arrangement. *(O.C.)*

13. A circular disc of 8 in. diameter has in it three circular holes each of 2 in. diameter arranged as in the sketch. Find the position of

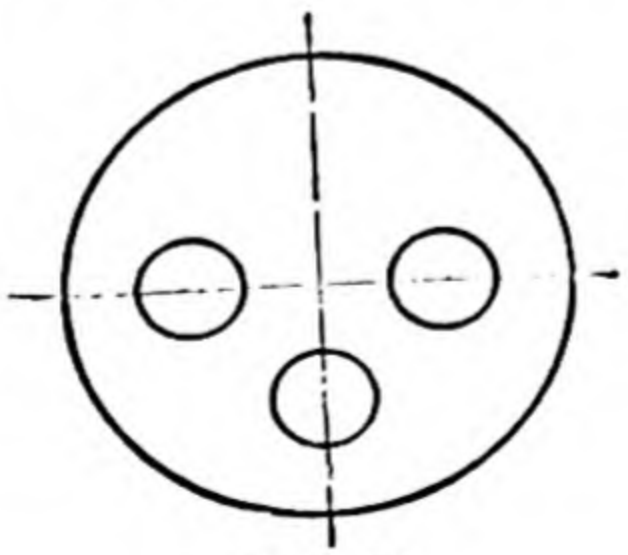


FIG. 97.

the centre of gravity of the disc. The centres of the holes are 2 in. from the centre of the disc. (*O.C.*)

14. ABCD is a square sheet of metal of side 8 in. The middle points of AB, AD are E, F. The triangular portion AEF is cut off. Find the distance from C of the centre of gravity of the remaining portion. (*O.C.*)

15. Where must a circular hole 2 in. in diameter be cut from a circular plate 10 in. in diameter in order that the centre of gravity of the plate may be 0·1 in. from the centre of the plate? (*L.*)

16. Define "centre of gravity."

Find the centre of gravity of a 6-ft. plank of uniform thickness but tapering from 6 in. width at one end to 4 in. thick at the other. (*G.S.*)

17. Describe two methods of finding the position of the centre of gravity of an irregularly shaped lamina.

A thin uniform metal plate ABCD is 2 ft. square. It is folded so that the side AB lies along the line joining the midpoints of AD and BC, and the folded part is pressed flat. Find the position of the centre of gravity of the folded plate. (*L.M.*)

54. Moments about an Axis.

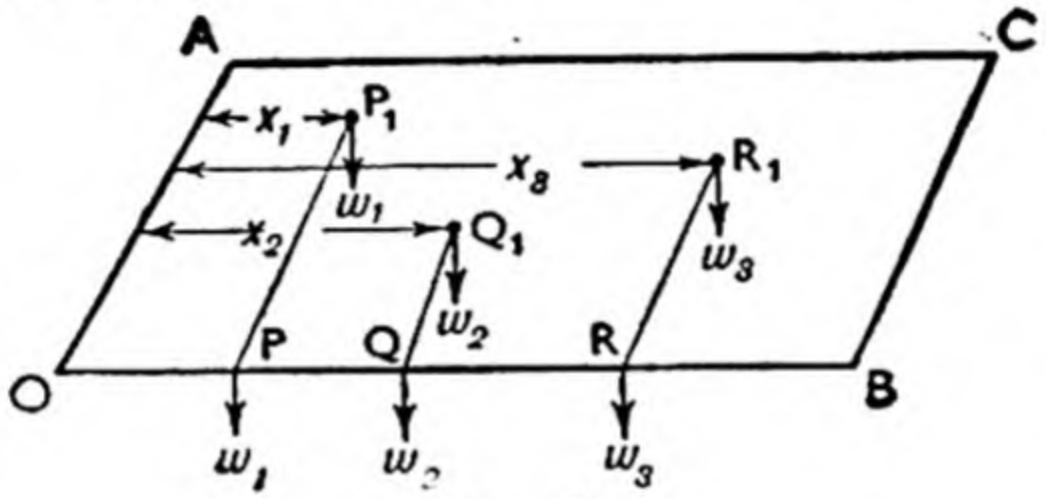


FIG. 98.

Weights w_1, w_2 . . . act at points P, Q, R . . . along the horizontal straight line OB at distances x_1, x_2 . . . as in paragraph 50.

Let OACB be a horizontal plane, and OA be perpendicular to OB.

Suppose that w_1, w_2 . . . be made to act at points in the plane P_1, Q_1, such that

$$PP_1 \parallel QQ_1 \parallel OA.$$

Then w_1 at P_1, w_2 at P_2, have the same moments about the axis OA as in their original positions they had about the point O.

55. Centre of Gravity of a System of Weights in One Plane.

Let the weights w_1, w_2 . . . act at points in a plane. O is a convenient point in the plane, and OY, OX two mutually

perpendicular straight lines from which the perpendicular distances of the weights w_1, w_2 . . . are (x_1, y_1), (x_2, y_2) . . .

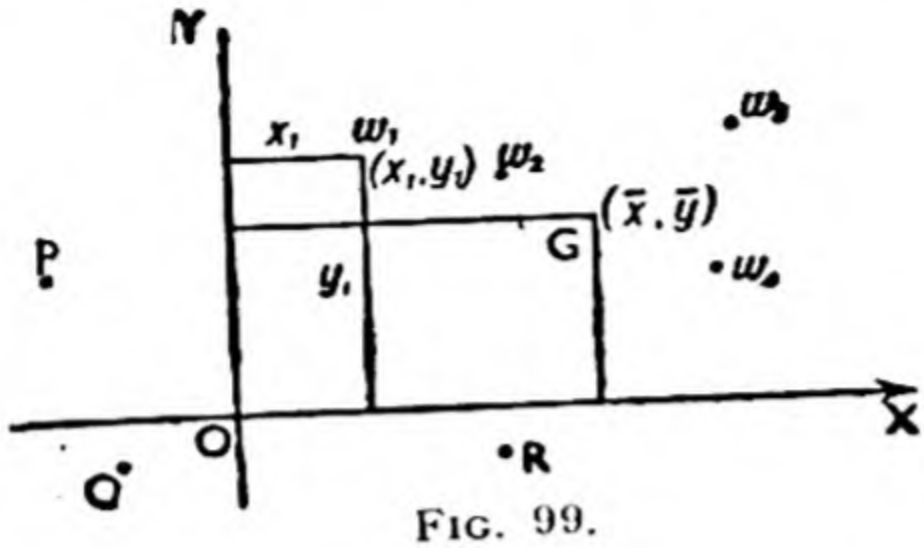


FIG. 99.

Let the centre of gravity of the weights be $(\bar{x}, \bar{y})$ from the axes OY, OX respectively.

If we imagine the weights carried by a light rigid frame attached to XOY, then we may consider this frame placed in a horizontal position. We proceed to take moments about the axis OY.

The perpendicular distance of w_1 from the axis OY is x_1, the perpendicular distance of w_2 from the axis OY is x_2, and so on for the other weights.

Hence, since the moment of the resultant of the weights about OY is equal to the sum of the moments of the weights about the same axis,

$$(w_1 + w_2 + \ . \ . \ .)\bar{x} = w_1x_1 + w_2x_2 + \ . \ . \ .$$

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ . \ . \ .}{w_1 + w_2 + \ . \ . \ .} = \frac{\Sigma wx}{\Sigma w}.$$

Similarly, by considering moments about the axis OX,

$$\bar{y} = \frac{\Sigma wy}{\Sigma w}.$$

It is very important to notice that the weights may be situated at points in the plane such as P, Q and R. In these cases the moments of P and Q about OY are negative, taking the moments of weights on the opposite side of this axis as positive moments. Similarly the moments of weights acting at Q and R about OX

will be negative if the moments of weights acting at points in the opposite side of OX are taken as positive.

Example.—ABCD *represents the plan of a square table-top of weight* 40 *lb. The four legs situated at* E, F, G *and* H *weigh* 8 *lb. At* Q *a* 24*-lb. wt. is placed on the table* 1 *ft.* 6 *in. from* AD *and* 1 *ft. from* AB. *Find the point in the table through which the line of action of the resultant weight passes. Find also the least weight supported at one edge of the table which would just cause the table to topple.*

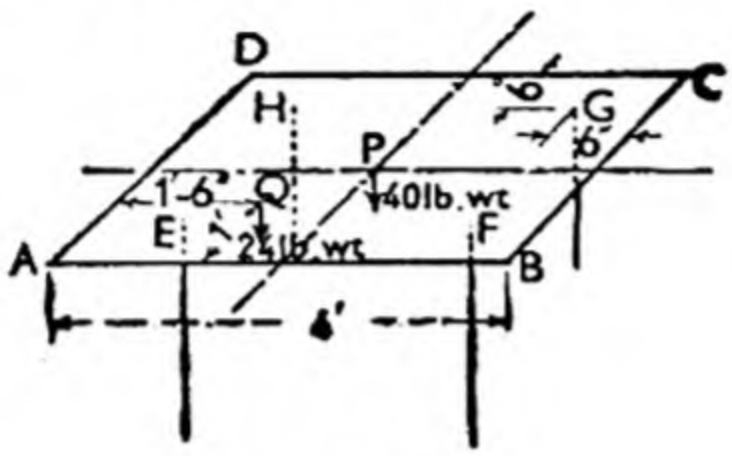


FIG. 100.

Let G be the position of the point of the table through which the line of action of the resultant weight passes. Suppose G is x ft. from AD and y ft. from AB. The weights of the four legs may be replaced by 32 lb. wt. acting at P. Hence we have 72 lb. wt. acting at P, the centre of the table-top, and the weight acting at Q.

Take moments about the axis AB,

$$24 \times 1 + 72 \times 2 = (72 + 24)y,$$

$$y = \frac{168}{96} = 1\tfrac{3}{4}.$$

Take moments about AD,

$$24 \times 1\tfrac{1}{2} + 72 \times 2 = 96x,$$

$$x = \frac{180}{96} = 1\tfrac{7}{8}.$$

G is 1 ft. 9 in. from AB and 1 ft. $10\frac{1}{2}$ in. from AD.

Since G is nearest to the axis through E and F, the moment of the resultant weight about this axis is less than that about FG, GH or HF. Hence the table will first begin to topple about the points of contact of the legs E and F with the ground.

If W lb. wt. is the least weight required, then, taking moments about EF,

$$W \cdot \tfrac{1}{2} = 96 \times 1\tfrac{1}{4},$$

$$W = \frac{96 \times 5}{2} = 240.$$

The table will be on the point of toppling when a 240-lb. wt. is suspended from the edge AB.

Note.—If the position of the centre of gravity of the table-top and legs is required, the legs may be replaced by an equivalent mass at their centre of gravity, and the table-top by an equivalent mass at its centre of gravity, then the centre of gravity of the complete table may be found by determining the position of the centre of gravity of these two masses.

56. Centre of Gravity of a Trapezium.

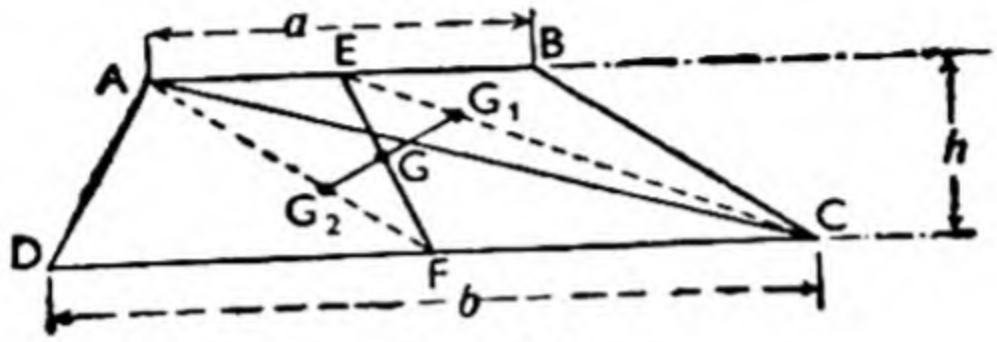


FIG. 101.

ABCD is a trapezium in which AB is parallel to DC. E and F are the midpoints of AB and CD respectively. If the trapezium is divided into strips parallel to AB, it is easily observed that its centre of gravity lies along the line EF on which the centres of gravity of all the strips lie. Suppose now that the trapezium is divided into two triangles ABC and ADC. CE and AF are the medians of the triangles ABC and ADC, and G_1 and G_2 the positions of their centres of gravity. Let AB = a units, CD = b units and h units the height of the trapezium.

$$\triangle ABC = \frac{ah}{2} \text{ units of area,}$$

$$\triangle ADC = \frac{bh}{2} \text{ units of area,}$$

$$\text{Trapezium } ABCD = \tfrac{1}{2}(a+b)h \text{ units of area.}$$

The perpendicular distance from G_1 to AB $= \frac{h}{3}$.

" " " " G_2 " $= \frac{2h}{3}$.

" " " " G " $= \bar{x}$.

Taking moments about the axis AB,

$$\frac{ah}{2}\cdot\frac{h}{3}+\frac{bh}{2}\cdot\frac{2h}{3}=\bar{x}\cdot\frac{(a+b)\cdot h}{2},$$

$$h^2(a+2b)=3h(a+b)\cdot\bar{x},$$

$$\frac{\bar{x}}{h}=\frac{(a+2b)}{3(a+b)}.$$

Since $\frac{\text{EG}}{\text{EF}}=\frac{(a+2b)}{3(a+b)}$, $\frac{\text{EG}}{\text{GF}}=\frac{a+2b}{2a+b}$.

EXAMPLES XIII

Centre of Gravity

1. The floor of a room ABCD is a rectangle in which AB = 22 ft. and AD = 18 ft. Weights of 50, 100, 250 lb. may be considered to be supported at points on the floor whose distances from AB and AD are respectively 3, 4 ft.; 8, 10 ft.; 16, 12 ft. Find the distances of the centre of gravity of the weights from AB and AD respectively.

2. A square board ABCD of side 2 ft. weighs 10 lb. Weights of 2, 3, 5 and 8 lb. are placed at the vertices A, B, C and D of the board. Find the position of the centre of gravity of the board and the weights which it carries.

3. A triangular board ABC of weight 5 lb. is in the form of an isosceles triangle in which AB = AC = 6 in. and $\angle BAC = 90°$. Weights of 3, 4 and 5 lb. are placed at A, B and C respectively. Find the position of the centre of gravity of the board and weights.

4. ABCD is a trapezium in which AB is parallel to CD; AB = 3 in., CD = 5 in., $\angle ABC = 90°$ and BC = 2 in. Calculate the distances of the centre of gravity of the trapezium from BC and CD.

5. A sheet of tin of uniform thickness is in the form of a square ABCD 20 cm. by 20 cm. The triangular piece of tin formed by the line joining the midpoints E and F of two adjacent sides AB, AD of the square is folded about EF until it rests flat upon the remaining part of the tin. Find the position of the centre of gravity of the tin in its new shape.

6. A square board ABCD, 8 in. by 8 in., is divided into sixty-four equal squares by lines parallel to the sides of the board. If squares of sides 1, 2, 3 and 4 in. are removed at the corners A, B, C and D, find the position of the centre of gravity of the remainder of the board.

7. A circular sheet of metal has a diameter of 14 in. Two holes, each of diameter 2·8 in., are punched out of the metal so that their centres are at the same distance from that of the sheet of metal and the line joining the centres of the holes, which are 4 in. apart, is 3 in. from the centre of the sheet of metal. Find the position of the centre of gravity of the remainder of the metal.

8. A triangular framework consists of uniform metal rods of lengths

3, 4, and 5 ft. and of weights 20, 40 and 80 lb. Find the position of the centre of gravity of the rods.

9. A framework of rods consists of seven rods of lengths and weights as shown in the figure. Find the distance of the centre of gravity of the framework from AB.

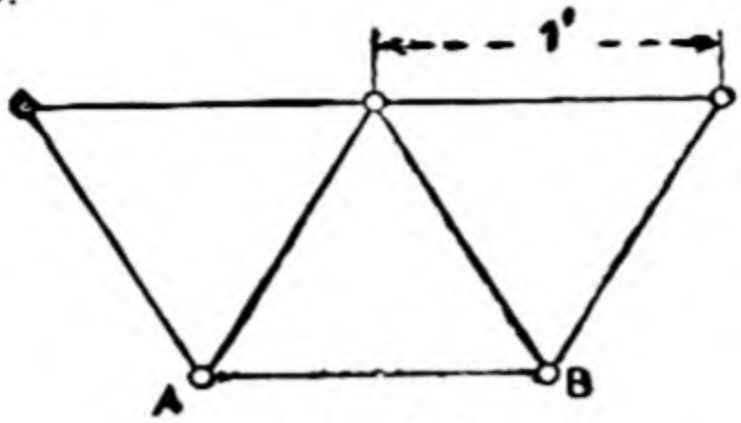


FIG. 102.

10. Two uniform rods AB and AC, of lengths 2 and 4 ft., are fastened rigidly together at A so that $\angle BAC = 90°$. If their weights are proportional to their lengths, find graphically or by calculation the angle made by AC with the vertical if the rods are freely suspended from C.

11. A table consists of a square top of side 4 ft. and weight 48 lb. The table is supported by four legs at the four corners of a square of side 3 ft. 6 in., the points of support being on the diagonals of the table. The legs are each of length 3 ft. and weight 15 lb. Find the position of the centre of gravity of the table.

12. In example 11 what is the greatest weight which can be supported at the midpoint of one edge of the table? The table-top may be assumed to be horizontal.

13. The table in example 11 is placed with one edge of the top in contact with the floor and resting on two legs. What force applied horizontally to the midpoint of the top horizontal edge will just lift the lower edge from the ground? Assume that the legs do not slip. (Assume the position of the centre of gravity of the table to be 10 in. below the centre of the table-top when the latter is horizontal.)

14. Define "centre of gravity."

A uniform board ABC in the form of an equilateral triangle of 12 in. side weighs 3 lb. and has weights of 4 and 5 lb. hanging from A and B respectively. Find a point from which the board may be suspended so that it sets in a vertical plane with AB horizontal and C pointing down. (*G.S.*)
Is there more than one such point?

15. Explain what is meant by the centre of gravity of a body.

A square piece of cardboard ABCD, of side 10 cm., has masses of 2, 3, 4 and 6 gm. at the corners A, B, C and D respectively. Find the centre of gravity of the cardboard and draw a diagram showing how it will hang when suspended from the corner A. (Neglect the mass of the cardboard.) (*L.M.*)

16. What is meant by the centre of gravity of a body?

A uniform rectangular metal sheet 9 in. × 6 in. is marked out into six squares of 3-in. side, and the centre square along one of the 9-in. sides is cut out of the sheet. At what point must the remainder be placed on a vertical pin so that it balances horizontally? (*L.M.*)

17. A circular table of radius 2 ft. weighs 15 lb., and its centre of gravity is vertically below the centre when the top of the table is horizontal. It is supported by three legs arranged at the corners of an equilateral triangle, each leg being 1 ft. 8 in. from the centre of the table. Determine the least weight which, placed at the edge of the table, will cause it to topple, and state exactly the positions where the weight may be placed. (*L.M.*)

18. Explain how you would find the centre of gravity of a body made up of two different parts, the centre of gravity of each being known. ABCDEF is a hexagonal plate of uniform thickness and density; O is the centre of the figure, and the sides are each 2 in. long. The triangles OAF and ODE are cut away. Where is the centre of gravity of the remainder of the plate? (*L.M.*)

19. Three uniform rods AB, BC, CD are each 6 in. long and of mass 2 lb. They are arranged to form three sides of a square, and the system is suspended freely at A with the plane of the framework vertical. Determine either graphically or by calculation the point along BC which is vertically below A.

What mass must be attached at B in order that the framework may rest with the midpoint of BC vertically below A? (*L.M.*)

20. The figure represents a section of a small gable which is carried

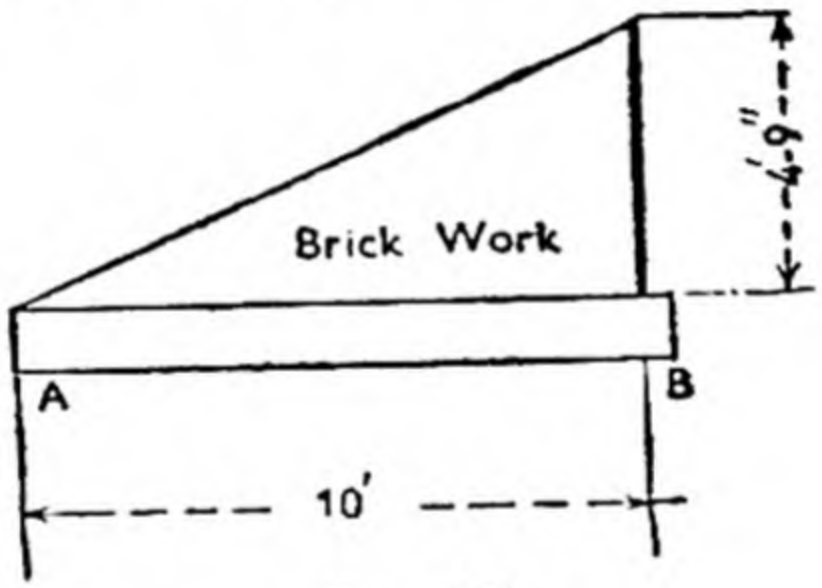


FIG. 103.

by the beam AB. If the brickwork of which it is composed weighs 118 lb. per cub. ft. and is 9 in. thick, (*a*) calculate the total weight carried by the beam, (*b*) find the position of the centre of gravity of the brickwork, and (*c*) find the reactions at A and B due to this load.

57. Stability.

A body is in **stable** equilibrium if when it is slightly displaced and then released it returns or tends to return to its original position.

A body is in **unstable** equilibrium if when it is slightly displaced and then released it moves or tends to move still farther from its original position.

A body is in **neutral** equilibrium if when it is slightly displaced it remains in equilibrium in its new position.

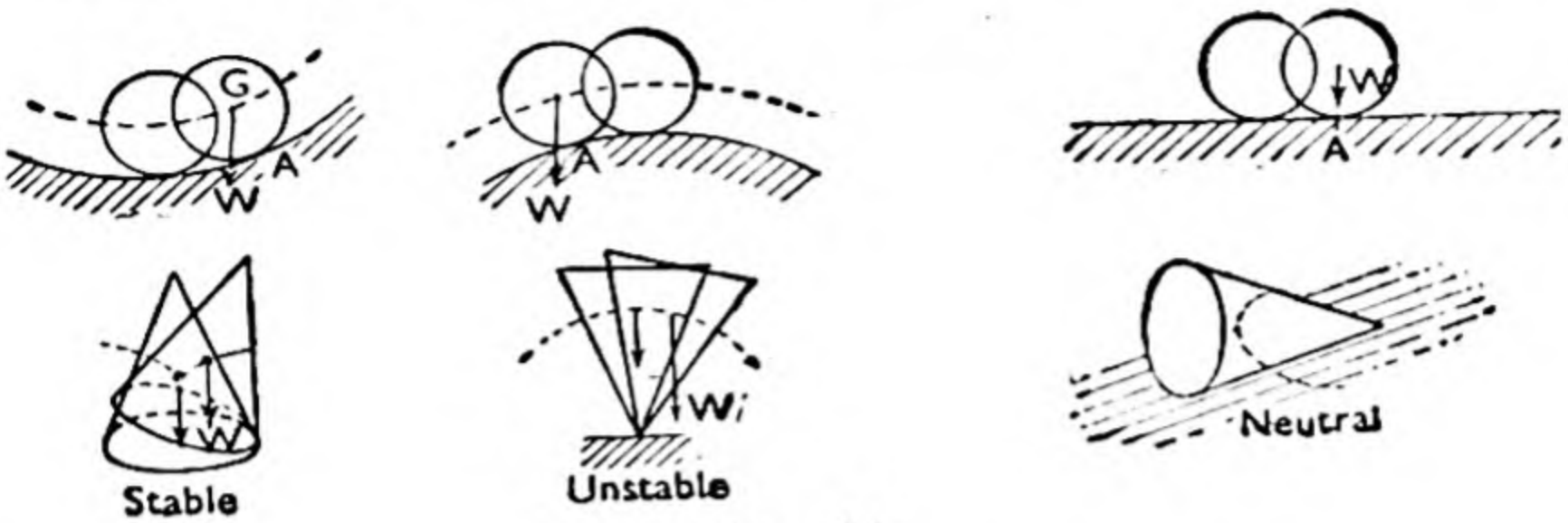


FIG. 104.

Consider the two sets of figures above.

In the first set of figures a sphere whose centre of gravity is at its centre rests in turn on a curved surface concave upwards, on a horizontal plane and on a curved surface concave downwards. When the sphere is slightly displaced let A be the new point of contact with the surface upon which it rests.

In the first figure, in the displaced position, G is to the left of the vertical through A, therefore the weight of the sphere has a turning effect about A which tends to bring it back into its original position. This is true for any displacement of the sphere so long as it remains in contact with the curved surface. The sphere is therefore in stable equilibrium.

In the second figure, when the sphere is slightly displaced, G is again to the left of A, and the weight of the sphere tends to turn it away from its original position. In this case the sphere was in unstable equilibrium before it was disturbed.

In the third figure, when the sphere is slightly displaced, G is vertically above A, and it is therefore again in a position of equilibrium. The sphere in this case was in its original position in neutral equilibrium.

Consider in a similar manner the behaviour of the right circular cone, also shown in three positions on a horizontal plane marked respectively as stable, unstable and neutral equilibrium. The dotted lines in the figure show the path of the

centre of gravity of the body as it takes up new positions during its displacements. It will be noticed that when the body is in stable, unstable, neutral equilibrium the centre of gravity rises, falls, remains on the same horizontal level respectively when the body is slightly displaced.

58. Toppling.

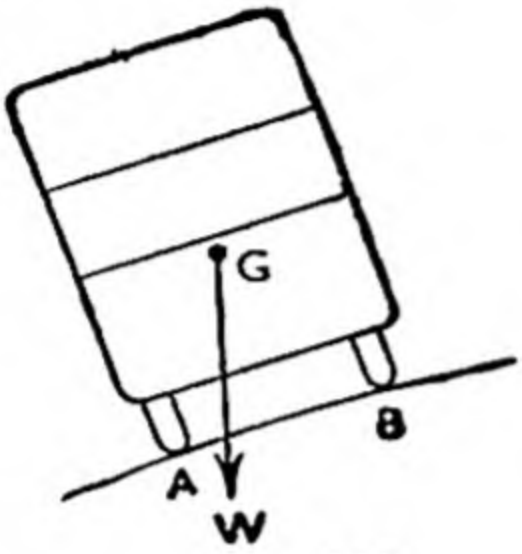


Fig. 105.

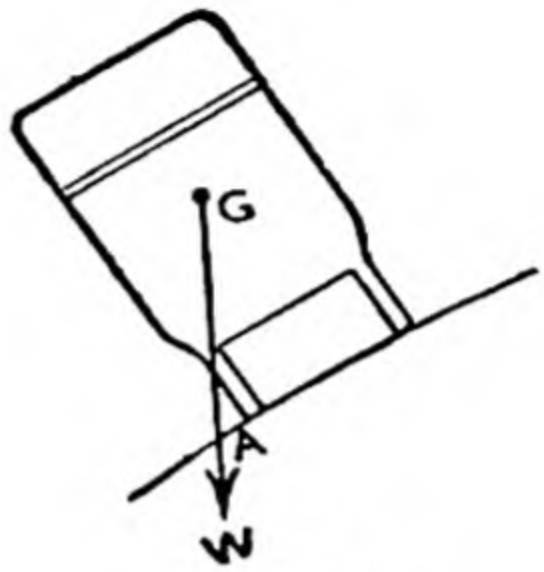


Fig. 106.

In Fig. 105 a motor bus is shown in elevation upon a cambered road. G is the position of the centre of gravity of the bus, and through G the weight of the bus acts vertically downwards. The line of action of the weight of the bus meets the road between the two wheels, which make contact with the road at A and B.

If either wheel is imagined to be just raised above the road the weight of the bus exerts a turning moment about the point of contact of the other wheel with the ground which tends to bring the first wheel again into contact with the ground.

Consider Fig. 106, in which we suppose the road camber to be large and the passengers to be on the top deck, standing up and concentrated to the left of the bus. We now see that the line of action of the weight meets the road outside the wheel track, and there is therefore a moment tending to turn the bus about the point of contact A of one of the wheels with the ground. Since there is no force exerting a moment to balance that of the weight, the bus would overturn. Clearly, toppling is made more difficult by having a wide wheel track and a low

centre of gravity of the bus. Again, we can appreciate the reason why no standing is allowed on the top deck of a bus.

If a body rests upon a plane without slipping, then it can rest without toppling, provided that the line of action of its weight meets the plane within the contour of the base of the body, as found by passing a string round the outside of the base of the body and drawing it tight.

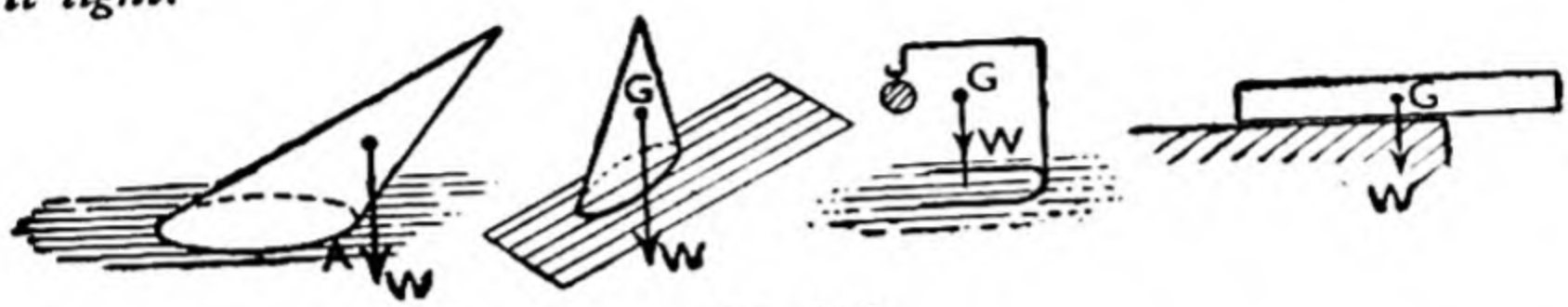


FIG. 107.

It is left as an exercise for the reader to say in which of the above figures toppling will occur.

Example (1).—*A motor car has a wheel span of 4 ft. 6 in., and its centre of gravity is symmetrically placed at a height of 3 ft. above the ground. What is the greatest angle of slope of an inclined plane upon which it can rest with its axles parallel to the line of greatest slope?*

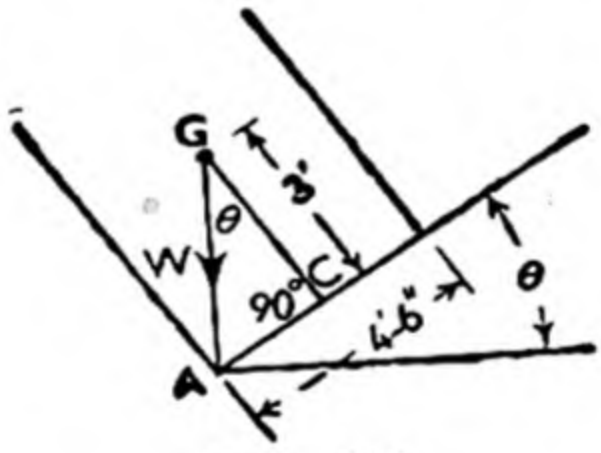


FIG. 108.

G is the centre of gravity of the car. A is the point of contact of one of the lower wheels with the ground.

Let the angle of the inclined plane be increased until the vertical through G passes through A. The car will topple about A if the angle of the plane is increased.

In the position shown in the figure,

$$\angle GAC = 90^\circ - \theta,$$

$$\angle AGC = \theta.$$

therefore

From the right-angled triangle ACG,

$$\tan\theta = \frac{AC}{GC} = \frac{2\frac{1}{4}}{3} = 0{\cdot}75,$$

$$\theta = 36^\circ\ 52'.$$

Example (2).—*Four bricks (each 9 in. × 4½ in. × 3 in.) are formed into a pile, as shown in the figure, each projecting the same amount, x in., beyond the brick immediately beneath it, and the lowest brick projecting the same amount beyond the edge of a horizontal table. Find the value of x so that the bricks may just be on the point of toppling about the edge of the table.*

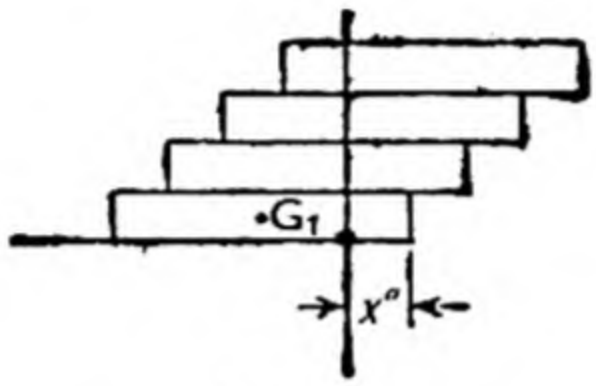


FIG. 109.

The centre of gravity G_1 of the first brick is $(4\frac{1}{2} - x)$ in. to the left of the vertical line through the edge of the table. Thus the centres of gravity of the second, third and fourth bricks are $(4\frac{1}{2} - 2x)$ in., $(4\frac{1}{2} - 3x)$ in., $(4\frac{1}{2} - 4x)$ in. to the left of this vertical line.

Any one of these numbers will be negative in the case of a brick whose centre of gravity lies to the right of the vertical line.

Let w be the weight of a brick.

The algebraic sum of the moments of the weights of the bricks about the horizontal edge of the table is zero when they are on the point of toppling about this edge, and therefore

$$w(4\tfrac{1}{2} - x) + w(4\tfrac{1}{2} - 2x) + w(4\tfrac{1}{2} - 3x) + w(4\tfrac{1}{2} - 4x) = 0,$$

$$18 - 10x = 0,$$

$$x = 1\tfrac{4}{5}.$$

Thus each brick projects 1·8 in. over the one immediately below it.

Note.—It may readily be shown that:

For 5 bricks each can project 1/6th of the length of a brick;

" 6 " " " 1/7th " " "

" n " " " $\frac{1}{n+1}$ " " "

EXAMPLES XIV

STABILITY AND TOPPLING

1. Explain what you understand by stable, unstable and neutral equilibrium, and illustrate your answer by diagrams.

2. A walking-stick consists of a straight cane surmounted at one end by a heavy knob. Explain why it is easier to keep it balanced with the

lighter end on the tip of a finger and the knob uppermost than with the lighter end uppermost and the knob in contact with the finger.

3. Why is it impossible to balance a fresh egg with either of its ends in contact with a horizontal table ?

4. Two forks are stuck into a cork and the cork is then balanced on a

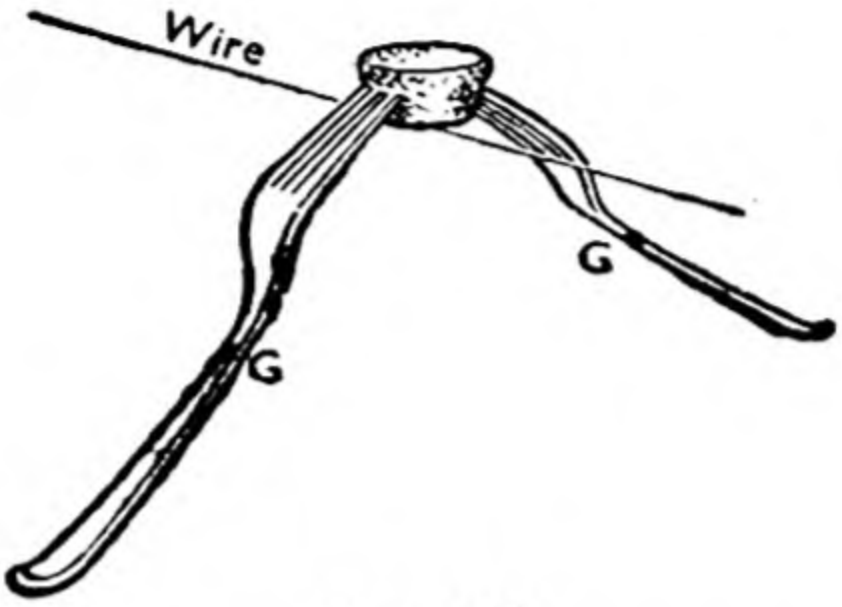


FIG. 110.

stretched wire, as shown in the figure. Explain why the body consisting of the cork and the forks is in stable equilibrium. The centres of gravity of the corks are marked G.

5. Explain (*a*) how a tight-rope walker maintains his balance by the use of a large paper parasol; (*b*) how a boy on stilts maintains his balance when not moving the points of contact of the stilts with the ground.

6. Explain how a flower vase may be in unstable equilibrium when water and flowers are placed in it. How can you make a flower vase less likely to topple when water and flowers are placed in it ?

7. A light, hollow, egg-shaped celluloid body and some lead shot are provided. Explain by their use the meaning of the terms unstable, neutral and stable equilibrium.

8. A solid cylinder of uniform density is placed on a rough inclined plane of angle 30° with the horizontal. If the diameter of the cylinder is 10 cm., find by calculation (or an accurate drawing) the length of the cylinder so that it may just be on the point of toppling about its lowest point.

9. A motor car has a wheel span of 5 ft. 4 in., and the centre of gravity of the car is symmetrically placed 2 ft. above the ground. If the car rests on an inclined plane with the axles parallel to the lines of greatest slope, determine the angle which the plane makes with the horizontal when the car is about to topple about the points of contact of the lower wheels with the plane.

10. Explain with diagrams how a motor omnibus may be made less liable to toppling and why passengers are not allowed to stand on the top deck of a double-decker bus.

11. A block of wood is in the form of a square prism 1 ft. × 1 ft. × 2 ft. and of weight 100 lb. It rests on a horizontal plane on one of its

square ends. What is the least force that must be applied horizontally at the midpoint of one of its upper edges to cause it to topple about an edge of its base? What is the least force which will cause the block to topple, where should it be applied, and at what angle to the horizontal?

12. Six bricks are built into a pile, the first rests on a horizontal plane, the second overlaps the first by the same amount that the third overlaps the second, and so on. If the bricks are on the point of toppling about the upper edge of the bottom brick, find the amount by which each brick overlaps the one below it. A brick may be taken as a rectangular block 9 in. × 4½ in. × 3 in.

13. If a heavy body is supported in equilibrium at a single point, what positions with regard to this point are possible for the centre of gravity of the body? Discuss the different cases.

A bar 4 ft. long, which balances about a point 1 ft. from one end, has a string 8 ft. long attached to its ends. The string passes over a rough nail and the bar hangs from the nail in a horizontal position. Find the proportion in which the nail divides the string. (*O.C.*)

14. From a body, whose mass and centre of gravity are known, a portion, whose mass and centre of gravity are known, is removed. Show how to find the centre of gravity of the remainder.

ABCD is a square and E and F are points on CB and CD respectively, such that CE = $\frac{1}{3}$ CB and CF = $\frac{2}{3}$ CD. A uniform solid prism has for its cross-section ABEFD. Determine whether it can rest with the face whose section is DF in contact with a horizontal plane. (*O.C.*)

15. What do you understand by the terms centre of gravity and stable equilibrium? A reading lamp is supported on a single central pillar from a square base of 4 in. side and 2 in. thick. If the base is uniform and weighs 2 lb. and the pillar, etc., has its centre of gravity 12 in. above the centre of gravity of the base, what is the mass of the pillar, etc., if the whole is just unstable when tilted through an angle of 45° about one of the sides of the base? (*L.M.*)

16. Point out the conditions which determine whether the equilibrium of a body is stable, unstable or neutral. Illustrate your answer by considering the equilibrium of an egg on a horizontal table. (*L.M.*)

17. Discuss the conditions which determine whether the equilibrium of a rigid body at rest is stable, unstable or neutral.

A cylindrical tube 14 cm. long is partly filled with lead and placed on a rough inclined plane, which is gradually tilted until the cylinder topples over. If the heavy end of the cylinder rests on the plane this happens when the gradient of the plane (*i.e.*, the ratio of its height to its base) is 1 in 2; but if the other end of the cylinder rests on the plane it falls as soon as the gradient becomes 1 in 5. Find the position of the centre of gravity of the cylinder, assuming it to be on the axis. (*L.M.*)

CHAPTER VII

WORK AND MACHINES

59. Work.

We associate WORK with "movement" and "doing." If we watch a man digging we say that he is working, and we judge this from his movements. When he remains at rest for even a short time we say that during this time he is not working. A crane is "working" when it is raising a load, and when the load is at rest the crane is no longer working. This idea of work is one which we retain in mechanics. In fact, NO MOVEMENT implies that NO WORK is being done.

Before considering how we are to measure an amount of work we have to notice that work may be done slowly or quickly without affecting the amount of work done. Suppose that two workmen each have the same number of bricks to carry up through the same vertical height to bricklayers working on a scaffolding. The first workman may do his work without interruption in twenty minutes. The second workman may take three hours to finish his work owing to his attention having been called away. The two men do the same amount of work. The time which each takes to do the work has no effect on the amount of work which each has done. Again, the first workman may exhaust himself by working quickly and the second be quite unaffected, and each still has done the same amount of work.

Now let us consider how one particular force acting upon a body does work. A 5-lb. weight rests on the floor. The upward force of the floor on the weight balances the weight. No work is being done by either of these forces. Suppose that the weight is slowly raised from the floor to a bench through a height of 4 ft. and is then placed on the bench, not being raised any higher than is necessary to do this. The force exerted vertically upwards in raising the weight is 5 lbs. wt. How much work has this force

done ? As a comparison we may say, twice as much as if it had raised the weight 2 ft., and four times as much as if it had raised it 1 ft. The comparison at once suggests a *unit of work*, viz., the work done when a force of 1 lb. wt. acts through a distance of 1 ft.

A crane which is capable of travelling horizontally picks up a load, raises it to a certain height above the ground, travels horizontally through a certain distance, and finally lowers the load into its appointed position. During the first part of this operation work is done BY the tension in the rope which raises the load. Whilst the crane is moving slowly in a horizontal direction the tension in the rope does no work upon the load, which remains at the same horizontal level during the motion. Here we see that when the crane is moving steadily and the load is moving horizontally the MOVEMENT of the load does NOT imply that work is being done on it. Finally, when the load is lowered the load itself does work against the tension in the rope.

60. Units of Work.

When a force of 1 lb. wt. moves its point of application through a distance of 1 ft. in the direction of the force, the work done is 1 **ft. lb.**

When a force of 1 ton wt. moves its point of application through a distance of 1 ft. in the direction of the force, the work done is **1 ft. ton.**

These units are the British Engineer's units of work.

Example (1).—*In loading a vessel fifty packages, each weighing* 5 *cwt., have to be raised through a vertical height of* 20 *ft. Find the work done.*

The work done in raising one package through a vertical height of 20 ft. is

$$5 \times 20 \text{ ft. lb.} = \frac{5 \times 112 \times 20}{2240} \text{ ft. tons} = 5 \text{ ft. tons.}$$

The work done in raising fifty packages through a vertical height of 20 ft. is

$$50 \times 5 \text{ ft. tons} = 250 \text{ ft. tons.}$$

Example (2).—*A horse draws a cart through a distance of* $\frac{1}{4}$ *mile along a horizontal road. The average total horizontal tension in the traces is* 80 *lb. wt. Find the work done.*

The work done is against the resistance to motion produced by friction mainly at the bearings of the wheels.

Work done = Force × Distance through which the point of application of the force is moved in the direction of the force.
= 80 × 440 × 3 ft. lb.
= 105,600 ft. lb.
= 47½ ft. tons.

61. In transferring a mass of earth, water or any substance from one level to another the shape of the substance may be changed. In this case the total amount of work done is found by determining the change in vertical height of the centre of gravity of the mass in its original and final positions.

Thus if a wire rope of length 100 ft. is originally hanging vertically from a drum whose axis is horizontal and is then wound on the drum, the work done is the same as if the total mass of the rope were imagined concentrated at a point 50 ft. below the point of suspension and then raised from this position to the new position of the centre of gravity when wound on the drum. If the top of the rope was originally at the same horizontal level as the axis of the drum, and the weight of the rope per foot-length is 2 lb., the total amount of work done in raising the rope is

Weight of rope × Height through which centre of gravity is raised is (2 × 100) × 50 = 10,000 ft. lb.

EXAMPLES XV

WORK

1. A crane lifts twenty packages, each of weight 15 lb., through a height of 12 ft. How much work is done by the crane? (*a*) In foot-pounds, (*b*) in foot-tons.

2. A horse draws a load of coal at a uniform speed through a horizontal distance of 1 mile. If the horizontal pull of the horse on the cart is 100 lbs. wt. and is assumed to be constant, how much work is done by the horse on the cart?

3. A lift of weight 5 cwt. carries a man of weight 12 st. How much work is done by a wire rope which draws the lift through a height of 50 ft.?

4. A cyclist rides a distance of ¼ mile against a head wind along a horizontal road. The force exerted by the wind against the cyclist is 4 lb. wt., and he has to overcome frictional resistances amounting to 3 lb. wt. Calculate the work done by the cyclist against these forces.

5. To raise loads from the bottom of a mine it is arranged that an empty cage is let down when a full cage is raised. How much work must be done to raise a 2-ton load 300 ft. if the two cages are exactly equal in

weight? If the weight of a cage is ½ ton, and no counterbalancing cage is employed, how much additional work has to be done?

6. The surface of the water in a well remains at a constant depth of 15 ft. below the level of the ground. How much work is done in raising 1,500 gals. of water to the level of the ground?

7. A train is stopped in a distance of 100 yds. by a force parallel to the track due to the brakes of 230 lb. wt. per ton weight of the train, together with the additional force due to frictional effects also parallel to the track of 15 lb. wt. per ton. The total weight of the train is 320 tons. Calculate the work done in bringing the train to rest.

8. A machine does 1,000 ft. lb. of work in raising a load of 80 lb. vertically. Through what height is the load raised?

9. A wire rope of uniform cross-section weighs 1 lb. per foot. If 180 feet of this rope, which is originally hanging vertically, is wound on to a drum, how much work has been done?

10. A pony is used to draw coal tubs out of a mine. If the average force exerted by the pony is 80 lb. wt. and the distance through which a tub is drawn is ½ mile, find the work done by a pony which makes five journeys. Only the outward journeys are to be considered.

11. The piston of a steam engine has a diameter of 14 in. and the length of its stroke is 12 in. The mean effective pressure during a stroke is 120 lb. per sq. in. Find the work done in one minute if the piston makes 35 strokes per minute.

12. The cross-sectional area of a well shaft is 30 sq. ft., and the shaft is 56 ft. deep. The earth removed has an average density of 160 lb. per cu. ft. Calculate the work done in bringing the soil removed to the surface of the well shaft.

13. Forty tons of water flow over a waterfall every second. How much work is the water which flows over the fall in one minute capable of doing due to its having fallen vertically through 30 ft.?

14. In excavating a canal bed 10 ft. deep and 20 ft. wide the earth is built into an embankment whose cross-section is a trapezium of base 30 ft. and sloping sides inclined at 45° to the horizon, the width of the top side of the trapezium being 10 ft. Calculate the work done per foot-length of the canal. (1 cu. ft. of earth removed weighs 150 lb.)

62. Machines and Work.

The machines which are to be described are used to do work which it is not possible to do conveniently without their use. If we attach a " load " to the appropriate part of a machine and use the machine to move this load, thus doing work ON the load, we have to apply an " effort " to another part of the machine which does work ON the machine. The work done on the load by the machine is referred to as the OUTPUT OF WORK, and that done on the machine by the effort the INPUT OF WORK. The input of work done on a machine is in practice greater than the output of work done by the machine. The perfect machine would

therefore be one which could give an output of work equal to the input of work.

The "Principle of Work" applied to machines states that the input of work is equal to the output of work plus the work done against friction, plus the work done in moving parts of the machine.

Example (Class Discussion).—*Suppose that a machine could be invented which was able to give a greater output of work than the input of work done on the machine, what would be some of the consequences?*

63. Efficiency, Mechanical Advantage (Force Ratio) and Velocity Ratio.

The efficiency of a machine is measured numerically as follows :—

$$\text{EFFICIENCY} = \frac{\text{Output of work}}{\text{Input of work}}.$$

In addition to the efficiency, two other quantities are of importance, these are :—

$$\text{FORCE RATIO} = \frac{\text{Load moved}}{\text{Effort applied}} = \text{MECHANICAL ADVANTAGE}.$$

Mechanical Advantage is the name used by engineers.

$$\text{VELOCITY RATIO} = \frac{\text{Distance through which the effort is applied}}{\text{Distance through which the load is moved}}.$$

Thus,

$$\frac{\text{FORCE RATIO or MECHANICAL ADVANTAGE}}{\text{VELOCITY RATIO}}$$

$$= \frac{\text{Load moved}}{\text{Effort applied}} \div \frac{\text{Distance through which effort is applied}}{\text{Distance through which load is moved}}$$

$$= \frac{\text{Load moved} \times \text{Distance through which load is moved}}{\text{Effort applied} \times \text{Distance through which effort is applied}}$$

$$= \text{EFFICIENCY}.$$

The definition of velocity ratio indicates that it is independent of the load moved or the effort applied, in fact it depends only upon the geometrical properties of the machine.

The mechanical advantage of a machine evidently depends upon the velocity ratio, but it also depends upon the frictional resistances in the machine, and it must therefore be determined experimentally. Moreover, the frictional resistances in a machine are not constant, in fact we may anticipate that they increase with the load upon the machine.

If these two ratios are determined for a machine, then the efficiency can be found as above by dividing the mechanical advantage by the velocity ratio. It has already been stated that the perfect machine is one which is able to give an output of work equal to the input of work. Thus from the above definition of efficiency we see that unity is the greatest numerical value of the efficiency of a machine. Efficiency is sometimes expressed as a percentage, thus an efficiency of $\frac{3}{4}$ is equivalent to 75 per cent.

64. We are able, by various mechanical devices such as are to be described, to arrange for the velocity ratio of a machine to be as great as we please. If a machine has a high velocity ratio the effort must be applied through a relatively large distance as compared to the distance through which the load is moved.

By the principle of work as applied to machines,

Work done by effort = Work done on the load + Work done against frictional resistance + Work done in moving parts of the machine.

Thus when the velocity ratio is high the load moved is also many times the effort applied, *i.e.*, the mechanical advantage is high, and as we have already seen the velocity ratio is equal to the mechanical advantage in a perfect machine, for which the work done by the effort is equal to the work done on the load.

65. Pulleys.

A simple pulley which is used in experiments in mechanics consists of an aluminium wheel of about 2 in. diameter, the bearings of which are carried by an aluminium framework. The groove in the wheel carries the string, which is prevented from slipping out of the groove by the framework. If we attach the

framework to a fixed support and fasten equal weights to the ends of a string which passes over the groove in the pulley, these weights will hang at rest. When one of the weights is tapped

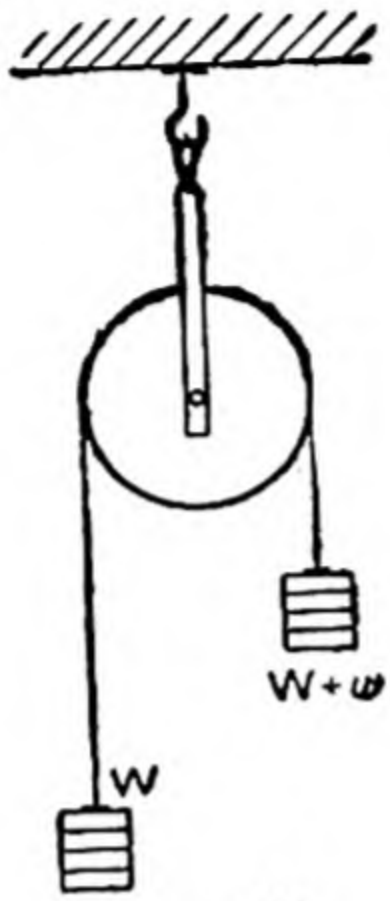


FIG. 111.

lightly downwards the weights move through a short distance before coming to rest. They are brought to rest by the friction at the bearings of the pulley. Suppose that we now add a "rider" weight to one of the weights, such that when this weight is touched lightly downwards it continues to move as long as it meets no obstacle without any change in speed. This condition of motion has to be judged as nearly as possible by careful observation. We may have to be satisfied by getting the weights to move through as great a distance as possible before again coming to rest. To calculate the work on the machine, let each weight be W lb., and w lb. the weight of the rider, then when the weights each move through a distance x ft. the work done on the machine by the effort $(W + w)$ lb. wt. is

$$(W + w)\ x \text{ ft. lb.}$$

The work done by the machine is

$$W\ x \text{ ft. lb.},$$

and this is done in raising the load W lb. wt.

The part of the input of work which is done in counteracting the effect of friction at the bearings of the pulley is wx ft. lb.

In earlier experiments it was assumed that the tension throughout a light string passing over a number of pulleys, such as that referred to above, was constant for practical purposes. We now observe that in the above experiment the difference of the tensions on either side of the pulley $(W+w)$ lb. wt. and W lb. wt., viz., w lb. wt., although small in comparison with either tension, is yet sufficient to balance the effect of the friction. The smaller we can make w, the more perfect will the pulley become as a machine.

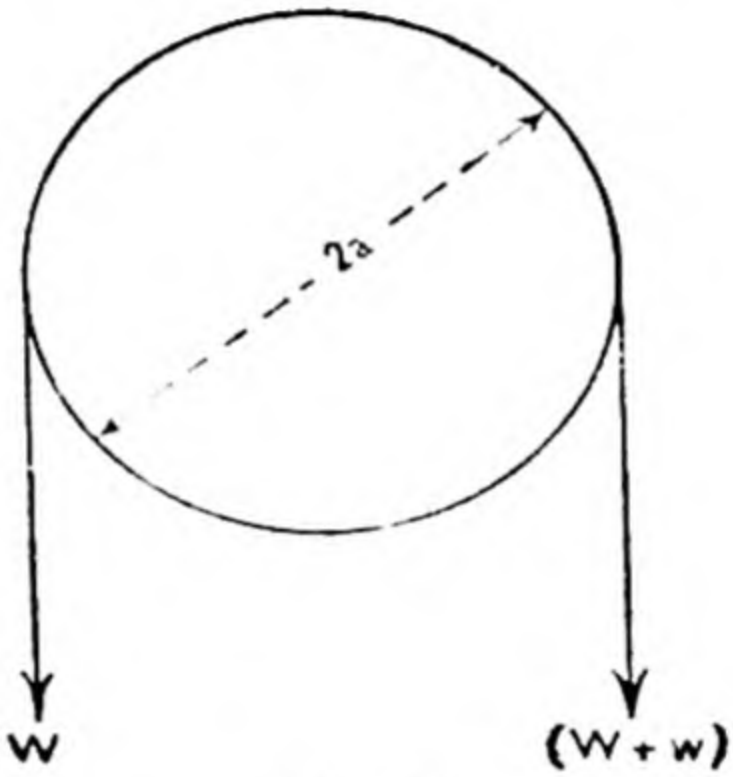


FIG. 112.

$$\text{Efficiency of pulley} = \frac{\text{Output of work}}{\text{Input of work}}$$

$$= \frac{Wx}{(W+w)x}$$

$$= \frac{W}{(W+w)} \text{ or } \frac{100W}{(W+w)} \text{ per cent.}$$

$$\text{Mechanical advantage} = \frac{\text{Load moved}}{\text{Effort applied}} = \frac{W}{(W+w)}.$$

$$\text{Velocity ratio} = \frac{\text{Distance through which effort is applied}}{\text{Distance through which load is moved}} = \frac{x}{x} = 1.$$

In the diagram the pulley is on the point of turning in a clockwise direction about its bearings. Thus, taking moments about the centre of the pulley,

$$(W + w)a - Wa - \text{Turning effect of friction} = 0.$$

Hence
$$wa = \text{Turning effect of friction.}$$

$$w = \frac{\text{Turning effect of friction}}{a}.$$

Thus the larger a is made the smaller does w become, and referring to the expression for the efficiency of the pulley, the more closely does this approach unity, and consequently the more perfect is the machine. Probably the best example of large pulleys used in practice is to be found at a pithead where the pulleys are used for raising and lowering the cage.

Example (Mental).—*A single pulley is used to raise a load of weight* 200 *lb. through a vertical height of* 20 *ft. If the average vertically downwards pull on the pulley rope required to raise a load is* 210 *lb. calculate:*

FIG. 113.

(a) *The output of work.*
(b) *The input of work.*
(c) *The efficiency of the pulley.*
(d) *The mechanical advantage.*
(e) *The velocity ratio of the pulley.*

66. The Wheel and Axle.

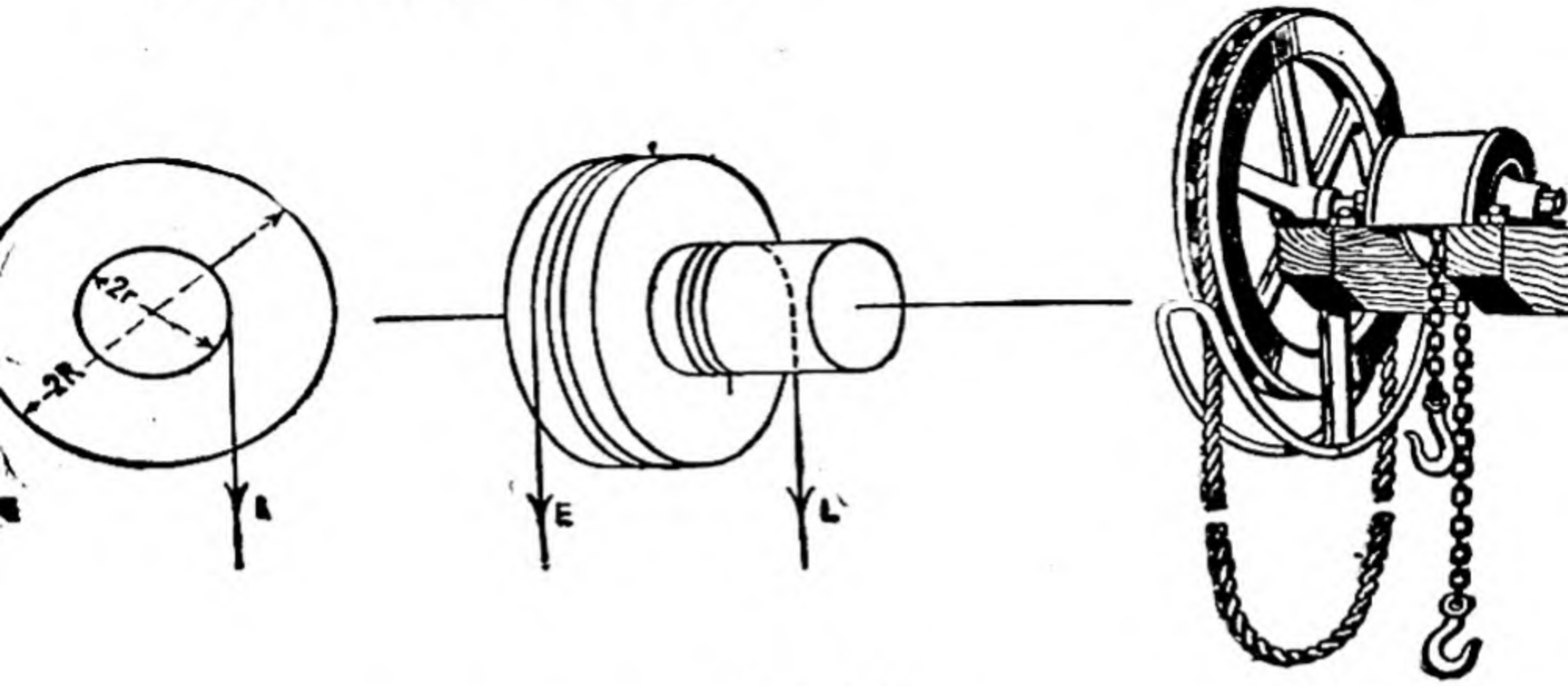


FIG. 114.

It has been shown that by means of this machine a load can be raised by a comparatively small effort.

The distance moved through by the effort applied at the rim of the wheel in one rotation is $2\pi R$ ft. (R ft. is the radius of the wheel.)

If E lb. wt. is the effort applied and L lb. the load raised, the input of work is $2\pi RE$ ft. lb.

Similarly, the output of work is $2\pi r$. L ft. lb.

Therefore, neglecting the effect of friction,

$$\text{Efficiency} = \frac{2\pi r \,.\, L}{2\pi RE} = \frac{rL}{RE}.$$

$$\text{Mechanical advantage} = \frac{L}{E}, \quad \text{Velocity ratio} = \frac{2\pi R}{2\pi r} = \frac{R}{r}.$$

Now if there is no friction by taking moments about the centre of the wheel (see diagram), $rL = RE$,

and $$\frac{L}{E} = \frac{R}{r}, \quad \therefore \text{ Mechanical advantage} = \text{Velocity ratio.}$$

Hence the efficiency of the wheel and axle is unity. This is

only what we expect under these ideal conditions of no friction, for in this case the output of work is equal to the input of work.

If the rope or string used in the machine has a diameter comparable with that of the axle it should be taken into account. Thus half the diameter of the rope should be added to the radius of either the wheel or axle as the case demands. By taking the thickness of the rope into account in one of the calculations, it is easy to find out whether or not it can be neglected.

EXPERIMENT.—Attach a load to the rope passing round the axle of a wheel and axle, and apply an effort to the rim of the wheel so that it is on the point of moving the load, and when touched lightly downwards it continues to move without gaining speed. Repeat the experiment a number of times with increasing loads and tabulate the results.

Below is a table of results obtained in an experiment similar to the above :—

Velocity Ratio = 4

Effort.	Load.	Mechanical Advantage.	Efficiency.
Lb. wt.	Lb.		
0·25	0	0·00	0·000
0·55	1	1·82	0·455
0·85	2	2·35	0·588
1·15	3	2·61	0·653
1·50	4	2·67	0·668
1·80	5	2·78	0·695
2·10	6	2·86	0·715
2·40	7	2·92	0·730
2·70	8	2·96	0·740
3·00	9	3·0	0·750
3·30	10	3·03	0·758

It should be noticed that the table is completed by first calculating the mechanical advantage, and by dividing the mechanical advantage by the velocity ratio the efficiency is obtained. When the efficiency is plotted against a load base, it will be noticed that *the efficiency increases rapidly with the load raised at first*, i.e., *for comparatively small loads, but that as the load increases the efficiency increases more and more slowly until it remains almost constant for the last three loads.* The shape of the graph is similar to that obtained for a wide variety of machines.

Example (1).—*A wheel and axle is used for raising a load of 2 cwt. The radius of the wheel is 10 in. and that of the axle 2 in. Neglecting friction, what effort is required?*

Let E lb. wt. be the effort required.

Work done by the effort in one turn of the wheel $= 2\pi \cdot 10 \cdot E$ in. lb.
Work done in raising the load in one turn $= 2\pi \cdot 2 \times 224$ in. lb.
Since friction is to be neglected,

$$2\pi \cdot 10 \cdot E = 2\pi \cdot 2 \times 224,$$

$$E = \frac{2\pi \cdot 2 \times 224}{2\pi \cdot 10} = 44{\cdot}8.$$

The effort required is 44·8 lb. wt.

Otherwise, since for a perfect machine

$$\text{Mechanical advantage} = \text{Velocity ratio},$$

$$\frac{224}{E} = \frac{2\pi \cdot 10}{2\pi \cdot 2} = 5.$$

$$224 = 5\,E.$$

$$E = 44{\cdot}8 \text{ as above.}$$

Example (2).—*The diameters of wheel and axle in a wheel-and-axle machine are 2 ft. and 6 in. respectively. A load of 200 lb. is raised by an effort of 60 lb. wt. What is the efficiency of the machine for this load and effort? What additional load could have been raised if the machine had been frictionless?*

$$\text{Mechanical advantage} = \frac{\text{Load raised}}{\text{Effort applied}} = \frac{200}{60} = \frac{10}{3}.$$

$$\text{Velocity ratio} = \frac{2\pi \cdot 1}{2\pi \cdot \frac{1}{4}} = 4.$$

$$\text{Efficiency} = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}} = \frac{10}{3} \div 4 = \frac{5}{6}$$

$$= 0{\cdot}833 \text{ or } 83{\cdot}3 \text{ per cent.}$$

Otherwise $\quad \text{Efficiency} = \dfrac{\text{Output of work}}{\text{Input of work}}.$

Output of work $= 2\pi \cdot \frac{1}{4} \cdot 200$ ft. lb. in one turn of the axle.
Input of work $= 2\pi \cdot 1{\cdot}60$ ft. lb. in one turn of the wheel.

$$\text{Efficiency} = \frac{2\pi \cdot \frac{1}{4} \cdot 200}{2\pi \cdot 1{\cdot}60} = 0{\cdot}833 \text{ or } 83{\cdot}3 \text{ per cent.}$$

If there had been no frictional effect an effort of 60 lb. wt. would have raised a load of 60 × velocity ratio lb. $= 60 \times 4$ lb. $= 240$ lb.
The additional load raised is therefore 40 lb.

EXAMPLES XVI

Wheel and Axle, Windlass, Capstan, Mechanical Advantage, Velocity Ratio, Efficiency

1. In a wheel and axle the diameter of the wheel is 1 ft. 6 in. and that of the axle 6 in. What is the velocity ratio of the machine? What load is raised by an effort of 4 lb. wt. if friction is neglected?

2. By means of a wheel and axle for which friction is negligible an effort of 8 lb. wt. raises a load of 48 lb. Calculate the mechanical advantage. What is the velocity ratio? If the diameter of the wheel is 3 ft. what is the diameter of the axle?

3. The velocity ratio of a wheel and axle is 5, and a load of 320 lb. is raised by an effort of 80 lb. wt. What is the efficiency of the machine? What fraction of the effort of 80 lb. wt. is used in overcoming the friction in the machine?

4. The efficiency of a wheel and axle is 0·84 when a load of 168 lb. is raised by an effort of 56 lb. wt. The radius of the wheel is 1 ft. 6 in. What is the velocity ratio of the machine and the radius of the axle?

5. A windlass has a barrel of diameter 8 in., and is turned by a handle of radius 1 ft. 6 in. What effort is required to raise a load of 450 lb. when friction is neglected? If the actual effort required is 120 lb. wt., what is the efficiency of the machine for this effort and load?

6. (*a*) If 25 per cent. of the input of work for a machine is used in overcoming frictional effects, etc., what is the efficiency of the machine?

(*b*) If 25 per cent. of the actual load raised by a lifting machine is equal to the additional load which could be raised by the same effort applied to the machine if there were no frictional effect, what is the efficiency of the machine?

7. A capstan is used to haul a load. Four men apply equal efforts of 40 lb. wt. each perpendicular to levers at equal distances of 5 ft. from the axis of the capstan barrel. Find the work done by the men in fifteen complete turns of the barrel. If the efficiency of the capstan is 60 per cent. and the barrel has a diameter of 1 ft., calculate the tension in the rope being wound on the barrel.

8. In a wheel-and-axle experiment the following results were obtained:—

Effort. Lb. wt.	Load. Lb.	Effort. Lb. wt.	Load. Lb.
2·0	10	21·0	60
5·8	20	25·0	70
9·6	30	28·6	80
13·5	40	32·5	90
17·2	50	36·4	100

Calculate the efficiency at the various loads and draw the graph of the efficiency on a load base. (Velocity ratio of machine = 6.)

9. A lifting machine is used to raise a load of 500 lb. When the effort has moved through 100 ft. the load is raised 24 ft. and the efficiency of the machine is 75 per cent. Find the effort applied.

67. Systems of Pulleys.

The single pulley is extensively used for lifting purposes, but since the effort applied must be greater than the load moved

it is evident that it is at a disadvantage when a heavy load is to be moved in spite of its relatively high efficiency.

Systems of more than one pulley for lifting and hauling purposes used in cases where the load is large are easily designed, so that by arranging for a high velocity ratio the effort required to move the load is relatively small. Such systems of pulleys have usually greater internal frictional resistances and are less efficient than the single pulley.

It is now proposed to examine some of these systems of pulleys, for they can readily be built up in experimental form for demonstration purposes in a mechanics laboratory.

68. One Movable Pulley.

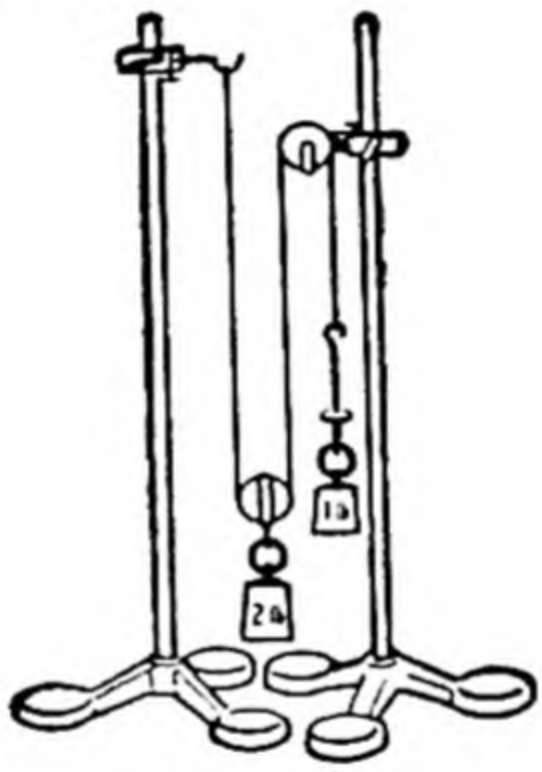


FIG. 115.

Attach a load to a pulley, which in turn is supported by a string passing round the pulley and fastened at its two ends to two vertical spring balances, so that the parts of the string on both sides of the pulley are vertical. When the load is at rest the tensions registered by the spring balances are each equal to one-half of the load and the weight of the pulley together, as one might expect, since the pulley acts as a beam with supports at its ends and the load at its midpoint.

Remove the right-hand spring balance from its support and attach it to a vertical string which passes over a fixed pulley and

carries a weight E at its other end. E is adjusted until the load L is on the point of rising. There is now a slight difference in the tensions registered by the two spring balances, caused by the fact that the friction at the bearings of the pulley must be overcome before the load can be raised. Remove both spring balances, leaving only the pulley system, consisting of one movable pulley and one fixed pulley. We have now a machine for raising L, and the effort applied only slightly exceeds half of the weight of the movable pulley and the load raised.

$$\text{Mechanical advantage} = \frac{\text{Load raised}}{\text{Effort applied}} = \frac{L}{E}.$$

This ratio is equal to 2 only under ideal conditions of negligible friction and negligible weight of the movable pulley.

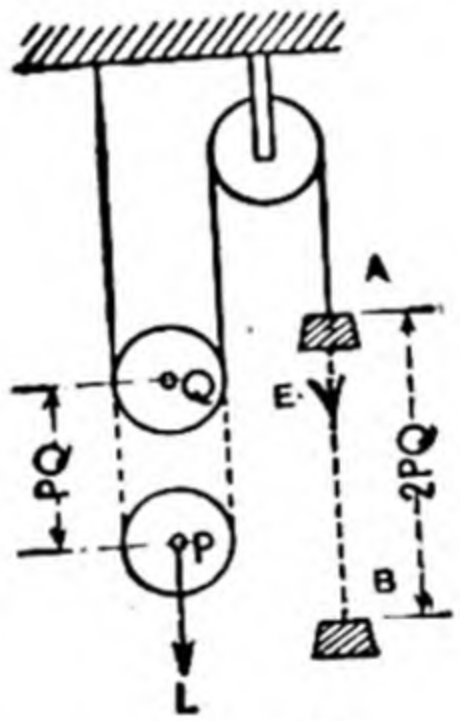


FIG. 116.

Let A be the initial position of the end of the string and P that of the axle of the movable pulley. When P is raised from P to Q the string must be taken up a length PQ on both sides of the pulley, and, therefore, if the end A is finally at B,

$$AB = 2 \,.\, PQ.$$

$$\text{Velocity ratio} = \frac{\text{Distance through which effort is applied}}{\text{Distance through which load is moved}} = \frac{AB}{PQ} = 2.$$

$$\text{Efficiency} = \frac{\text{Output of work}}{\text{Input of work}} = \frac{L \,.\, PQ}{E \,.\, AB}.$$

Under ideal conditions of negligible friction and negligible weight of movable pulley the efficiency becomes unity.

69. Effect of Frictional Resistance on Efficiency.

The work done against frictional resistance in a machine depends upon the place at which the friction is exerted. Thus in the machine which has just been considered, twice as much work must be done in overcoming frictional resistance at the fixed pulley as at the movable one. Why is this ? It is important, therefore, if a machine is to be made more efficient to avoid frictional resistance where it is most effective. It should also be noticed that frictional effects always oppose motion in whichever direction it takes place or tends to take place. Thus if the load on a machine tends to " take charge " of the machine, as in a lifting machine when the effort is removed, the frictional resistances are completely reversed and oppose motion with much the same effect when the load is moving the machine as when the effort is moving it.

70. Archimedes System of Pulleys.

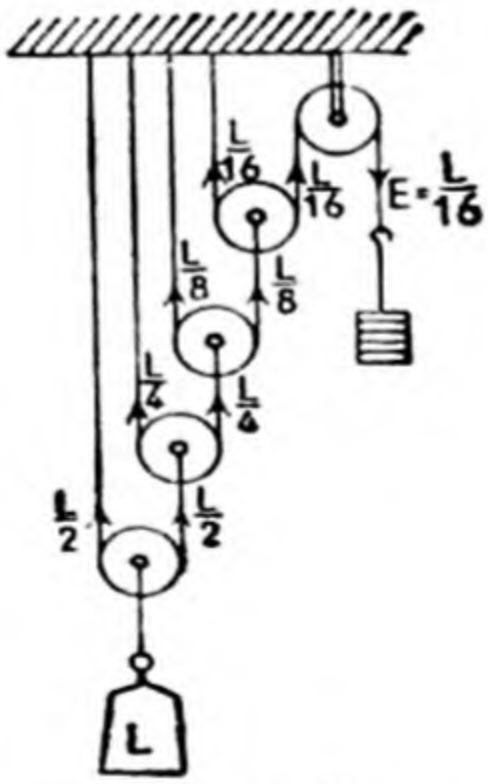


FIG. 117.

If the string supporting the movable pulley is attached to the axle frame of a second movable pulley supported in a similar way to the first, it is readily seen that the first system of pulleys

may be extended to include as many movable pulleys as is desired, before the string supporting the last movable pulley is passed over the single fixed pulley. If we neglect friction and the weights of the movable pulleys,

The tension in the first string supporting the load is $\frac{1}{2}$L,

" " " second string is $\frac{1}{4}$L,

" " " third " $\frac{1}{8}$L,

" " " nth "

$$\frac{L}{2\times2\times2\times \ldots n \text{ factors}} = \frac{L}{2^n}.$$

Thus with n movable pulleys the effort applied to raise a load L is $\frac{L}{2^n}$.

$$\text{Mechanical advantage} = L \div \frac{L}{2^n} = L \cdot \frac{2^n}{L} = 2^n.$$

Under ideal conditions a system of three movable pulleys has a mechanical advantage equal to $2\times2\times2=8$.

If the first pulley is raised 1 in. (*i.e.*, the load is raised 1 in.); the second pulley is raised 2 in.; the third 4 in.; the fourth 8 in.; the nth pulley is therefore raised

$$2\times2\times2 \ldots (n-1) \text{ factors} = 2^{(n-1)} \text{ in.}$$

The distance through which the effort is applied is therefore

$$2\times2\times2 \ldots n \text{ factors} = 2^n \text{ in.}$$

$$\text{Velocity ratio} = \frac{2^n}{1} = 2^n.$$

71. The Single-rope System of Pulleys.

This system of pulleys consists of two pulley blocks, one movable and one fixed. The effort is applied by means of a rope, which passes in turn round a pulley in the fixed block and then round the pulley in the movable block, and so on until the rope has passed once round each pulley in the two blocks. The rope is finally attached to the frame of one or other of the two blocks. In practice the pulleys on any one block are arranged side by

side with their axle bearings on the same shaft. In order to show how the system works the pulley blocks are often arranged for

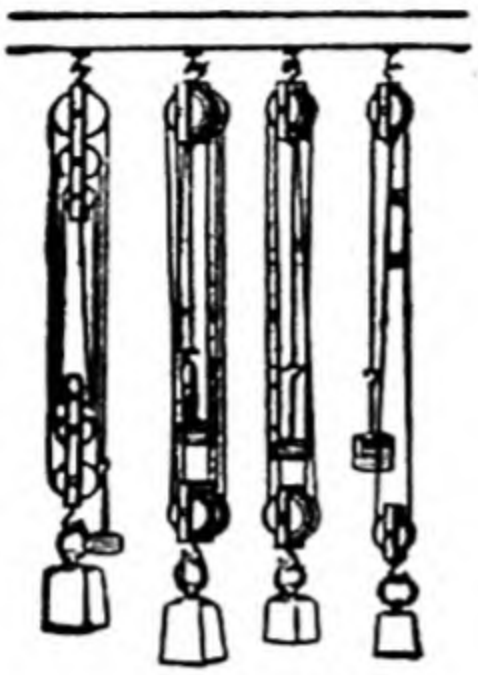


FIG. 118.

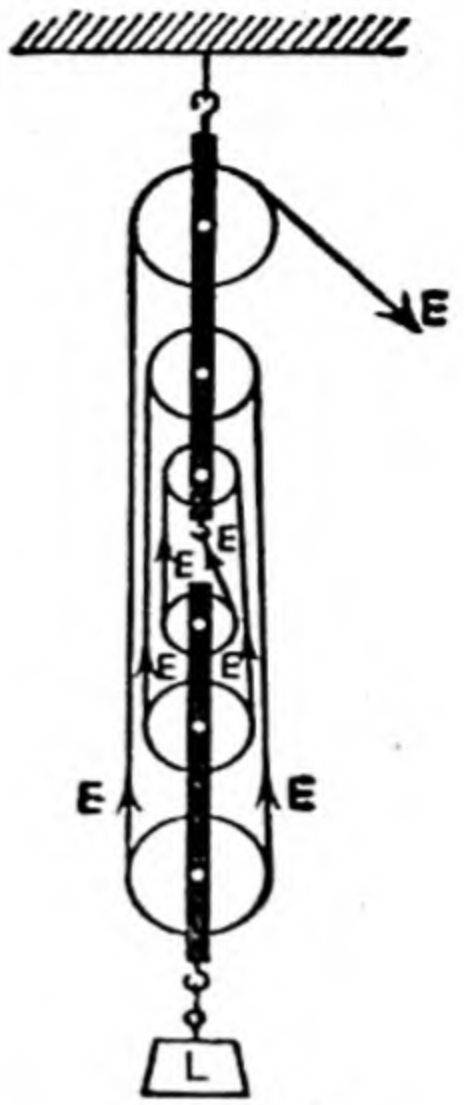


FIG. 119.

FIG. 120.

laboratory experiments as they are illustrated in Fig. 119, whereas in practice they are used in the form shown in Fig. 120. It is usual for the movable block to carry one less pulley than the fixed one, and in this case the rope is finally attached to the frame of the movable block. If the blocks carry equal numbers of pulleys the rope is finally attached to the fixed block.

Consider the system of six pulleys in which the upper block has three pulleys and the lower block three. The number of vertical ropes which support the lower block and its load is equal to the number of pulleys, *i.e.*, six. If we neglect friction the tension in each of these six vertical ropes is equal to the effort applied, E lb. wt. shall we say; hence the total upward force which supports the load L lb. and the weight w lb. of the lower pulley block is 6E lb. wt. Therefore,

$$L + w = 6E.$$

$$\text{Mechanical advantage} = \frac{L}{E} = \frac{6E - w}{E} = 6 - \frac{w}{E}.$$

This expression is only correct when the pulleys are used for raising a vertical load. If they are used for haulage in a horizontal direction the mechanical advantage neglecting friction is not affected by the weights of the pulleys. Thus in the system we are considering the mechanical advantage when used for hauling in a horizontal direction is equal to 6.

Suppose that the effort is applied through a distance d ft., then this length of rope is equally taken up by each of the six ropes supporting the movable pulley block, each rope is therefore shortened in length by $d/6$ ft.; this is the distance through which the load is moved. Thus the effort is applied through six times the distance moved by the load, and therefore

$$\text{Velocity ratio} = 6.$$

Similar results to those obtained above hold for the system of five pulleys illustrated in the figure.

If the system has n pulleys the total force exerted on the movable pulley block, neglecting friction, is n times the effort applied, and the velocity ratio is n.

Example (1).—*In an Archimedes system of three movable pulleys the effort required to raise a load of 64 lb. is 8 lb. wt. The efficiency under these conditions is ½. Calculate the velocity ratio of the system.*

$$\text{Mechanical advantage} = \frac{\text{Load raised}}{\text{Effort applied}} = \frac{64}{8} = 8.$$

Now, Efficiency = Mechanical advantage ÷ Velocity ratio.

$$\tfrac{1}{2} = \frac{8}{\text{Velocity ratio}}.$$

$$\tfrac{1}{2} \times \text{Velocity ratio} = 8.$$

$$\text{Velocity ratio} = 16.$$

Example (2).—*In a single rope system of five pulleys the lower block weighs 2 lb., and the effort required to raise a load of 60 lb. is 15 lb. wt. Calculate the efficiency of the system. If there is no friction, calculate the efficiency when the load raised is 60 lb.*

$$\text{Mechanical advantage} = \frac{\text{Load raised}}{\text{Effort applied}} = \frac{60}{15} = 4.$$

Velocity ratio = Number of pulleys = 5.

Efficiency = Mechanical advantage ÷ Velocity ratio

= 4/5 = 0·8, or 80 per cent.

If there is no friction, then

5 × Effort = Load raised + Weight of movable pulley block

= 62 lb. wt.

Effort = 12·4 lb. wt.

Efficiency = Mechanical advantage ÷ Velocity ratio

$$= \frac{60}{12{\cdot}4} \div 5$$

$$= \frac{60}{12{\cdot}4 \times 5}$$

$$= \frac{60}{62}$$

= 0·97, or 97 per cent. to the nearest 1 per cent.

Example (3).—*An Archimedes system of pulleys has three movable pulleys and one fixed pulley. Each movable pulley weighs 2 lb. Calculate the effort required to raise a load of 10 lb. and the efficiency of the system for this load. Friction is neglected.*

The tension in the first string is ½(10 + 2) lb. wt. = 6 lb. wt.
" " " second " ½(6 + 2) " = 4 "
" " " third " ½(4 + 2) " = 3 "

The effort required is therefore 3 lb. wt.

$$\text{Mechanical advantage} = \frac{\text{Load raised}}{\text{Effort applied}} = \frac{10}{3}.$$

$$\text{Velocity ratio} = 2^3 = 2 \times 2 \times 2 = 8.$$

$$\text{Efficiency} = \text{Mechanical advantage} \div \text{Velocity ratio}$$

$$= \frac{10}{3} \div 8$$

$$= \frac{10}{3} \times \frac{1}{8} = \frac{5}{12} = 0{\cdot}42, \text{ or } 42 \text{ per cent.}$$

Correct to the nearest 1 per cent.

72. The Perfect Machine.

The definition of the efficiency of a machine shows that this ratio would become unity (or 100 per cent.) if the machine could give an output of work equal to the input of work, *i.e.*, if the machine were perfect. The efficiency therefore compares the output of work of a machine with its output under ideal conditions of no frictional resistances or weights of moving parts which may have to be raised. It is also readily understood that if a machine is perfect its mechanical advantage is equal to its velocity ratio, for in this case

$$\text{Efficiency} = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}} = 1,$$

i.e., Mechanical advantage = Velocity ratio.

Now the effort which raises a given load

= Load raised ÷ Mechanical advantage.

Therefore in the case of the perfect machine,

Effort applied = Load raised ÷ Velocity ratio.

Since the efficiency of a machine is equal to

$$\frac{\text{Actual output of work}}{\text{Output of work if the machine were perfect}},$$

and both actual load and the theoretical load are moved through the same distance, therefore efficiency

$$= \frac{\text{Actual load raised}}{\text{Load raised by the same effort if the machine were perfect}}.$$

Again, the efficiency may be measured by

$$\frac{\text{Input of work to raise a given load if the machine were perfect}}{\text{Actual input of work to raise the load}},$$

and since both the actual effort and the theoretical effort move through the same distance, therefore efficiency

$$= \frac{\text{Effort required to raise a load if the machine were perfect}}{\text{Actual effort required to raise the load}}.$$

Example (1).—*In a lifting machine the efficiency is* 30 *per cent. when a load of* 50 *lb. is being moved. If the velocity ratio of the machine is* 10, *find the effort required to move the load.*

Let E lb. wt. be the effort required.

On a perfect machine with a velocity ratio of 10, the effort required to raise a load of 50 lb. is $\frac{50}{10}$ lb. wt. $= 5$ lb. wt.

$$\text{Efficiency} = \frac{\text{Effort required if the machine were perfect}}{\text{Actual effort for the same load}}.$$

$$\frac{3}{10} = \frac{5}{E}$$

$$E = \frac{50}{3} = 16\tfrac{2}{3}.$$

The effort required is $16\frac{2}{3}$ lb. wt.

Example (2).—*A one-rope block and tackle has two pulleys in each block. An effort of* 10 *lb. wt. is required to raise a load when the efficiency is* 0·8. *Find the load raised.*

Let L lb. be the load raised.

The velocity ratio of the block and tackle is 4.

The load raised by the effort of 10 lb. wt. if the machine were perfect $= 4 \times 10$ lb. $= 40$ lb.

$$\text{Efficiency} = \frac{\text{Actual load raised}}{\text{Load raised by the same effort if the machine were perfect}}.$$

$$0{\cdot}8 = \frac{40}{L}.$$

$$0{\cdot}8L = 40.$$

$$L = \frac{40}{0{\cdot}8} = 50.$$

Therefore the load raised is 50 lb.

EXAMPLES XVII

1. In an Archimedes system of pulleys, if the five movable pulleys are of negligible weight and there is no friction, calculate the effort required to raise a load of 256 lb. If each of the five movable pulleys weighs 2 lb., what is now the effort required? Calculate the mechanical advantage and the velocity ratio in the latter case.

2. A load of 120 lb. is moved by means of a one-rope system of six pulleys. Under ideal conditions calculate the effort required. State these ideal conditions. If the weight of each pulley block is 6 lb., calculate the effort required to raise the same load. Find also the mechanical advantage and efficiency in this latter case and the force exerted on the support of the stationary pulley block.

3. Calculate the mechanical advantage and efficiency of an Archimedes system of two movable pulleys which weigh 1 lb. each when it is used to raise a load of 20 lb. (Neglect frictional effects.)

4. In an Archimedes system of pulleys show that, if friction and the weights of pulleys are neglected, the vertically downward forces exerted by the successive strings carrying the pulleys (beginning at the first movable pulley next to the fixed pulley) on their support are E, 2E, 4E, 8E . . ., etc. Hence find the TOTAL downward force on the support when there are four movable pulleys, and show that it is equal to the load plus the effort. (Assume that the effort is applied vertically downwards.)

5. The efficiency of a system of pulleys which is used to raise a load of 50 lb. is 0·4 when the effort applied is 10 lb. wt. Show that the velocity ratio is 12·5. If the calculated mechanical advantage is 6 when friction is neglected but the weights of the pulleys are taken into account, how much of the effort, 10 lb. wt., is used in overcoming friction?

6. An Archimedes system of four movable pulleys is used to raise a load of 112 lb. The effort applied is 21 lb. wt. Show that the efficiency of the system is $\frac{1}{3}$.

7. An Archimedes system of three pulleys has an efficiency of $\frac{4}{7}$ when an effort of 6 lb. wt. is required to raise a load. Show that this load is 36 lb., and that, if the movable pulleys have equal weights, each weighs $1\frac{5}{7}$ lb. (Friction is neglected.)

8. Draw careful diagrams illustrating one-rope systems of pulleys, using six and seven pulleys respectively. In each case if the friction and weights of the movable blocks are neglected, calculate the velocity ratio and the mechanical advantage.

9. The weight of the movable block in a system of five pulleys, using one rope, is 2 lb. If friction is neglected, verify the following table as far as it goes and extend it up to efforts of 10 lb. wt. What do you notice about the way in which the efficiency changes as the load increases?

Load.	Effort.	Mechanical Advantage.	Velocity Ratio.	Efficiency.
0	0·4	0	5	0
2	0·8	2·5	5	0·5
4	1·2	3·33	5	0·667
6	1·6	3·75	5	0·75

GOVT GIRLS COLLEGE SRINAGA ACC. NO.....

10. A one-rope system of four pulleys is used to raise a 2-cwt. load, and an effort of 70 lb. wt. is required and is applied vertically downwards. Calculate the efficiency of the system under these conditions. How much of the effort is used in overcoming the effects of friction and the weight of the movable block ?

11. The efficiency of a one-rope system of three movable pulleys when used to move a load of 100 lb. is $\frac{1}{2}$. Calculate the effort required and the mechanical advantage under these conditions.

12. A one-rope system of movable pulleys has six pulleys and is used to raise a vertical load of 10 cwt. If the movable block weighs 38 lb., calculate the efficiency, neglecting friction, and show that the effort required is 193 lb. wt.

13. An effort of 10 lb. wt. is required to raise a load of 40 lb. when a one-rope system of five pulleys is used. Calculate the weight of the lower block of pulleys and the efficiency of the system.

14. Make a sketch of a three-sheave pulley block. If it is required to raise a load of 4 cwt. by means of such tackle, what effort must be applied, assuming it to have an efficiency of 32 per cent. ?

15. Show that the velocity ratio of the Barton system of pulleys shown in the figure is 4, and that no work is lost in moving the pulleys

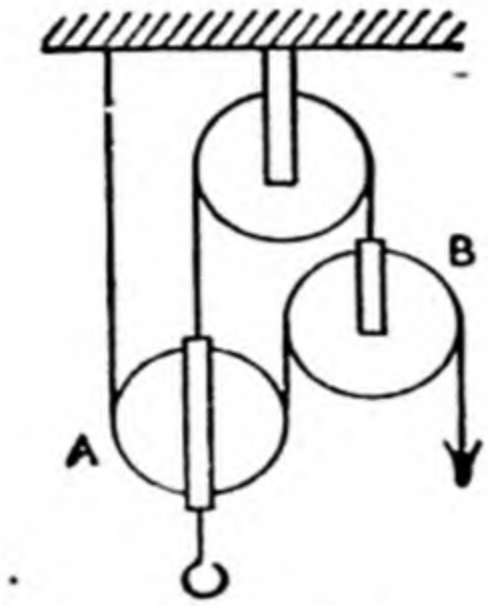


FIG. 121.

A and B vertically. If the weights of the movable pulleys A and B are 6 lb. each respectively, calculate the load which can be raised by an effort of 10 lb. wt. (Neglect friction.)

16. In the Barton pulley system, shown in question 15, a load of 260 lb. is raised by an effort of 100 lb. wt. Calculate the efficiency of the system. If there were no friction, what load could be raised by the effort of 100 lb. wt. ?

17. A 12-st. man pulls vertically downwards on the rope of a machine whose velocity ratio is 48 and efficiency 50 per cent. If the load raised is 600 lb., find the pressure of the man's feet upon the ground. (The forces acting ON the man are the reaction of the ground vertically upwards, his weight vertically downwards and the tension in the rope of the machine which acts vertically upwards.)

18. A one-rope pulley system of two blocks which have two and

three pulleys in them is used to haul a load horizontally. Which of the blocks should be attached to the load, and for what reasons ?

19. Distinguish between the mechanical advantage and efficiency of a machine. A pulley system consists of two blocks, each containing two pulleys, and the same string passes round each pulley. The upper block is suspended from a beam, and an effort of 20 lb. wt. is required to support a mass of 76 lb. suspended from the lower block. Find the velocity ratio, mechanical advantage and efficiency of the system. (*L.M.*)

20. What is meant by the terms velocity ratio, mechanical advantage and efficiency as applied to a machine ? Deduce the relation between them.

A system of pulleys consists of two blocks, the upper one, which is fixed, contains three pulleys, and the lower one, which is movable, two pulleys. Draw a diagram showing how the system is used for lifting a load. If an effort of 12 lb. wt. is required to lift a load of 50 lb. attached to the movable block, what is the efficiency of the system ? Why is the efficiency never 100 per cent. ? (*L.M.*)

73. The Differential (or Compound) Wheel and Axle and its application to the Weston Differential Pulley Block.

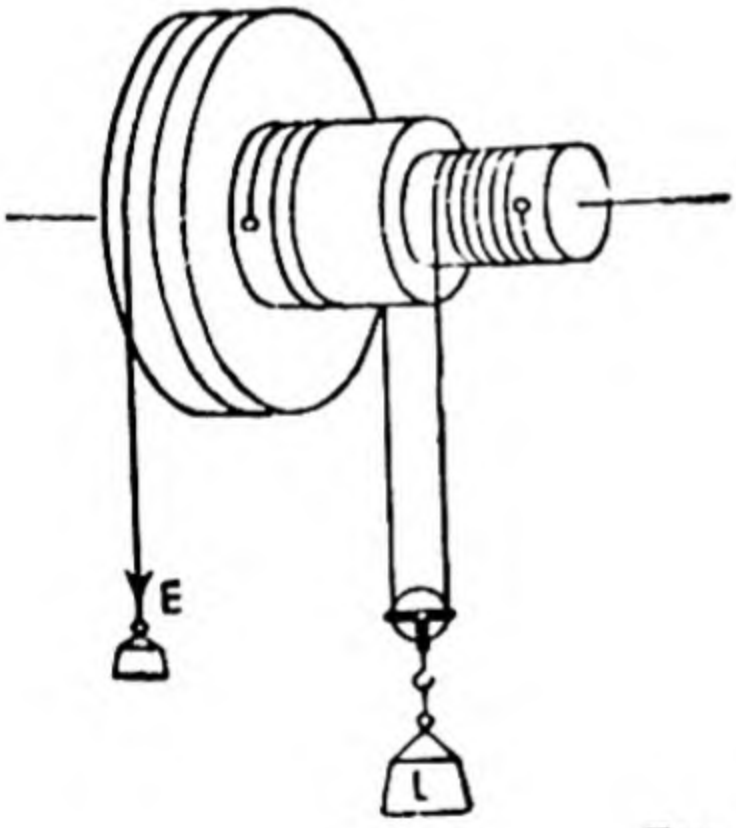
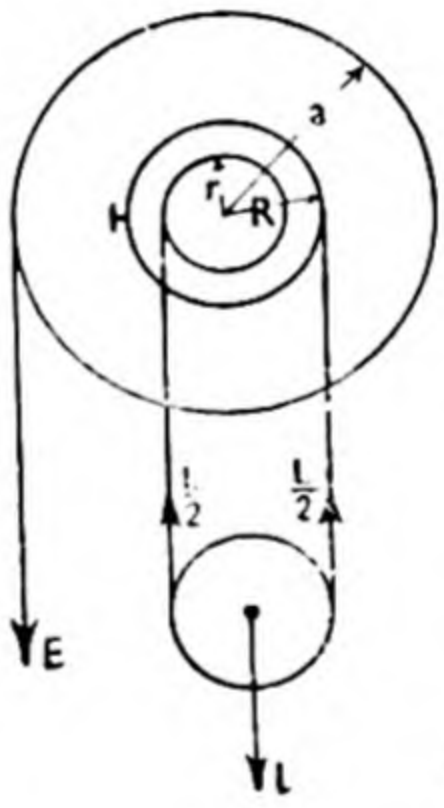


FIG. 122.



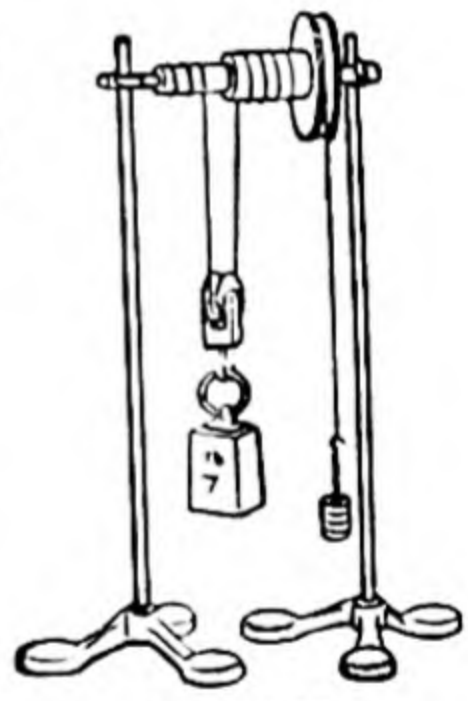


FIG. 123.

The axle of a differential wheel-and-axle arrangement is made up of two circular cylindrical parts rigidly connected, of different diameters, 2R and 2*r* in. A rope fastened to the part of the axle which has the larger diameter passes round this part several times, and after passing round a single pulley, which it supports, is finally wound IN THE SAME DIRECTION round the smaller part of the

axle to which it is finally secured. The load to be raised is supported by the pulley, and the effort is applied at the wheel as in the ordinary form of the wheel and axle, and turns the wheel in the opposite sense to that in which the rope is wound round the axle.

Suppose that the whole wheel and axle makes one complete rotation. The rope on one side of the pulley is drawn upwards $2\pi R$ in. The rope on the other side of the pulley is let down $2\pi r$ in. The string supporting the pulley is shortened by $2\pi(R-r)$ in., and therefore the pulley is raised $\pi(R-r)$ in.

If a in. is the radius of the wheel, the effort moves through a distance of $2\pi a$ in. in one rotation of the wheel.

$$\text{Velocity ratio} = \frac{\text{Distance through which the effort is applied}}{\text{Distance through which the load is moved}}$$

$$= \frac{2\pi a}{\pi(R-r)}$$

$$= \frac{2a}{(R-r)}.$$

It is thus easily seen that as the radii of the two parts of the axle are made more nearly equal, the denominator of this fraction is made smaller and the fraction correspondingly large. In this way a very large velocity ratio can be obtained.

If we neglect the weight of the pulley and the friction, the tension in each of the vertical ropes supporting the pulley is $\frac{1}{2}L$.

Take moments about the centre of the wheel and axle (referring to the end elevation),

$$Ea - \frac{L}{2}R + \frac{L}{2}r = 0$$

$$E = \frac{L(R-r)}{2a}$$

$$2aE = L(R-r)$$

$$\frac{2a}{(R-r)} = \frac{L}{E} = \text{Mechanical advantage.}$$

Thus as the radii of the parts of the axle are made more nearly equal the mechanical advantage becomes larger. The

machine is therefore capable of lifting very large loads with a small applied effort.

74. The Weston Differential Pulley Block.

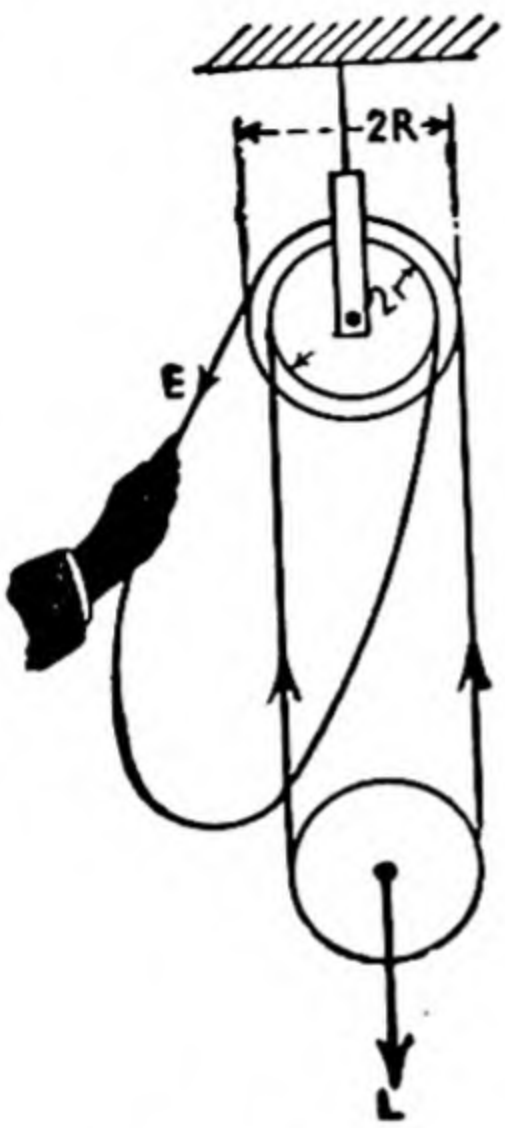


FIG. 124.

FIG. 125.

The principle of the differential wheel and axle finds its practical application in the form of the Weston differential pulley block. Two steel wheels of slightly different radii are rigidly fastened on the same shaft. An endless chain passes in turn round the upper parts of the two wheels in the same direction, and the links are held from slipping by properly formed grooves in the wheels. One of the loops of chain carries a pulley to which the load is attached. The effort is applied to the larger wheel by pulling the part of the chain which does not carry the pulley. As in the case of the differential wheel and axle, the calculated velocity ratio $= \frac{2R}{R-r}$, since the effort is applied to the wheel of

radius R instead of to the wheel of radius *a*. As in the case of the differential wheel and axle, it is advisable in a machine of this type to check the velocity ratio by actual experiment, *i.e.*, to measure the height through which the load is raised and the corresponding distance through which the effort is applied.

Overhauling.

It is often desirable in practice that a machine should be capable of remaining at rest when the effort is temporarily removed. If the load on a machine is capable of reversing the operation and overcoming the internal frictional resistances, when no effort is applied it is unsafe to remove the effort. Since the efficiency of a machine is equal to the mechanical advantage divided by the velocity ratio, it is clear that the mechanical advantage may be satisfactory for moving loads when the efficiency is small by virtue of the machine having a very high velocity ratio.

We are thus able to construct a machine with a low efficiency yet capable of giving any required mechanical advantage. Suppose that the efficiency is 50 per cent. corresponding to a certain effort and load, then since in a perfect machine the same effort would lift twice this load, the effect of friction (neglecting weights of moving parts) is equivalent to a load equal to that already on the machine. Thus if the effort is removed this frictional load reversed is capable of balancing the load on the machine, and the machine does not therefore overhaul. If a machine has an efficiency of less than 50 per cent., it follows that it will not overhaul. In a good deal of modern lifting apparatus the efficiency is much higher than 50 per cent., but the machine is provided with automatic braking, which prevents overhauling when the load tends to take charge.

EXAMPLES XVIII

Compound Wheel and Axle, Weston Pulley Block

1. The rim of the wheel of a compound wheel-and-axle arrangement has a diameter of 1 ft. 6 in., and the diameters of the parts of the axle are 5 and 3 in. Calculate the velocity ratio of the machine, showing how you obtain your answer.

If an effort of 3 lb. is required to raise a load of 36 lb., calculate the efficiency of the wheel and axle for this load.

2. Calculate the velocity ratio of a Weston pulley block in which the diameters of the compound pulley are 8 and 7 in. respectively.

3. What is the velocity ratio of a Weston pulley block in which the diameters of the compound pulley are (*a*) 9 and 7 in.; (*b*) 8 and $7\frac{1}{2}$ in.; (*c*) 8 and $7\frac{3}{4}$ in.; (*d*) 10 and 9 in.; (*e*) 10 and $9\frac{1}{2}$ in.; (*f*) 10 and $9\frac{3}{4}$ in.?

4. In a Weston pulley block the diameters of the compound pulley are 8 and 7 in. (*a*) Calculate the load raised for an effort of 20 lb. wt. (i) neglecting friction, (ii) when the efficiency is 65 per cent. (*b*) What is the mechanical advantage in the latter case? (*c*) What effort would be necessary to raise a load of $\frac{1}{4}$ ton, (i) neglecting friction, (ii) when the efficiency is 35 per cent.

5. Sketch an arrangement of pulleys for which the velocity ratio is 4. A Weston pulley block has a velocity ratio of 16, and it is found that an effort of 18 lb. wt. is required to lift a load of 1 cwt. Express the efficiency as a percentage and find how much work is done against friction. Does this work serve any useful purpose?

6. In a differential wheel and axle the diameter of the wheel is 2 ft. and the diameters of the parts of the axle are 6 and 4 in. Show that the velocity ratio of the arrangement is 24. If a load of 40 lb. is raised and the efficiency is then $33\frac{1}{3}$ per cent., calculate the corresponding effort.

7. In a compound wheel and axle the diameter of the wheel is 2 ft. 6 in. and the diameters of the parts of the axle are 9 and 8 in. Calculate the velocity ratio of the machine. If a load of 96 lb. is raised by an effort of 4 lb. wt., calculate the efficiency of the machine for this load. What part of the effort is required to overcome frictional effects in the machine?

8. The diameters of the compounds pulley of a Weston pulley block are 8 and $7\frac{1}{2}$ in. Show that the velocity ratio of the arrangement is 32. An effort of 6 lb. wt. is required to raise a load of 90 lb.; calculate the efficiency for this load. Show that the efficiency is the same as if an extra load of 102 lb. were added and there were no frictional resistances. Show that overhauling cannot take place if the efficiency of the machine is less than 50 per cent.

75. The Inclined Plane as a Machine.

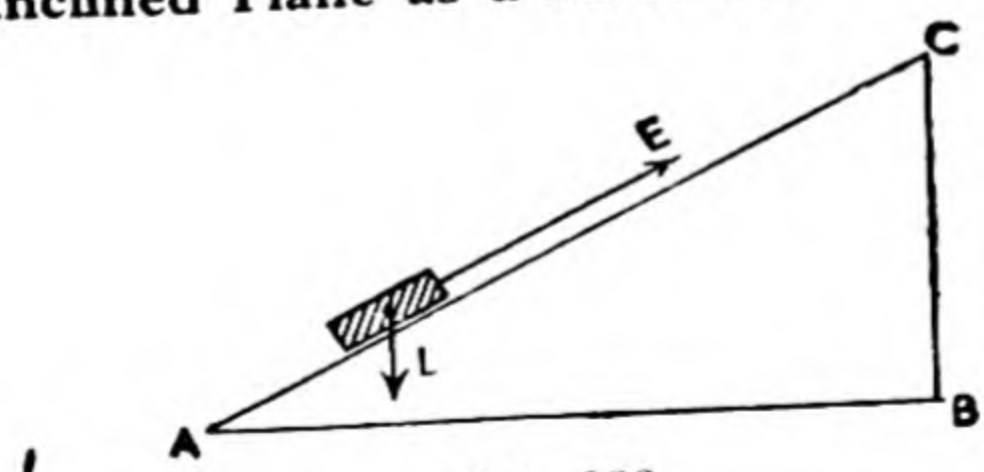


FIG. 126.

Suppose that a load rests upon an inclined plane AC and that it is acted upon by a force E parallel to the plane. By the

principle of work applied to machines, the work done by E is equal to the work done in raising the load vertically plus the work done against frictional forces.

(*a*) *Frictional Forces Neglected.*—The work done by the effort E is equal to the work done in raising the load vertically.

Referring to the figure,

$$E \cdot AC = L \cdot BC.$$

$$\frac{L}{E} = \frac{AC}{BC} = \text{Mechanical advantage.}$$

Hence the mechanical advantage of the machine can be made as large as we please by making AC large in comparison with BC.

(*b*) *Frictional Forces taken into Account.*—Let E_1 be the effort now required to raise the load L. E_1 is greater than E, the effort required to draw the same load L up the plane when there is no friction.

$$\text{Efficiency} = \frac{L \cdot BC}{E_1 \cdot AC} = \frac{E \cdot AC}{E_1 \cdot AC} = \frac{E}{E_1}$$

$$= \frac{\text{Effort required when there is no friction}}{\text{Actual effort}}.$$

The work done against friction

$$= \text{Input of work} - \text{Output of work}$$

$$= E_1 \cdot AC - L \cdot BC.$$

$$\text{Velocity ratio} = \frac{AC}{BC}.$$

Since the velocity ratio can be made very large, the inclined plane may be used to raise very big loads with comparatively small efforts.

The inclined plane finds a most important practical application in the use of screws as lifting machines, for the screw thread may be regarded as an inclined plane wound round and round in the form of a helix.

Example.—*A load of 2 tons is dragged 400 ft. up an inclined plane of 1 in 40 by a force of ½ ton wt. applied parallel to the plane. Find the efficiency of the machine and the work done against friction.*

$$\text{Mechanical advantage} = \frac{2}{\frac{1}{2}} = 4.$$

Velocity ratio = 40. (Correct to 1 per cent. accuracy.)

$$\text{Efficiency} = \frac{4}{40} = 0{\cdot}1, \text{ or } 10 \text{ per cent.}$$

Work done in raising the load = 2 × 10 ft. tons.
" " by the effort = ½ × 400 ft. tons.
" " against friction = (200 − 20) ft. tons.
= 180 ft. tons.

EXAMPLES XIX

1. A load of 80 tons is drawn 400 yds. up an incline of 1 in 50 at uniform speed by a force of 2 tons wt. applied parallel to the track. Calculate the work done against friction and the efficiency of the arrangement.

2. Two carriages are used for conveying passengers up and down a slope whose upper end is 400 yds. vertically above the lower. Part of the load in the carriage moving up the incline is counterbalanced by the load in a down-moving carriage. The carriages each weigh ½ ton, and the work done by the engine driving the arrangement during one journey is 2,400 ft. cwt. when the carriages are empty. Calculate the work done by the engine when the down-going carriage carries an 8-cwt. load and the up-going a load of 18 cwt. (Assume the work done against friction to be the same as when the carriages are empty.) What is the efficiency of the arrangement in this case?

3. A moving staircase carries passengers through a vertical height of 20 yds. If a load of 8 cwt. is conveyed up the staircase, and the efficiency of the mechanism driving the staircase is then 30 per cent., calculate the total work done against friction while the load is being raised.

4. The coal in a container used for supplying locomotives is conveyed in trucks up a slope 40 ft. long, inclined at 75° to the horizontal, and is then tipped into the container. A counterbalancing weight of 2 tons moves vertically. When a truck of weight 2½ tons carries a load of 10 tons, the work done by the machine which raises the truck is 500 ft. tons. Calculate the work done against friction and the efficiency of the arrangement for this load. (A solution by graphical process may be given.)

5. State and explain the " principle of work " applied to a simple machine.

In an experiment on an inclined plane it was found that a force of 6½ lb. wt., applied up the plane and parallel to it, was required to haul a 10-lb. mass slowly up the plane. If the inclination of the plane is 30° and its length 6 ft., find (*a*) the work done against gravity, (*b*) the work done against friction, when the mass is pulled the whole length of the plane. What is the efficiency of this plane considered as a simple machine? (*G.S.*)

6. A cask weighing 3 cwt. is rolled up a plank 6 ft. long, inclined at 30° to the horizontal. How many foot-pounds of useful work have been done? Use the principle of work to determine the minimum value of the force required to do this if the force is applied (*a*) up the plane, (*b*) horizontally. (*G.S.*)

76. Screw Threads.

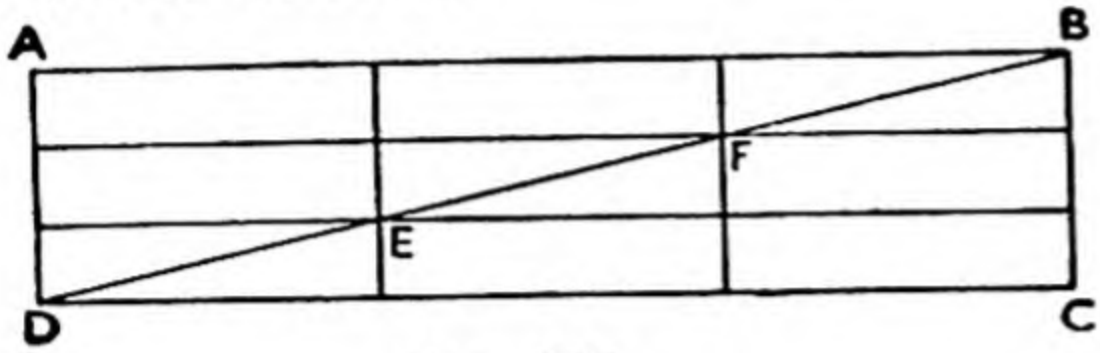


FIG. 127.

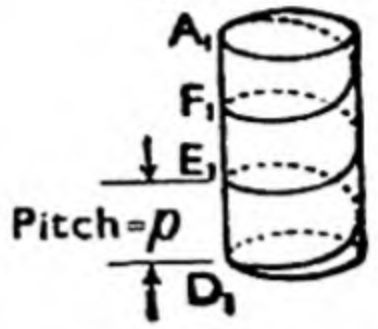


FIG. 127*a*.

On a rectangular sheet of paper ABCD draw the diagonal BD and cut away the triangle DBC. Wrap the remaining triangle DAB around a circular cylinder of suitable radius so that DA is parallel to the axis of the cylinder and AB passes completely round the cylinder more than once.

If we trace the edge DB of the paper round the cylinder it is found to describe a SPIRAL CURVE on the surface of the cylinder. The thread of a screw winds round the screw in a similar way to that in which the spiral curve winds round the cylinder. If the tracing point begins at D_1, then the second time it meets the line A_1D_1, namely, at E_1, it has moved parallel to the axis of the cylinder the distance D_1E_1. If the spiral curve represents an edge of a screw thread, the distance D_1E_1 is called the PITCH of

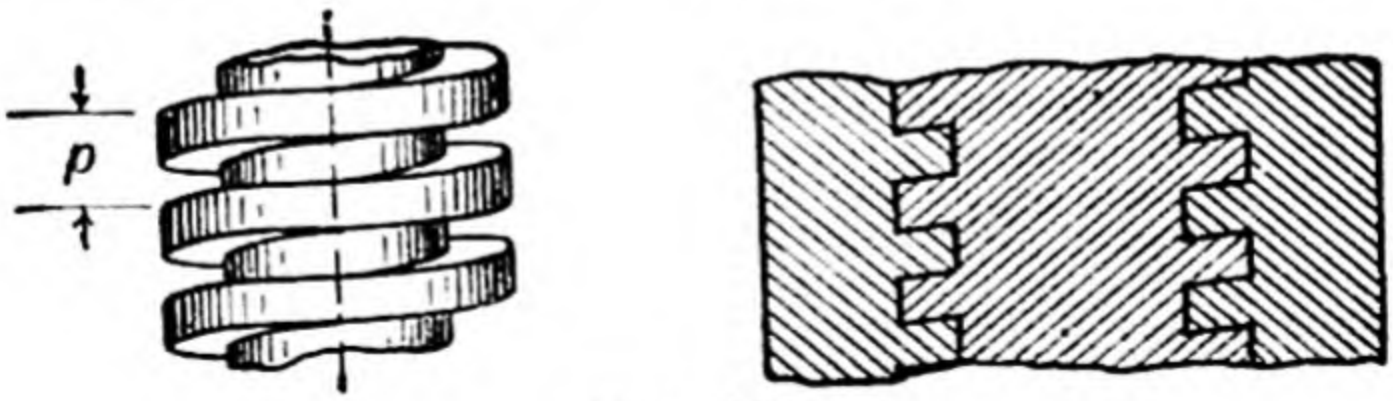


FIG. 128.

the thread. Fig. 128 shows a screw with a square thread in sectional elevation with a nut, it is seen that if the screw is held firm and the nut is given one complete turn, the nut moves along

the screw a distance equal to the pitch of the thread parallel to the axis of the screw. If the nut is held firm and the screw is given one complete turn, the screw moves parallel to its axis a distance equal to the pitch of the thread.

77. The Screw-jack.

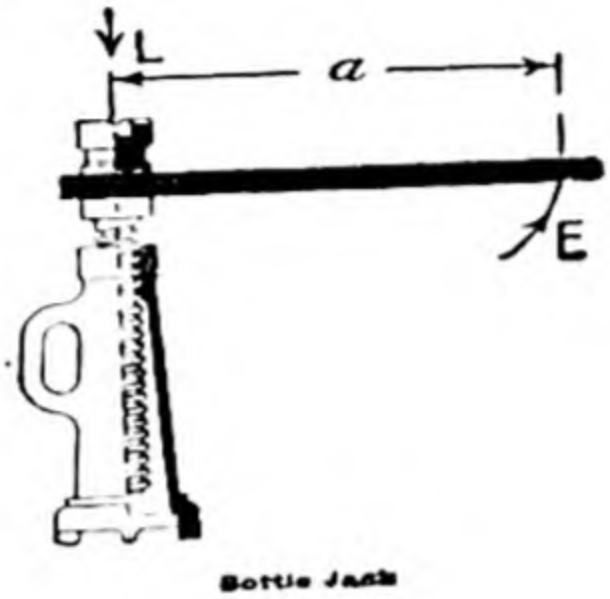


FIG. 129.

The drawing shows a common screw-jack. A lever passes through a hole in the upper part of the screw, which is screwed into the " bottle " part of the jack. If the axis of the screw is vertical when the lever is turned horizontally, the screw moves vertically and so raises the load, which is supported on a metal cap on top of the screw. The cap does not turn with the screw. Such a jack may be used for " jacking up " a car in order that a wheel may be changed, the cap of the jack being placed under the axle to be raised.

Suppose that a load L lb. is to be raised vertically.

Let E lb. wt. be the effort applied perpendicular to the lever, horizontally and at a distance a ft. from the axis of the screw. If the lever makes one complete turn,

$$\text{Input of work} = (2\pi a)\ \text{E ft. lb.}$$

$$\text{Output of work} = p\text{L ft. lb.}$$

Since the load is raised a distance equal to the pitch p of the thread for every complete rotation of the lever,

$$\text{Efficiency} = \frac{p\text{L}}{2\pi a\text{E}}.$$

In certain cases the same effort is applied at both ends of a lever which passes completely through the hole in the screw, *e.g.*, a book press or rail bender, in which case E in the above formulæ is replaced by 2E.

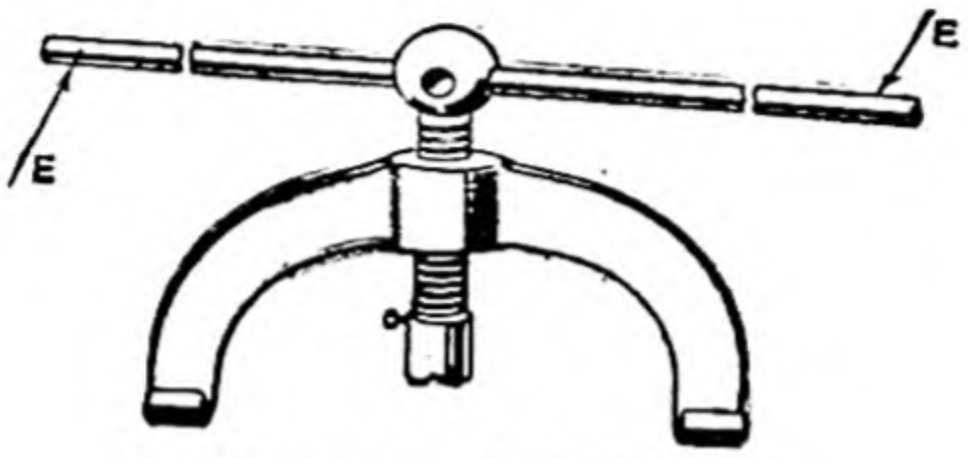


FIG. 130.—Rail Bender.

In both cases,

$$\text{Velocity ratio} = \frac{2\pi a}{p}.$$

If p is made small in comparison with a, the velocity ratio becomes very large, and so does the mechanical advantage unless the efficiency is very small, since

$$\text{Efficiency} = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}}.$$

Example.—*A screw-jack has a thread of pitch ½ in. A load of* 1 *ton is to be raised vertically by applying an effort to the effort lever at a distance of 2 ft. from the axis of the thread in a horizontal direction and perpendicular to the lever.*

(a) *What is the velocity ratio of the jack?*
(b) *Neglecting friction and the weight of the screw itself, what effort would be required?*
(c) *If an effort of 5·6 lb. wt. is actually required to raise the load, calculate the efficiency of the jack for this effort and load.*

(*a*) $\text{Velocity ratio} = \dfrac{\text{Distance through which effort is applied}}{\text{Distance through which load is moved}}$.

For one complete rotation of the lever the effort moves $2\pi \times 2$ ft.

Height through which the load is raised when the screw rotates once completely is equal to the pitch of the screw, $\frac{1}{48}$ ft.

Therefore, $\text{Velocity ratio} = \dfrac{4\pi}{1/48} = 4\pi \times 48$

$= 603$ to the nearest unit.

(*b*) Let E lb. wt. be the effort required.
Work done by the effort when the lever rotates once

$$= 2\pi \times 2 \times E \text{ ft. lb.}$$
$$= 4\pi \,.\, E \text{ ft. lb.}$$

Work done in raising the load through $\frac{1}{48}$ ft. $= 2240 \times \frac{1}{48}$ ft. lb.

Since it is assumed that negligible work is done against frictional effects or in raising moving parts of the machine,

Input of work = Output of work.

$$4\pi E = \frac{2240}{48}.$$

$$E = \frac{2240}{4\pi \times 48} = 3{\cdot}71.$$

The effort required is 3·71 lb. wt.

(*c*) The effort required to raise 2,240 lb. is 5·6 lb. wt., hence for this load mechanical advantage $= \frac{2240}{5{\cdot}6}$,

but

$$\text{Velocity ratio} = 192\pi.$$

$$\text{Efficiency} = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}}$$

$$= \frac{2240}{5{\cdot}6} \times \frac{1}{192\pi}$$

$$= 0{\cdot}663.$$

Otherwise :—
The effort required to raise the load of 2,240 lb. is 5·6 lb. wt.

$$\text{Output of work} = 2240 \times \frac{1}{48} \text{ ft. lb.}$$

$$\text{Input of work} = 2\pi \times 2 \times 5{\cdot}6 \text{ ft. lb.}$$

$$\text{Efficiency} = \frac{\text{Output of work}}{\text{Input of work}}$$

$$= \frac{2240}{48} \div \left(4 \times \frac{22}{7} \times 5{\cdot}6\right)$$

$$= \frac{2240}{48} \times \frac{7}{88 \times 5{\cdot}6}$$

$$= 0{\cdot}663.$$

The efficiency is 0·663 correct to three significant figures.

78. The Velocity Ratio of a Train of Wheels.

Suppose that two wheels are arranged so that they are in the same plane and have their axles parallel to each other. Let the rims of the wheels be pressed into contact, or if they are toothed

wheels, suppose that they gear properly, then one ot these wheels may be used to drive the other by friction in the first case and

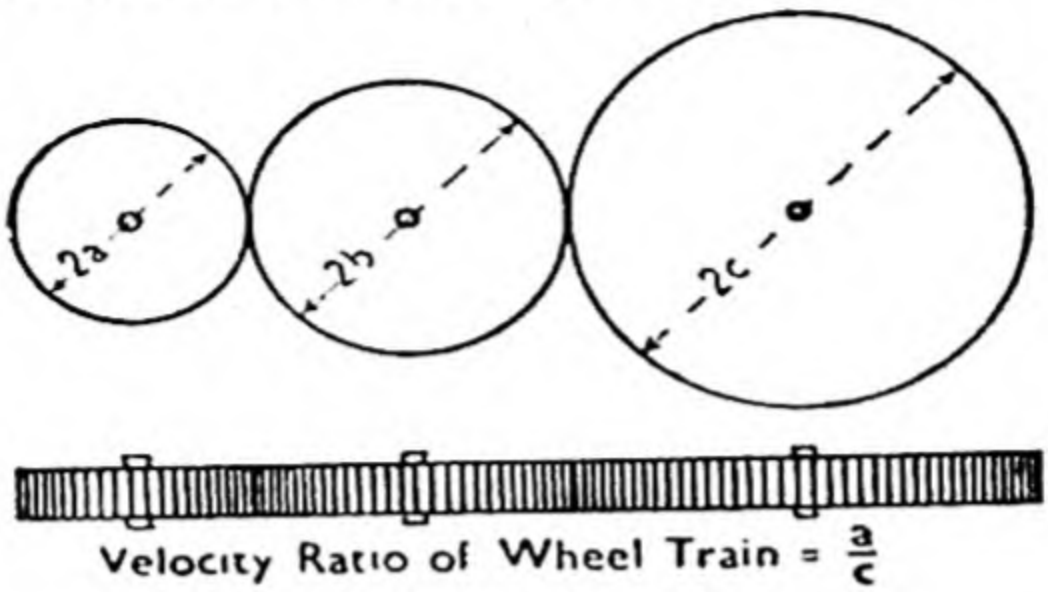


FIG. 131.

by the force exerted by the teeth of one upon the teeth of the other in the second case. Consider the case of the friction drive. If there is no slipping at the contact when the first wheel (radius a) rotates once, its whole circumference, $2\pi a$, has been in contact with the same length of the circumference of the other, and therefore the second wheel makes $\frac{2\pi a}{2\pi b} = \frac{a}{b}$ rotations.

(Thus if the radius of the first wheel is 2 in. and that of the second 6 in., for every complete rotation of the first wheel the second wheel rotates

$$\frac{2\pi \times 2}{2\pi \times 6} = \tfrac{1}{3} \text{ of a complete rotation.)}$$

When the first wheel makes N rotations the second wheel makes $N\frac{a}{b}$ rotations.

If n is the number of complete rotations of the second wheel,

$$n = N\frac{a}{b},$$

and

$$\frac{n}{N} = \frac{a}{b}.$$

$$\frac{\text{Number of rotations of the second wheel}}{\text{Number of rotations of the first wheel}} = \frac{\text{Radius of the first wheel}}{\text{Radius of second wheel}}.$$

If a third wheel of radius c is placed in contact with the second wheel, the three wheels form a wheel train.

When the second wheel makes 1 rotation, the third wheel makes $\frac{b}{c}$ rotations, therefore when the second wheel makes $\frac{a}{b}$ rotations, the third makes $\frac{a}{b} \times \frac{b}{c} = \frac{a}{c}$ rotations.

The velocity ratio of a wheel train

$$= \frac{\text{Number of rotations of the last wheel}}{\text{Number of rotations of the first wheel}}$$

$$= \frac{\text{Radius of the first wheel}}{\text{Radius of the last wheel}}.$$

If the wheels are toothed wheels.

$$\text{Velocity ratio} = \frac{\text{Number of rotations of last wheel}}{\text{Number of rotations of first wheel}}$$

$$= \frac{\text{Radius of first wheel}}{\text{Radius of last wheel}}$$

$$= \frac{\text{Circumference of first wheel}}{\text{Circumference of last wheel}}$$

$$= \frac{\text{Number of teeth in first wheel}}{\text{Number of teeth in last wheel}}.$$

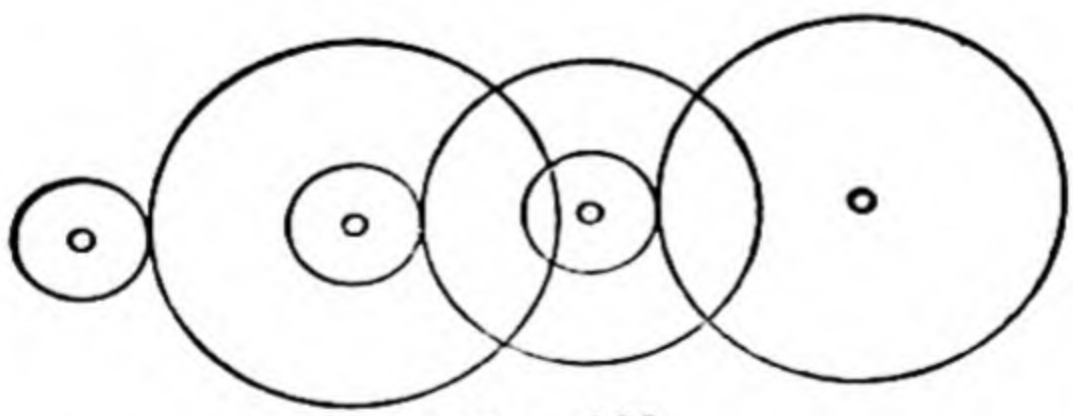


FIG. 132.

Consider the train of wheels shown in the above figure. Let a, b, c be the radii of the driving wheels, and B, C and D the radii of the driven wheels.

When the first driving wheel makes 1 rotation, the first driven wheel makes $\frac{a}{B}$ rotations.

When the second driving wheel makes 1 rotation, the second driven wheel makes $\frac{b}{C}$ rotations.

Thus when the first driving wheel makes 1 rotation, the second driven wheel makes $\frac{a}{B} \cdot \frac{b}{C}$ rotations.

The velocity ratio of the whole wheel train if given by

$$\text{Velocity ratio} = \frac{a}{B} \cdot \frac{b}{C} \cdot \frac{c}{D} = \frac{p}{Q} \cdot \frac{q}{R} \cdot \frac{r}{S},$$

where p, q, r, Q, R and S are the numbers of teeth in the wheels.

Suppose that the numbers of teeth are 12, 60, 15, 45, 12 and 84,

$$\text{Velocity ratio} = \frac{12}{60} \times \frac{15}{45} \times \frac{12}{84} = \frac{1}{105}.$$

Hence the number of revolutions of the first wheel in a given time × velocity ratio of the wheel train = number of revolutions of the last wheel in the same time. If the first wheel in the above train makes 600 revs. per min., then the last wheel makes

$$600 \times \frac{1}{105} = 5{\cdot}71 \text{ revs. per min.}$$

79. The Crab Winch.

The crab winch is a simple machine which makes use of toothed wheels. In a single-purchase winch (see Example 7, p. 134) the effort is applied to turn a small toothed wheel, which in its turn drives a larger toothed wheel. The winding drum is fixed to the shaft of the larger wheel. In a double-purchase winch the first toothed wheel drives a second larger wheel, which is secured to the same shaft as a smaller toothed wheel, and this in its turn drives the toothed wheel on the drum shaft. In Fig. 133 below a model crane is shown which makes use of a double-purchase winch.

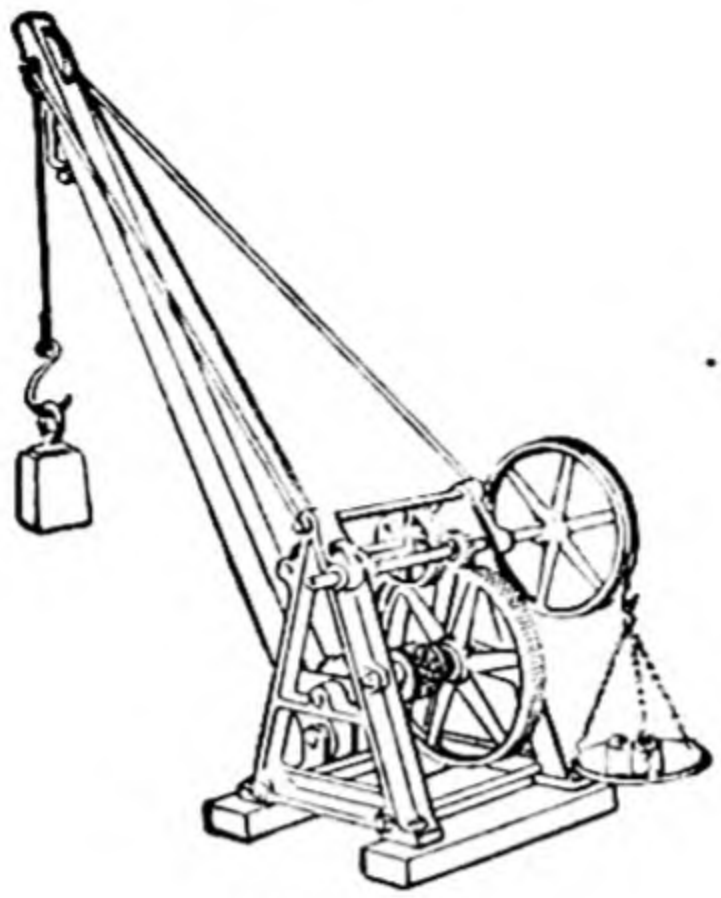


FIG. 133.

Example.—*In the model crane shown above the effort is applied at the rim of a wheel of radius* 1 *ft.* 6 *in. The numbers of teeth in the wheel train of the winch are* 16, 48, 12, 144. *The diameter of the drum is* 6 *in. Calculate the velocity ratio of the crane and the efficiency when a load of* 150 *lb. is raised by an effort of* 8 *lb. wt.*

When the first toothed wheel makes one rotation the first driven wheel makes $\frac{16}{48}$ rotations.

When the second driving wheel makes one rotation, the second driven wheel makes $\frac{12}{144}$ rotations.

Hence when the second driving wheel makes $\frac{16}{48}$ rotations, the second driven wheel makes $\frac{16}{48} \times \frac{12}{144}$ rotations $= \frac{1}{36}$.

Thus when the effort wheel makes 36 rotations the drum makes 1 rotation, and the velocity ratio of the crane is therefore

$$\frac{\text{Distance moved by effort}}{\text{Distance moved by load}} = \frac{36 \times 2\pi \times 1\frac{1}{2}}{1 \times 2\pi \times \frac{1}{4}} = 216.$$

$$\text{Efficiency} = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}}$$

$$= \frac{150}{8} \div 216$$

$$= \frac{25}{288}, \text{ or } 8{\cdot}7 \text{ per cent.}$$

80. Cranes.

The Jib Crane.—In a jib crane the jib can usually be raised and lowered, and the crane itself can be rotated about a vertical axis.

The Hammer-head Crane.—A hammer-head crane consists of a horizontal portion in the shape of a hammer-head which is balanced on a central supporting tower. The load is supported from the long portion of the "hammer-head" and it can be moved along it horizontally or raised vertically, and the whole hammer-head can rotate horizontally on the central tower. The centre of gravity of the load and hammer-head is kept over the central tower by adjusting a balancing weight on the part of the hammer-head behind the crane-operator's cabin. The whole crane can be moved horizontally on rails.

81. Worm Shafts and Worm Wheels.

The machine-cut teeth on a worm shaft and its associated worm wheel are frequently used for lifting purposes, since when properly designed they make an efficient combination in a highly convenient form, and enable a high velocity ratio to be obtained with a minimum of machine parts.

Worm drives are used on motor cars. The worm shaft is driven at one end by the engine of

FIG. 134.

FIG. 135.

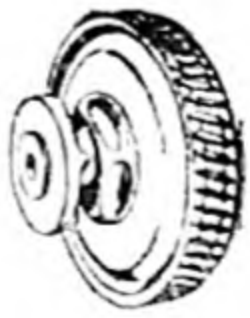


FIG. 136.

FIG. 137.

the car, and the worm which is on the opposite end of the shaft drives the worm wheel, which is fixed to the back axle of the car.

Figs. 134, 135, 136 and 137 show the method of using a worm and worm wheel in a lifting machine. The efficiency is high, and a brake, which acts automatically, is necessary to prevent the load from taking charge when the effort is removed.

EXAMPLES XX

Efficiency of Machines, Velocity Ratio, Etc.

1. A screw-jack is operated by means of a lever whose length measured from the centre line of the jack is 27 in. The screw has a pitch of $\frac{1}{2}$ in. What is the velocity ratio for the machine? What effort applied at the end of the lever would be required in order to raise a load of 36 cwt. assuming the efficiency of the apparatus to be only 28 per cent.?

2. The effective length of the handle of a screw-jack is 24 in. and the pitch of the screw thread is $\frac{1}{8}$ in. What force, applied at the end of the handle, would be required to raise a load of 15 cwt. if the efficiency of the jack is 15 per cent.?

3. In the lifting tackle illustrated on the opposite page, if the worm is single-threaded and the worm wheel has 36 teeth, calculate the height through which the load is raised when the effort wheel makes 36 complete turns. (Diameter of load chain sheave, 5 in.)

4. A worm wheel with 50 teeth is mounted and forms the nut of a screw $\frac{1}{4}$ in. pitch. Guides prevent the screw from turning, and the worm wheel is driven by a single thread worm (see Fig. 138). Calculate:

(i) The number of turns of the worm necessary to cause 1/1000th in. axial travel of the screw.

(ii) The travel of the screw that would result from $3\frac{1}{2}$ revolutions of the worm.

(iii) The velocity ratio of the arrangement.

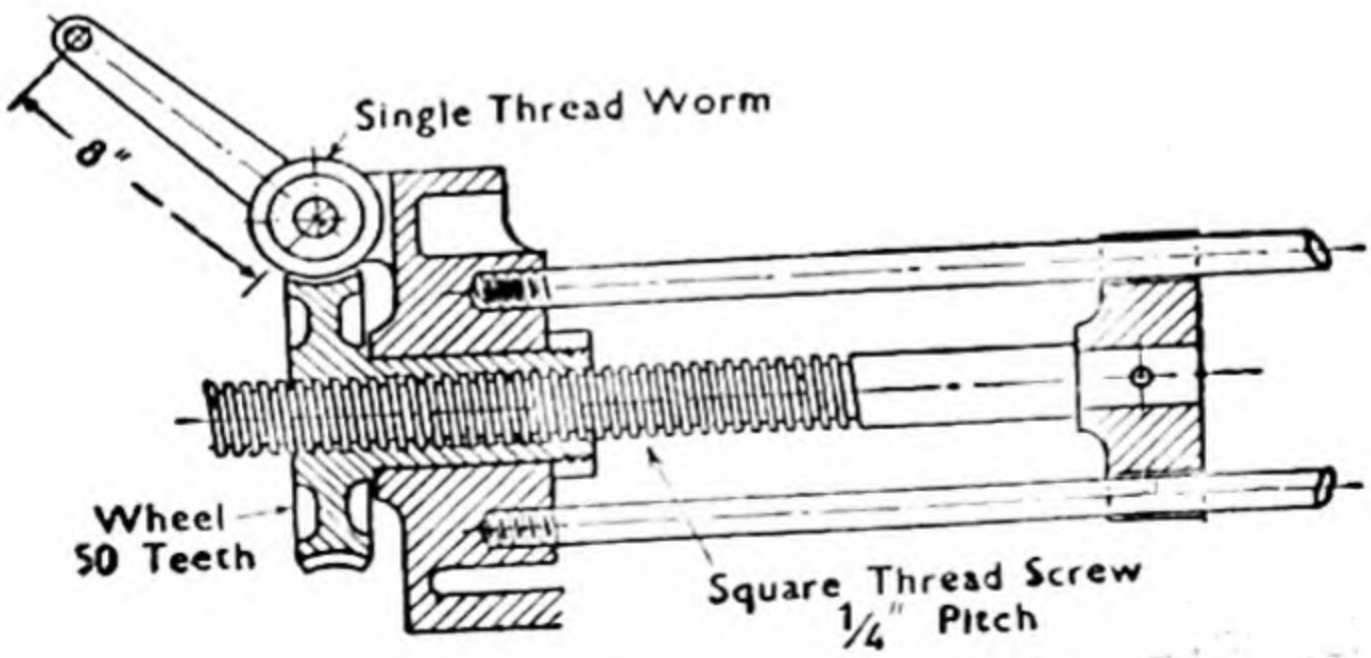


FIG. 138.

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5. Calculate the velocity ratio of the following wheel trains :—

(*a*) Driver, 20 teeth ; driven wheel, 45 teeth.
(*b*) Driver, 20 teeth ; first driven wheel, 30 teeth ; second driven wheel, 45 teeth.

6. Calculate the velocity ratio of the wheel trains shown in the figure.

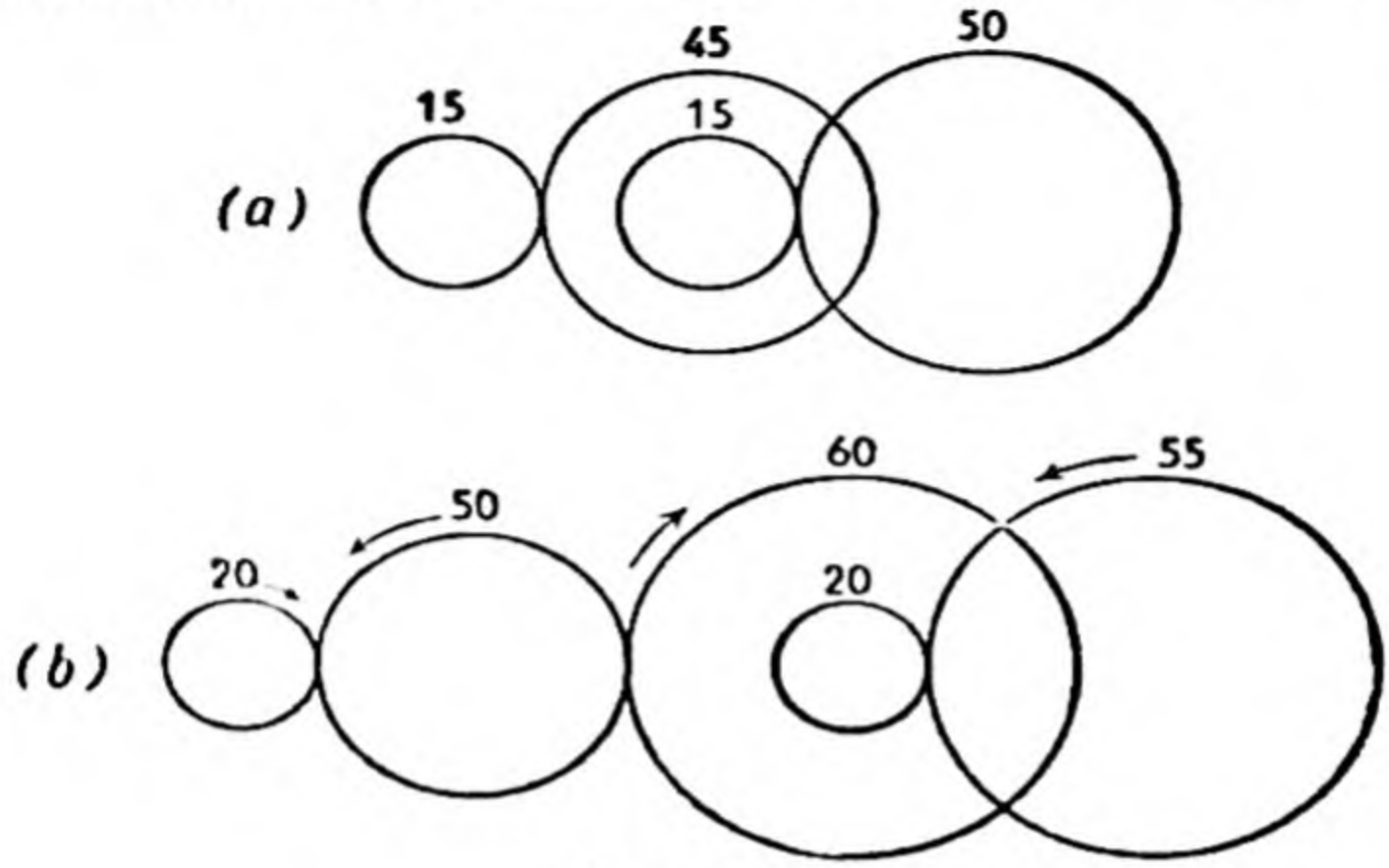

FIG. 139.

7. In the hoisting crab winch illustrated, calculate the velocity ratio. The diameter of the barrel is $4\frac{1}{2}$ in., the toothed wheels have 16 and 80 teeth respectively, the handle has a radius of 1 ft. 3 in. What load can

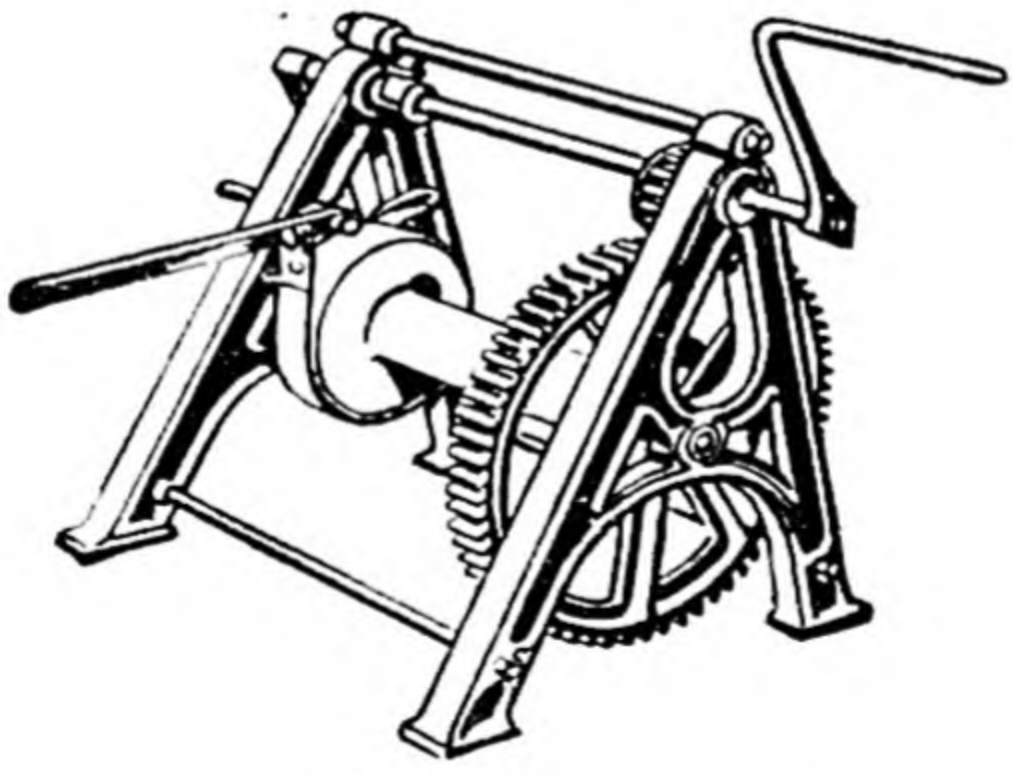

FIG. 140.

be moved with an effort of 30 lb. wt. applied perpendicular to the handle in the usual manner. The large toothed wheel and the barrel are fixed on the same shaft. The handle and small toothed wheel are fixed on the same shaft. (Neglect friction.)

8. A rack and pinion is used to raise a sluice gate. The pinwheel has 12 teeth, and is turned by a handle of radius 1 ft. 6 in. The weight

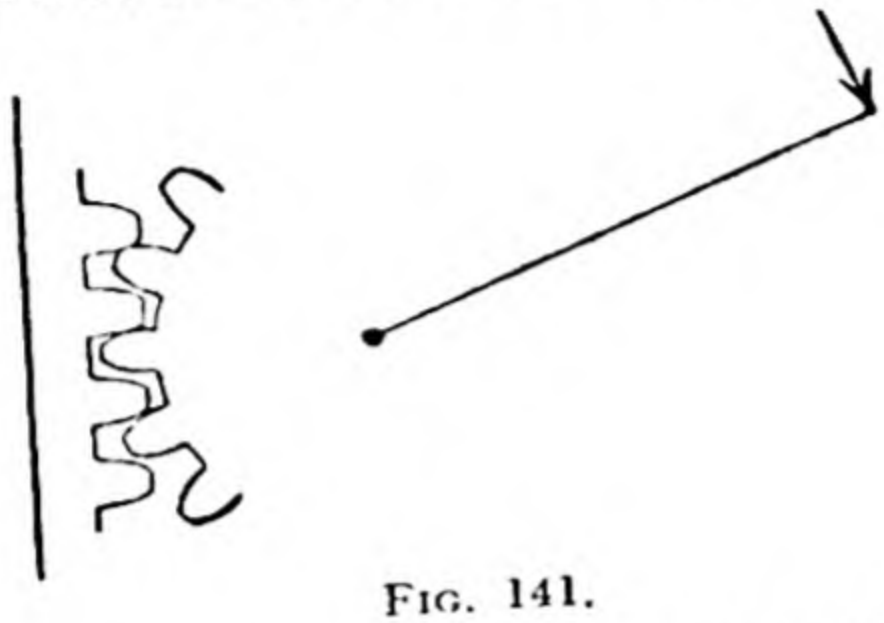

FIG. 141.

of the rack and sluice gate is 3 cwt. The pitch of the teeth in the rack is 1 in. If the efficiency of the arrangement is 0·35, calculate the effort required to raise the gate.

9. A machine for raising weights consists of a single-threaded worm which drives a worm wheel with 36 teeth, which is fixed to the barrel

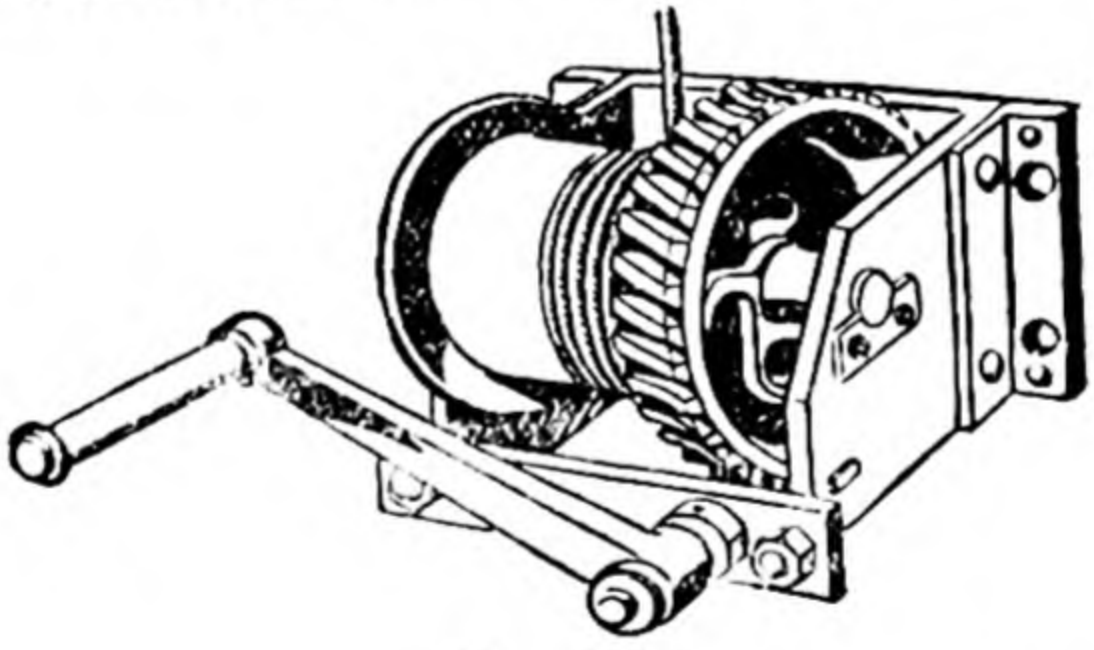

FIG. 142.

of the winding drum and has the same axis. If the barrel has a diameter of 8 in., and the handle turning the worm has an effective radius of 15 in., calculate the load raised by an effort of 40 lb. wt., the efficiency of the arrangement being 0·4.

10. Fig. 143 shows a double-purchase winch. The effort is applied at the end of a handle of length 1 ft. 6 in., and the drum has a diameter of 10 in. The number of teeth in the various wheels are : A, 12 ; B, 60 ; C, 15 ; D, 90. Calculate the velocity ratio of the winch. If the efficiency

of the machine is 25 per cent. when the effort applied is 50 lb., calculate the load.

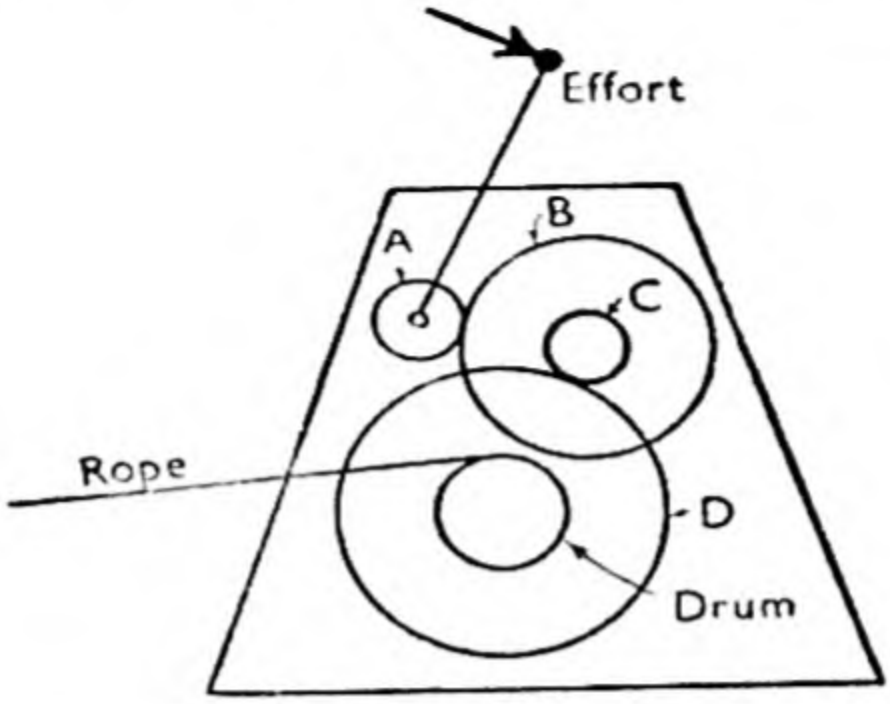

FIG. 143.

11. The figure shows the gearing of a machine. A is the motor shaft and runs at 480 revs. per min. The numbers of teeth in the wheels are

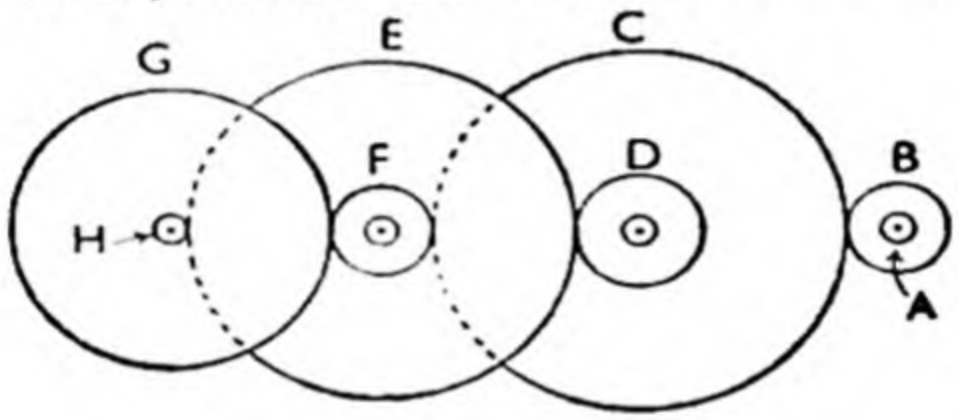

FIG. 144.

as follows: B, 20; C, 80; D, 25; E, 75; F. 20; and G, 60. G is keyed to shaft H. Determine the speed of shaft H.

12. A double-purchase winch (see figure of Example 10) consists of a barrel 5 in. in diameter, driven from a handle 20 in. long by a train of wheels: driver, 14 teeth; follower, 56 teeth; driver, 15 teeth; follower, 45 teeth. Find the velocity ratio for the machine. During a test on the machine the following observations were made :—

Load Lifted.	Force applied at End of Handle.
Lb.	Lb. wt.
160	6·8
200	7·3
240	7·8
280	8·3
320	8·8

Calculate the efficiency of the machine at each of these loads, and plot a graph to show how the efficiency varies.

13. In a screw-jack the effort E required to lift a load W is applied at the end of a handle which describes a circle of radius 7 in. The pitch of the screw is $\frac{1}{2}$ in. What is the velocity ratio? $\left(\pi = \frac{22}{7}\right)$. What is the efficiency if an effort of 5 lb. wt. will lift a load of 1 cwt.? If the efficiency for heavier loads rises to 33 per cent., what effort will lift a ton? Why must the efficiency be so low? (*O.C.*)

14. The velocity ratio of a screw-jack is 60. Its efficiency is 30 per cent. What effort is needed to lift a load of 1 ton with this jack? (*O.C.*)

15. A weight is raised by means of a rope coiled round a drum which is rotated by a handle connected with the drum by a series of cog-wheels. The velocity ratio is 10, and the effect of friction is equivalent to an additional load equal to two-thirds of the weight. Find the mechanical advantage and the efficiency. (*O.C.*)

16. A weight is drawn up by a rope wrapped round a vertical wheel A of 2 ft. radius. A is geared to a horizontal wheel B having 20 teeth,

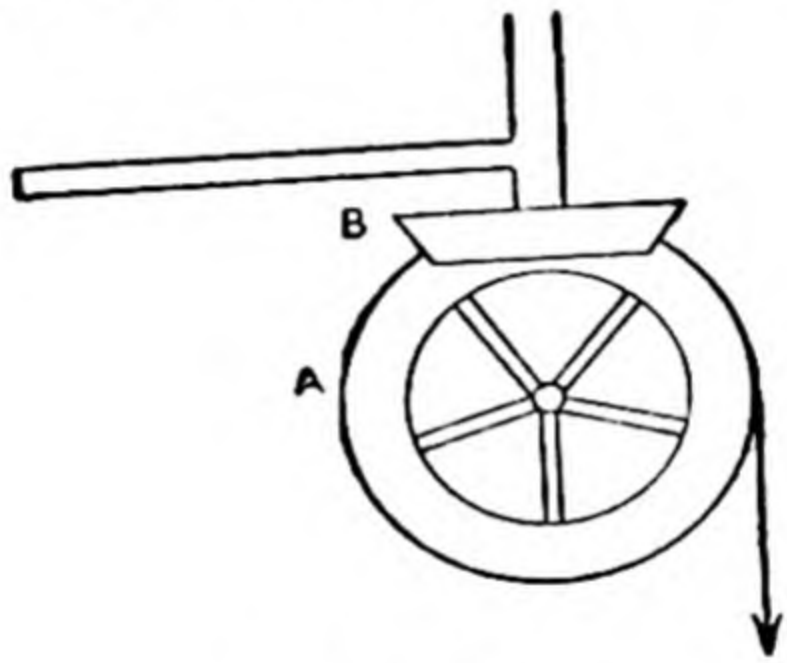

FIG. 145.

A having 70 teeth. B is rotated by a horizontal force applied at the end of an arm 6 ft. long, which is rigidly fixed to it. Find the velocity ratio.

When a weight of 1 cwt. is being lifted, friction is equivalent to an extra load of 2 cwt. Find the mechanical advantage and efficiency at this load. (*O.C.*)

17. The velocity ratio of a screw-jack is 50, and its efficiency for a load of half a ton is 26 per cent. What effort is wanted to lift this load, and how much work is done against friction in lifting it 1 in.? (*O.C.*)

18. Define the mechanical advantage of a machine.

A uniform lever AB is 30 in. long and weighs 3 lb. The load is placed at B, the effort at A and the fulcrum is 3 in. from B. Calculate the effort required to maintain the lever horizontal when the load has the following values: 30, 45, 60 and 75 lb. Calculate the mechanical advantage obtained with this machine at each load. (*L.M.*)

19. Describe two different types of machine in which an effort of 10 lb. wt. would be capable of lifting a load of 42 lb., the efficiency being 60 per cent. Illustrate your answer by carefully drawn diagrams. (*L.M.*)

20. Explain the meaning of the terms mechanical advantage, velocity ratio and efficiency as applied to a machine, and show the relation between them.

A screw has three threads to the inch, and the effort is applied at right angles to the end of an arm which is 1 ft. long. What force must be applied to lift a load of 2 cwt. if the efficiency is 25 per cent. ? (*L.M.*)

21. In a book press the effort is applied at the two ends of a lever of length 2 ft., and the screw has a pitch of $\frac{3}{8}$ in. The efficiency of the press is 42 per cent. when the efforts applied at the ends of the lever are each 25 lb. What is the pressure per square inch exerted by the table, which is rectangular and has the dimensions 12 in. by 11 in. ?

82. The Law of a Machine.

Using one of the machines which have now been described, such as the Weston pulley block, let us suppose that with no load on the machine we apply increasing efforts and note the effort which just begins to move the machine. If we repeat this a number of times we see that the effort which just moves the machine with no load cannot be determined very accurately, since only a small effort is required, and it is not easy to decide the precise effort which just moves the machine. If we apply an effort which is the average of the foregoing results, we may assume that we have a machine in which the frictional effects and weights of moving parts are just balanced on no load, thus an additional effort will enable us to raise a corresponding load. When additional efforts of 2, 4, 6 and 8 lb. wt. are applied in succession we notice that the loads raised also increase by equal amounts. Does this mean that once the frictional effects, etc., have been balanced for no load the machine behaves like a perfect machine for additional efforts, giving outputs of work equal to the *additional* inputs of work ? If this is so when we divide a load raised by the additional effort required to raise it and then divide this result by the velocity ratio of the machine, the answer should be UNITY.

Frictional resistances, etc., are balanced by 1·85 lb. wt. effort on no load.

The results of an experiment with a Weston pulley block are

Effort.	Additional Effort = A.	Load = L.	L ÷ A.
Lb. wt.	Lb. wt.	Lb.	
1·85	0	0	
3·85	2	18	9·0
5·85	4	38	9·5
7·85	6	57	9·5
9·85	8	77	9·6
11·85	10	96	9·6

Velocity ratio = 22.

given in the above table. It will be noticed that when the load raised is divided by the additional effort the result is VERY NEARLY CONSTANT, but that when this latter result is divided by the velocity ratio of the machine the quotient is very far from unity. Why is this? The figures quoted in the above table show that there is a simple law connecting the whole effort applied to the machine and the load which it moves, thus by dividing any load in the table by 9·5 and adding 1·85 to the result, we obtain to a satisfactory degree of accuracy the effort required to raise this load. The law referred to which connects the effort applied with the load raised is called THE LAW OF THE MACHINE. In the following experiment we shall determine the law of a machine and show that it can be expressed in a shortened form by means of a simple algebraic formula.

EXPERIMENT.—Apply increasing loads to a machine, and determine as accurately as possible the efforts required to move them. At least ten sets of readings should be taken, beginning with the effort required to move the machine on no load. It is usually possible to determine the velocity ratio of the machine by two methods: first, by calculation; and, second, by direct experiment. When finding the velocity ratio by experiment at least three results which agree fairly well with each other should be taken. Tabulate the results as below, and proceed to draw a graph in which the efforts applied are plotted against the loads which they raise.

Effort = E.	Load = L.	Effort = E.	Load = L.
Lb. wt.	Lb.	Lb. wt.	Lb.
2·50	7	6·50	45
3·25	14	7·00	52
4·00	21	8·25	63
5·25	32	9·75	77
5·75	38	11·25	98

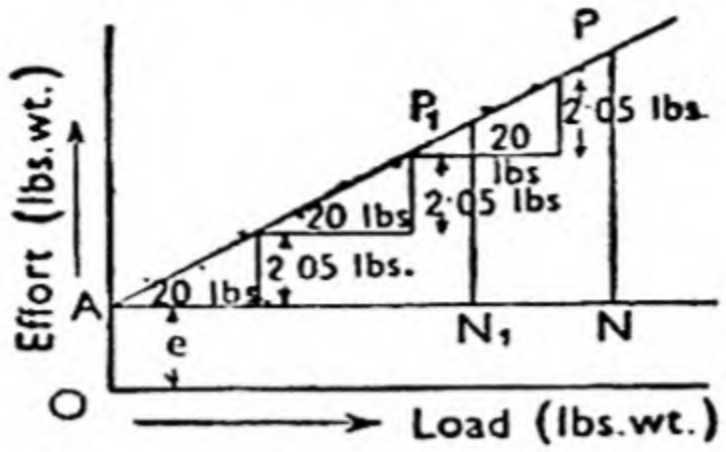

FIG. 146.

Once the points on the graph have been plotted, we observe that they lie fairly accurately upon a straight line, which we now draw to lie as evenly as possible through these points. This straight line need not pass exactly through any plotted point, but the points must lie equally distributed on either side of the line and as close to it as possible.

In the above graph the line meets the effort axis at A, and P is any point on it. A number of steps have been drawn on the graph to show that equal increases in the effort result in equal increases in the load. O is the origin corresponding to no load and no effort where the axes of the graph cut.

PN is drawn parallel to the effort axis.
AN is drawn parallel to the load axis to meet PN at N.
P_1 and N_1 are two points obtained in a similar manner to P and N.

Referring to the graph, AO represents the value of the effort required to move the machine on no load, and its value may be compared with that found by experiment.

Let AO represent e lbs. wt. (e is a constant number).

Using the similar triangles APN, AP_1N_1 . . .

$$\frac{PN}{AN} = \frac{P_1N_1}{AN_1} = \ldots$$

Therefore the ratio $\frac{PN}{AN}$ is constant in value no matter where P is chosen on the straight line.

If E lb. wt. represents the effort applied,
and L lb. represents the load moved,
then PN represents $(E - e)$ lb. wt.
and AN represents L lb.

Hence, $$\frac{PN}{AN} = \frac{(E-e)}{L} = \text{a constant number, say } a,$$

$$\frac{(E-e)}{L} = a$$

$$(E-e) = aL$$

$$E = aL + e.$$

This is the LAW OF THE MACHINE; it may be written:

The effort required to move a load
= a constant number × load + a second constant number.

First Method of Finding *a* and *e*.

It has already been shown that *e* is the effort which moves the machine on no load, and from the above graph, $e = 1{\cdot}85$.

To find *a* we choose AN to represent a convenient number of units on the load scale. In the present case,

AN represents 50 lb.,

and PN represents 5·15 lb. wt.

$$\frac{PN}{AN} = \frac{5{\cdot}15}{50} = 0{\cdot}103.$$

Thus the law of the machine is

$$E = 0{\cdot}103L + 1{\cdot}85.$$

From this law we can find the load L if we are given the effort E, and conversely we can find E if we are given L.

83. Second Method. To Find the Law of a Machine from Two Corresponding Pairs of Values of Effort and Load.

Referring to the above graph, suppose that we choose two corresponding values of the load and effort, and from these results try to find the law of the machine. It is readily seen that the law obtained by this method does not in general describe the behaviour of the machine as accurately as that found by the above methods; for the straight line joining any two points on the above graph will not as a rule coincide with that already drawn and which should give the best result we can obtain from the ten plotted points. As an example, consider the points:

Effort = 4 lb. wt.	Effort = 9 lb. wt.
Load = 21 lb.	Load = 70 lb.

Assuming that the law of the machine is a straight line law of the form

$$E = aL + e$$

$$4 = 21a + e \quad . \quad . \quad . \quad . \quad (1)$$

and

$$9 = 70a + e \quad . \quad . \quad . \quad . \quad (2)$$

Subtract equation (1) from equation (2),

$$5 = 49a$$

$$a = \frac{5}{49} = 0{\cdot}102.$$

Put 0·102 for a in equation (1),

$$4 = 21 \times 0{\cdot}102 + e.$$

$$e = 4 - 2{\cdot}14 = 1{\cdot}86.$$

The law of the machine is therefore

$$E = 0{\cdot}102L + 1{\cdot}86.$$

EXAMPLES XXI

1. Calculate the efficiency of the Weston pulley block from the results given in the table and draw a graph showing the efficiency for different loads. The velocity ratio is 20.

Effort.	Load.	Effort.	Load.
Lb. wt.	Lb.	Lb. wt.	Lb.
3·85	18	9·85	77
5·85	38	11·85	96
7·85	57		

2. From the given table of values determine the law of the machine in the form $E = aL + e$.

Effort, E.	Load, L.	Effort, E.	Load, L.
Lb. wt.	Lb.	Lb. wt.	Lb.
4·25	14	10·00	84
5·25	28	11·25	98
6·50	42	12·50	112
7·75	56	14·75	140
9·00	70		

What load will be raised by an effort of 7 lb. wt. ?

3. Assuming a law of the form $E = aL + e$. Calculate a and e, given that when $E = 5{\cdot}25$ lb. wt., $L = 28$ lb., and when $E = 12{\cdot}5$ lb. wt., $L = 112$ lb. From your law find what effort is required to move a load of 50 lb.

4. The law of a machine is found to be $E = 0{\cdot}02W + 1{\cdot}5$. Draw the graph showing the effort against a load base up to loads of 400 lb. If

the velocity ratio of the machine is 200, calculate efficiencies of the machine when the load is 50 and 250 lb.

5. Assuming a law of the form $E = aL + b$, where a and b are constants, find the values of a and b from the following table of results :—

Effort.	Load.	Effort.	Load.
Lb. wt.	Lb.	Lb. wt.	Lb.
7·0	10	26·0	60
10·8	20	30·0	70
14·6	30	33·6	80
18·5	40	37·5	90
22·2	50	41·4	100

6. Explain the terms velocity ratio and efficiency in connection with machines.

The relation between the force P and the load W in most machines is, except for extreme loads, of the form $P = a + bW$, where a and b are constants. If in a machine in which the velocity ratio is 15 to 1 a force of 30 lb. wt. will lift a load of 130 lb. and a force of 40 lb. wt. will lift a load of 200 lb., find the load that a force of 80 lb. wt. will lift and the efficiency of the machine under this load. (*O.C.*)

7. In a machine used for hoisting loads the velocity ratio is 42, and it is found that an effort of 32 lb. wt. is required to raise a load of 350 lb. at a steady speed. Find the efficiency at this load.

If in the above machine the effort P and the load w are connected by a law $P = a + bw$, and an effort of 2·2 lb. wt. is required to move the machine when there is no load, find a and b. (*O.C.*)

8. A Weston pulley has a velocity ratio of 16, and it is found that efforts of 9, 14 and 24 lb. wt. are needed to lift loads of 28, 56 and 112 lb. with this pulley. Find the effort necessary to lift 100 lb., and plot a graph showing how efficiency alters with the load. What is the probable efficiency with a load of 5 cwt., and how much work will be done against friction in lifting this load through 1 ft. ? (*O.C.*)

84. The Effect of Friction.

When the EFFECT OF FRICTION has been balanced on no load we have now seen that equal increases in the effort applied to a machine cause equal increases in the load moved. Moreover, the work done by the " additional effort " is greater than the work done in moving the load. This is due to the fact that the effect of friction is not constant. The load moved divided by the velocity ratio of a machine is the effort which would be required to move this load if there were no effect of friction, and therefore if this value of the effort is subtracted from its actual value the result

is the part of the effort used in overcoming the effect of friction.

Thus $$F = \text{Effect of friction} = E - \frac{L}{\text{Velocity ratio}}.$$

Now F is readily calculated, and it is found that there is a law connecting F with the load moved similar to that obtained for the law of a machine. In fact,

$$F = cL + d$$

where c and d are constant numbers, which may be determined by exactly the same procedure as that described when finding the law of a machine.

EXAMPLES XXII

Effect of Friction

1. The following results were obtained during an experiment with a lifting machine having a velocity ratio of 8.

Load Raised.	Effort.	Load Raised.	Effort.
Lb.	Lb. wt.	Lb.	Lb. wt.
5	4·8	30	14
10	6·0	40	18
20	10·0		

Assuming ideal conditions for the machine, calculate the loads that would be raised by the various efforts.

Set out on an effort base graphs of actual and ideal loads raised, and explain the reason for the differences between them.

2. An experiment on a lifting tackle gave the following values :—

Load.	Effort.	Load.	Effort.
Lb.	Lb. wt.	Lb.	Lb. wt.
20	15	110	37
50	22	120	41
80	30	160	49

What was the effort when there was no load and when the load was 150 lb., and what was the load which an effort of 35 lb. wt. could lift ?

If the velocity ratio of this machine were five, what would be the effort required to lift 100 lb. if there were no friction ? What effort would be

required to overcome the friction only of the machine when lifting a load of 100 lb. ?

3. Check the following table of values. Draw a graph in which F, the effect of friction, is plotted against the load L, and determine a law connecting F and L, assuming that it is of the form $F = cL + d$. Also on the graph plot the effort on a load base ($E = aL + e$). Compare the values of d and e when L is zero.

Effort.	Load.	$\frac{L}{V.R.}$	$F = E - \frac{L}{V.R.}$
Lb. wt.	Lb.		
2·50	7	0·318	2·18
3·25	14	0·636	2·61
4·00	21	0·955	3·05
5·40	32	1·460	3·94
6·10	38	1·730	4·37
7·00	45	2·050	4·95
8·00	52	2·360	5·64
9·25	63	2·860	6·39
11·00	77	3·500	7·50
13·75	98	4·460	9·29

4. Find the law connecting the effect of friction and the load, using the following table, given that the velocity ratio of the machine is 20.

Effort.	Load.	Effort.	Load.
Lb. wt.	Lb.	Lb. wt.	Lb.
4·25	14	10·00	84
5·25	28	11·25	98
6·50	42	12·50	112
7·75	56	14·75	140
9·00	70		

CHAPTER VIII

SPEED AND VELOCITY

85. A body which is changing its position is said to be in motion. When the motion of a BODY is being considered we may take it that, unless otherwise stated, the body is not spinning, and for convenience its path may be taken to be that of its centre of gravity. Thus if the motion of a football is being observed, its path may be taken as that of the centre of the ball.

A train is observed to travel four equal distances of $\frac{1}{4}$ mile in equal times of 15 sec. each. Suppose that the motion is quite steady, not changing in any manner, then the train travels in the following way during the 60 sec. :—

440 yds. in 15 sec.,

$\frac{440}{15}$ yds. in 1 sec.,

i.e., 88 ft. in 1 sec.,

8·8 ft. in 1/10th sec.,

0·88 ft. in 1/100th sec.,

and so on.

Do the observations of the motion of the train convince us that in any 1/100th sec. during the 60 sec. it travels 0·88 ft. ? Is it not possible for the train during a few seconds to travel a little more quickly than the steady motion which has been described, as well as a little more slowly during a few seconds, and yet travel $\frac{1}{4}$ mile in any one of the 15-sec. intervals ? To test for steady motion we can observe whether or not the train travels equal distances in equal times ; the shorter the times are made the better will be the test, provided that the distances and times can be accurately measured. It is more usual to refer to steady motion as UNIFORM SPEED.

Speed.

THE SPEED OF A BODY IS THE RATE AT WHICH IT DESCRIBES ITS PATH. Suppose that the distance described by a body is measured from a fixed point on its path, then the speed of the body is measured by the increase in this distance per unit time. If the path is curved, the increase in the distance is the length of a curved line. Again, if the speed is variable the increase in the distance described per unit time must be found upon the assumption that during unit time following the instant at which the speed is being measured *the speed remains constant.*

Uniform or Constant Speed.

A BODY TRAVELS WITH UNIFORM SPEED IF IT DESCRIBES EQUAL DISTANCES IN EQUAL TIMES HOWEVER SMALL THESE TIMES MAY BE.

This definition applies equally well to motion in a straight line or motion in a curved path ; in the latter case the distances are measured along the curve and will consist of equal lengths of curved line.

A uniform speed of 1 mile per minute is equal to a uniform speed of 60 miles per hour, *i.e.*,

$$1 \text{ mile per min.} = 60 \text{ m.p.h.}$$
$$= 60 \times 1760 \times 3 \text{ ft. per } 3600 \text{ sec.}$$
$$= \frac{60 \times 1760 \times 3}{3600} \text{ ft./sec.}$$

Therefore, $$\mathbf{60 \text{ m.p.h.} = 88 \text{ ft./sec.}}$$

Again, $$1 \text{ m.p.h.} = \frac{88}{60} \text{ ft./sec.} = \frac{22}{15} \text{ ft./sec.}$$

$$27 \text{ m.p.h.} = 27 \times \frac{22}{15} \text{ ft./sec.} = 39{\cdot}6 \text{ ft./sec.}$$

86. Variable Speed.

If a body does not describe equal distances in equal times during its motion it is said to travel with variable speed. Trains and motor cars travel with variable speeds, although during comparatively short times they may travel with uniform speeds.

When the speed of a body is changing, it is travelling at a particular speed at a certain instant only ; this speed is referred to as the instantaneous speed of the body at a certain time.

87. Average Speed.

We have seen that if a train is observed to travel 1 mile in 1 min., we cannot therefore assume that it travels at the *uniform speed* of 1 mile/min. Nevertheless, if the train were to travel at the uniform speed of 1 mile/min. it would travel the observed distance in the observed time. One mile per minute is in this case called the AVERAGE SPEED OF THE TRAIN during the 1 min. of time that it was under observation. *Thus, if a body completes a journey in a certain time its average speed is measured by the uniform speed with which it would have to travel to complete the same journey in the same time.* Since the speed of a body is to be expressed as the distance travelled in unit time,

$$\text{AVERAGE SPEED} = \frac{\text{Total distance travelled}}{\text{Total time taken}}.$$

Example (1).—*A motor car travels a distance of* 40 *miles in* 1 *hr.* 40 *min. What is its average speed* (a) *in miles per hour*, (b) *in feet per second ?*

(*a*) In $1\frac{2}{3}$ hrs. the car travels 40 miles.

In 1 hr. the car travels $\frac{40}{1\frac{2}{3}} = 24$ miles.

The average speed of the car is therefore 24 m.p.h.

Otherwise,

$$\text{Average speed} = \frac{\text{Total distance travelled (miles)}}{\text{Total time taken (hours)}} = \frac{40}{1\frac{2}{3}} = 24 \text{ m.p.h.}$$

(*b*) $$\text{Average speed} = \frac{\text{Total distance travelled (feet)}}{\text{Total time taken (seconds)}}$$

$$= \frac{40 \times 1760 \times 3}{100 \times 60}$$

$$= 35{\cdot}2 \text{ ft./sec.}$$

Otherwise,

$$60 \text{ m.p.h.} = 88 \text{ ft./sec.}$$

$$1 \text{ m.p.h.} = \frac{88}{60} \text{ ft./sec.}$$

$$24 \text{ m.p.h.} = \frac{88}{60} \times 24 = 35{\cdot}2 \text{ ft./sec.}$$

Example (2).—*Two boys run a race; the first boy gives the second 10 yds. start and runs 220 yds. in 30 sec., thereby losing the race by 2 sec. Which boy runs the race with the greater average speed?*

The second boy runs 210 yds. in 28 sec.

$$\text{Average speed of the second boy} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

$$= \frac{210 \times 3}{28} = 22{\cdot}5 \text{ ft./sec.}$$

The first boy runs 220 yds. in 30 sec.

$$\text{Average speed of the first boy} = \frac{220 \times 3}{30} = 22 \text{ ft./sec.}$$

The second boy has the greater average speed.

Example (3).—*A train travels from* A *to* D *stopping at* B *and* C. *The details of the journey are given in the table. Calculate the average speed of the train between the stations* A *and* B, B *and* C, C *and* D. *What is the average speed of the train for the whole journey?*

Station.	*Times.*	*Distances.*
A depart	1.00 P.M.	*A to B*, 30 *miles.*
B arrive	2.00 ,,	
B depart	2.05 ,,	*B to C*, 22 ,,
C arrive	2.45 ,,	
C depart	2.50 ,,	*C to D*, 49 ,,
D arrive	4.00 ,,	

$$\text{Average speed for the journey from A to B} = \frac{\text{Distance travelled}}{\text{Time taken}}$$

$$= 30 \text{ m.p.h.}$$

$$\text{,, ,, ,, ,, ,, B ,, C} = \frac{22}{\frac{2}{3}} = 33 \text{ m.p.h.}$$

$$\text{,, ,, ,, ,, ,, C ,, D} = \frac{49}{1\frac{1}{3}} = 42 \text{ m.p.h.}$$

For the whole journey,

$$\text{Total time taken} = 3 \text{ hrs.}$$

$$\text{Total distance travelled} = (30 + 22 + 49) = 101 \text{ miles.}$$

$$\text{Average speed} = \frac{\text{TOTAL DISTANCE TRAVELLED}}{\text{TOTAL TIME TAKEN}}$$

$$= \frac{101}{3}$$

$$= 33 \text{ m.p.h.}$$

EXAMPLES XXIII

1. At what uniform speed should a body travel if it is to describe 220 yds. in 10 sec. (*a*) in feet per second, (*b*) in miles per hour?

2. Two cyclists travel by different routes from a place A to a place B. The first cyclist travels 12 miles in 1 hr. 12 min. The second cyclist starts 20 min. after the first, and arrives at B at the same time after travelling $6\frac{1}{2}$ miles. Calculate the average speeds of the two cyclists in miles per hour.

3. Using the table given in the last paragraph, show that the average speeds for the journeys A to C and B to D are 29·7 and 37 m.p.h. respectively.

4. A belt is driven by a pulley of diameter 7 in. Find the speed in feet per second of a point on the rim of the wheel, and therefore the speed of the belt, given that the pulley rotates 100 times in 1 min. $\left(\pi = \frac{22}{7}\right)$.

5. The belt in the previous question is used to drive a smaller pulley of diameter 4 in. If there is no slipping of the belt on the pulleys, what is the speed of a point on the rim of the smaller pulley and how many times does it rotate in 1 min.?

6. The following is the time-table for the journey from London to Paris by the short sea route :—

Miles.				
—	London (Victoria)	depart	11.00	A.M.
78	Dover Marine	arrive	12.35	P.M.
		depart	12.55	,,
103	Calais Maritime	arrive	2.10	,,
		depart	2.30	,,
283	Paris	arrive	5.43	,,

Calculate the average speeds from London to Dover, Dover to Calais, Calais to Paris, and, finally, for the whole journey.

7. The records for the 100 yds. championships are held by J. Donaldson, Australia (Professional)—time, $9\frac{3}{8}$ sec.; F. Wykoff, U.S.A. (Amateur)—time, 9·4 sec. Calculate their average speeds in feet per second correct to three significant figures.

8. The greatest AIR RACE of 1934 from England to Melbourne was won by C. W. A. Scott and T. Campbell Black in 2 days, 23 hrs., the distance covered being 11,300 miles. Calculate the average speed for the journey correct to one decimal place. (The machine was a D.H. Comet.)

9. The *Silver Jubilee* train made four world records :

(*a*) Twenty-five miles between signposts 30 and 55 in 13 min. 57 sec.
(*b*) Hatfield to Huntingdon, 41 miles 15 chs., in 24 min. 34 sec.
(*c*) Forty-three miles at 100 m.p.h.
(*d*) Wood Green to Fletton Junction, 70 miles, in 45 min. 44 sec.

The first three were world records for either steam or Diesel traction and the fourth for steam traction. Calculate the record average speeds for (*a*), (*b*) and (*d*) correct to one decimal place.

10. Two ships A and B are 50 sea miles apart at noon, A being directly east of B. A steams directly south, B keeps a straight course in a direction 60° south of east, and the two ships meet at 8 P.M. Find graphically or otherwise the average speeds of the two ships.

11. A ship leaves a port P for another port Q which is 80 sea miles due north of P. Owing to fog the ship loses its bearings and finds that it is at R, 50 sea miles in a direction 20° east of north of P. If it has 5 hrs. in which to reach Q from R, and it maintains a straight course, at what average speed should it travel and in what direction from R ?

12. (i) Convert (*a*) 15 ft./sec., (*b*) 20 ft./sec., (*c*) 100 yds. in 10 sec. to miles per hour.

(ii) Convert (*a*) 72 m.p.h., (*b*) 18 miles 600 yds. in 56 min., (*c*) 5 fur. 3 chs. in 30 sec. to feet per second.

13. Taking 1 ft. as equal to 30·48 cm., prove that 1 m.p.h. =44·70 cm./sec.

14. If 1 ft. =30·48 cm., express a speed of 60 m.p.h. (*a*) in kilometres per hour, (*b*) in centimetres per second.

15. (*a*) The speed of sound is approximately 1,100 ft./sec. Express this in miles per hour.

(*b*) The speed of light is approximately 186,000 miles/sec. How many times as fast as sound does it travel ?

16. In D.G.A. Lowe's book for athletes he suggests that to gain a knowledge of pace an athlete should attempt :

(*a*) The first 300 yds. of a 440 yds. race in 33 sec.
(*b*) The first 600 yds. of a ½ mile race in 73 sec.
(*c*) The first 880 yds. of a 1 mile race in 2 min. 10 sec.
(*d*) The first two-thirds of a mile in a 1 mile race in 2 min. 55 sec.

These are first-class times. Express the average speeds for (*a*), (*b*), (*c*) and (*d*) in miles per hour correct to one decimal place.

17. In a quarter-miles race A can beat B by 20 yds. and C can beat B by 2 sec. If A and C were matched against each other for this distance, which would win and by what distance if A's time in each case is 50 sec. ? (*L.M.*)
(Answer to the nearest foot.)

88. Distance-Time and Speed-Time Graphs.

The two graphs on the next page show :

(1) The distance travelled by a tube train plotted on a time base.
(2) The speed of the train plotted on a time base.

The two graphs are so arranged that corresponding to a given time the total distance travelled by the train and the speed of the train at this time can be read at the same time.

When we consider the distance-time graph, as statements are

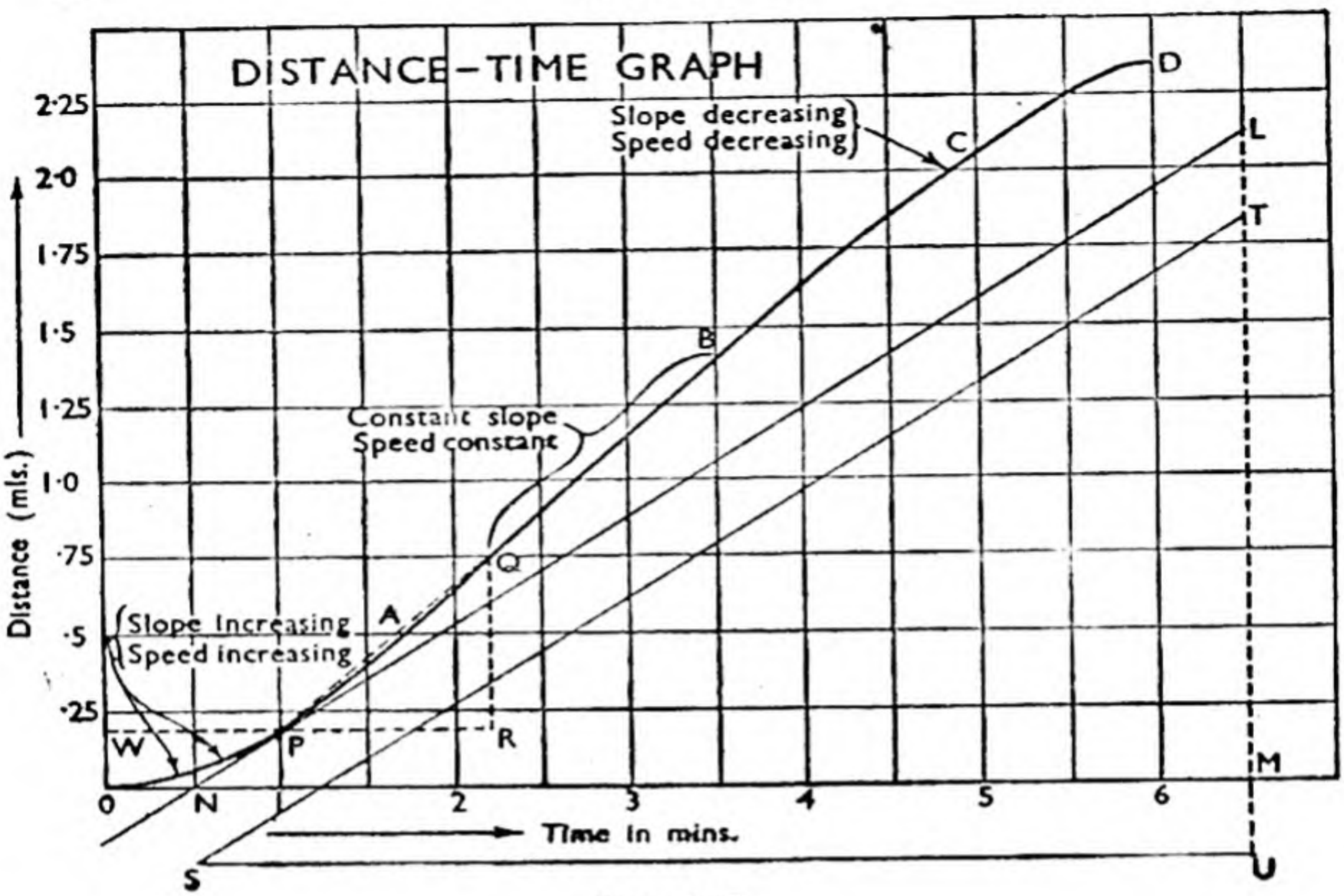

FIG. 147.

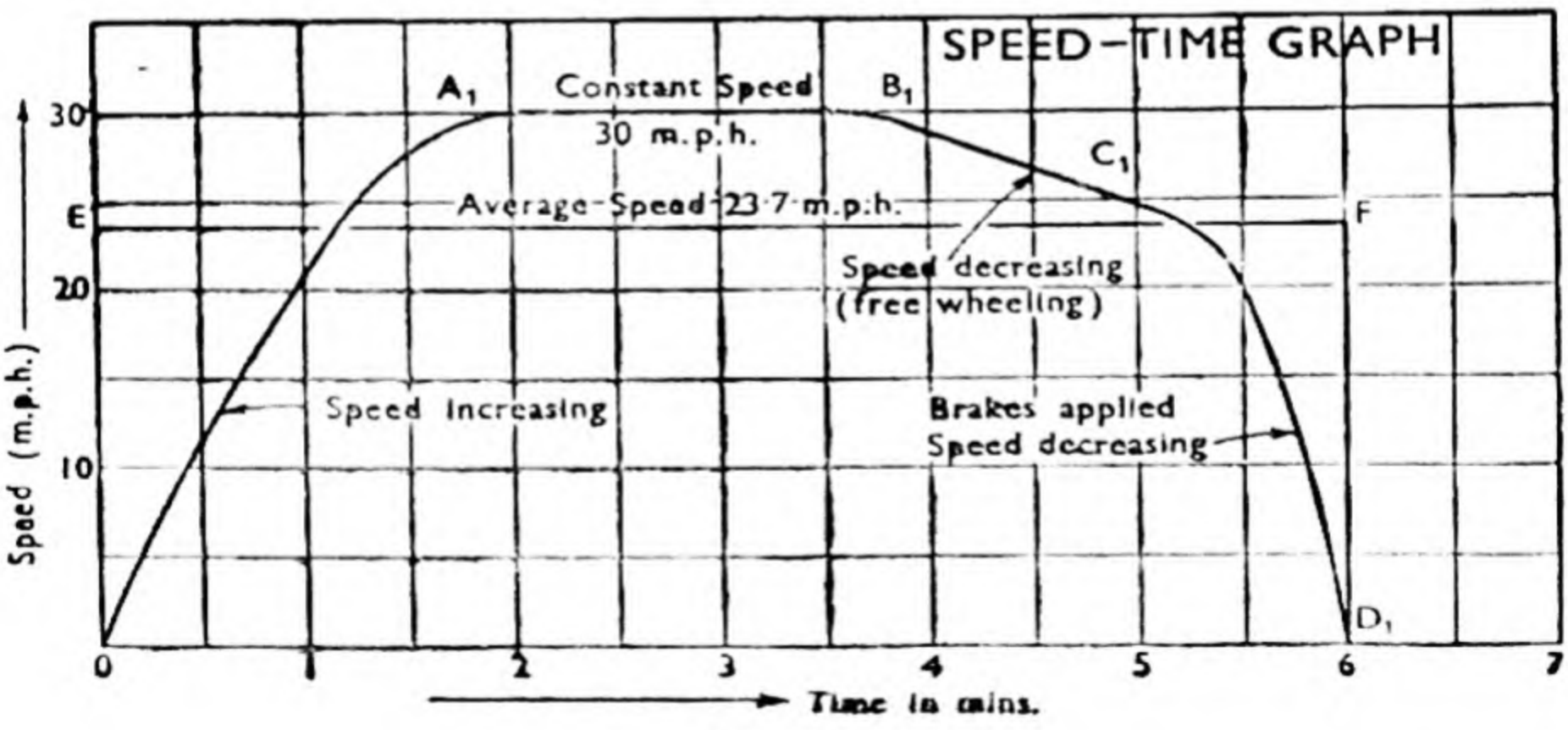

FIG. 148.

made about the speed of the train they should be checked by referring to the speed-time graph, and vice versa.

The graphs are divided into four main parts : $OA(O_1A_1)$, $AB(A_1B_1)$, $BC(B_1C_1)$ and $CD(C_1D_1)$.

OA and O_1A_1.—The part of the graph OA corresponds to the time during which the train is gaining speed.

Thus, in the first $\frac{1}{4}$ min. the train travels 0·02 miles.
„ second $\frac{1}{4}$ „ „ „ 0·03 miles.
„ third $\frac{1}{4}$ „ „ „ 0·05 miles.

These readings show that in equal times the distances travelled by the train are increasing. We also notice the very important property of the distance-time graph, viz., *the increasing speed of the train is shown by the increasing slope or gradient of the graph*, i.e., *where the distance-time graph is becoming steeper the train is moving with increasing speed.*

AB and A_1B_1.—At A the graph has become almost a straight line, and remains so until B is reached. If we examine the "steps" drawn on the graph we see that equal distances are travelled in equal times, moreover, if the steps are made smaller this remains true. AB therefore corresponds to a period during which the train is moving with uniform speed. On the distance-time graph constant slope indicates uniform speed. On the speed-time graph uniform speed is shown clearly, A_1B_1 being a straight line parallel to the time axis.

BC and B_1C_1, CD and C_1D_1.—The distance-time graph from B to E shows a continual decrease in the slope of the graph, and consequently a corresponding continual decrease in the speed of the train. The speed-time graph bears this out, but also gives information about the way in which the speed changes, *which cannot be readily seen from the distance-time graph.* Thus, from the *speed-time* graph we see that from B_1 to C_1 the speed is decreasing steadily, but at quite a slow rate in comparison with the rate at which it changes from C_1 to D_1. In fact this is due to the very different circumstances under which the train travels these two parts of its journey. From B_1 to C_1 the train is travelling under no motive power and is "free-wheeling," whereas from C_1 to D_1 the brakes are applied and, of course, the

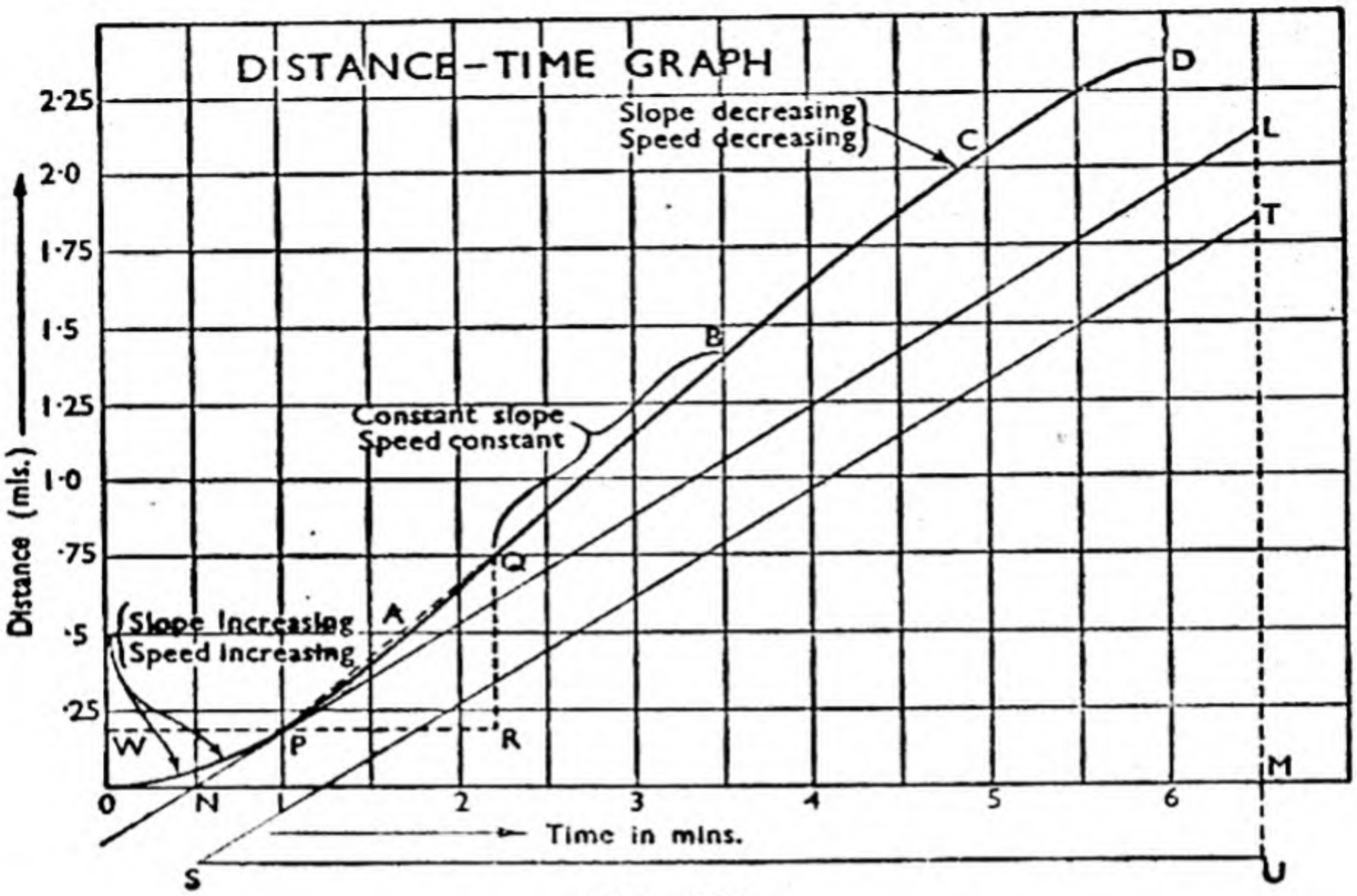

FIG. 147.

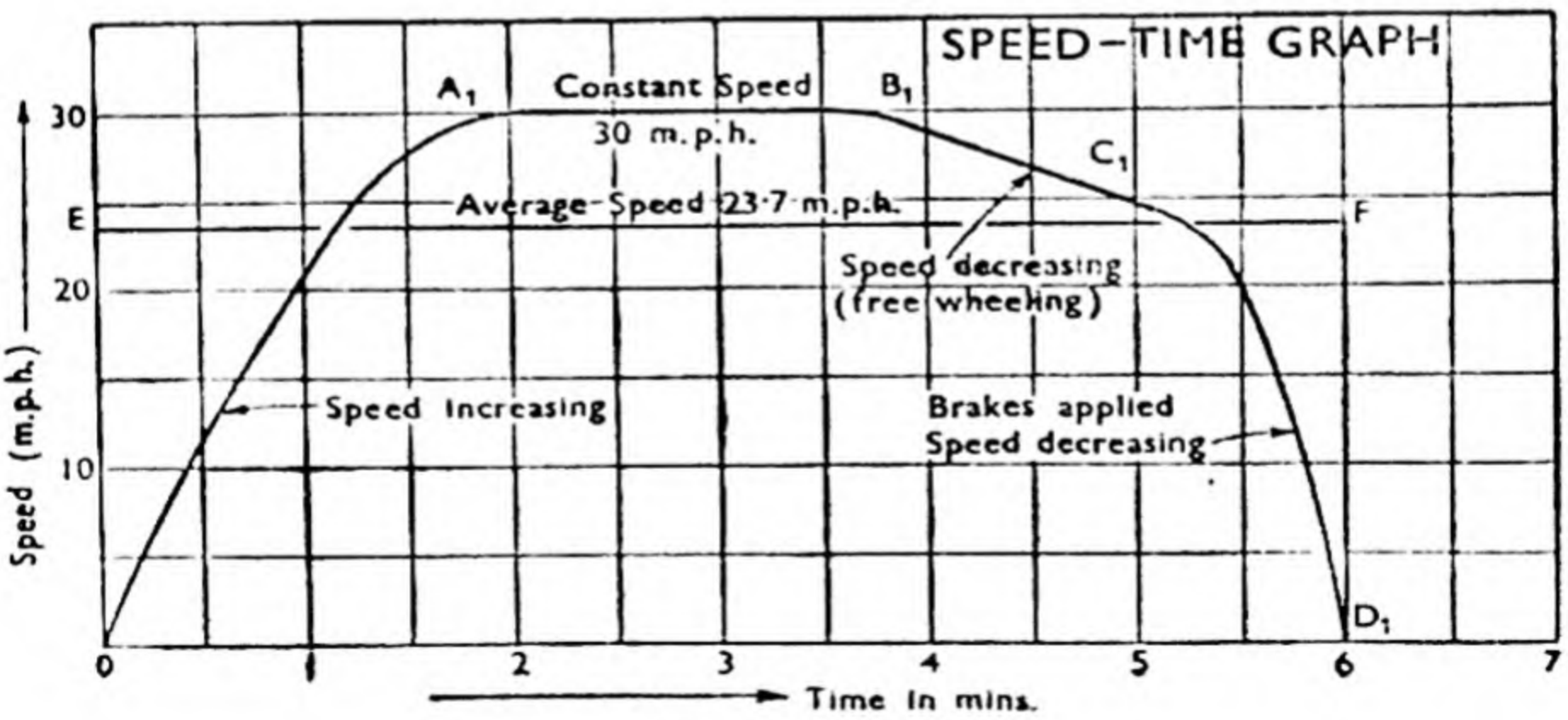

FIG. 148.

train loses speed at a much greater rate than when it is free-wheeling.

89. Determination of Speed from the Distance-Time Graph.

1. *Uniform Speed.*—AB is the part of the graph which corresponds to uniform speed. From the smaller "steps," which have been drawn, it will be seen that the train travels 0·26 miles in 0·5 min. This gives a speed of 0·52 miles per min. or 31·2 m.p.h. A better value of this uniform speed is obtained by taking the much larger step, for which in 2 min. the train travels 1·05 miles. This gives the more accurate speed 31·5 m.p.h.

2. *Instantaneous Speed.*—Let us suppose that the speed of the train is required at the point P. At P the speed is increasing and is therefore not to be found by the method used for uniform speed. At P a straight line is drawn to touch the curve of the graph at P. *PL is therefore a tangent to the curve at P.* If the speed of the train had been uniform at P the distance-time graph would have consisted of the straight line PL and LP continued. It has been stated that in order to find instantaneous speed we must determine the distance travelled in unit time following the instant at which the speed is required upon the assumption that the speed remains constant during this unit time. The instantaneous speed at P is therefore equal to the uniform speed, as given by the distance-time graph LPN. In general, on a graph the bigger the time base used in determining speeds the more accurate will be the results. In the case of the graph we are considering a time base of 5 sec. is used. MN is the time base, and LM represents the distance travelled.

$$\text{Instantaneous speed at P} = \frac{\text{LM}}{\text{MN}}$$

$$= \frac{2{\cdot}1}{6} \text{ miles per min.}$$

$$= 0{\cdot}35 \times 60 \text{ m.p.h.}$$

$$= 21 \text{ m.p.h.}$$

This result agrees with the speed as given by the speed-time graph.

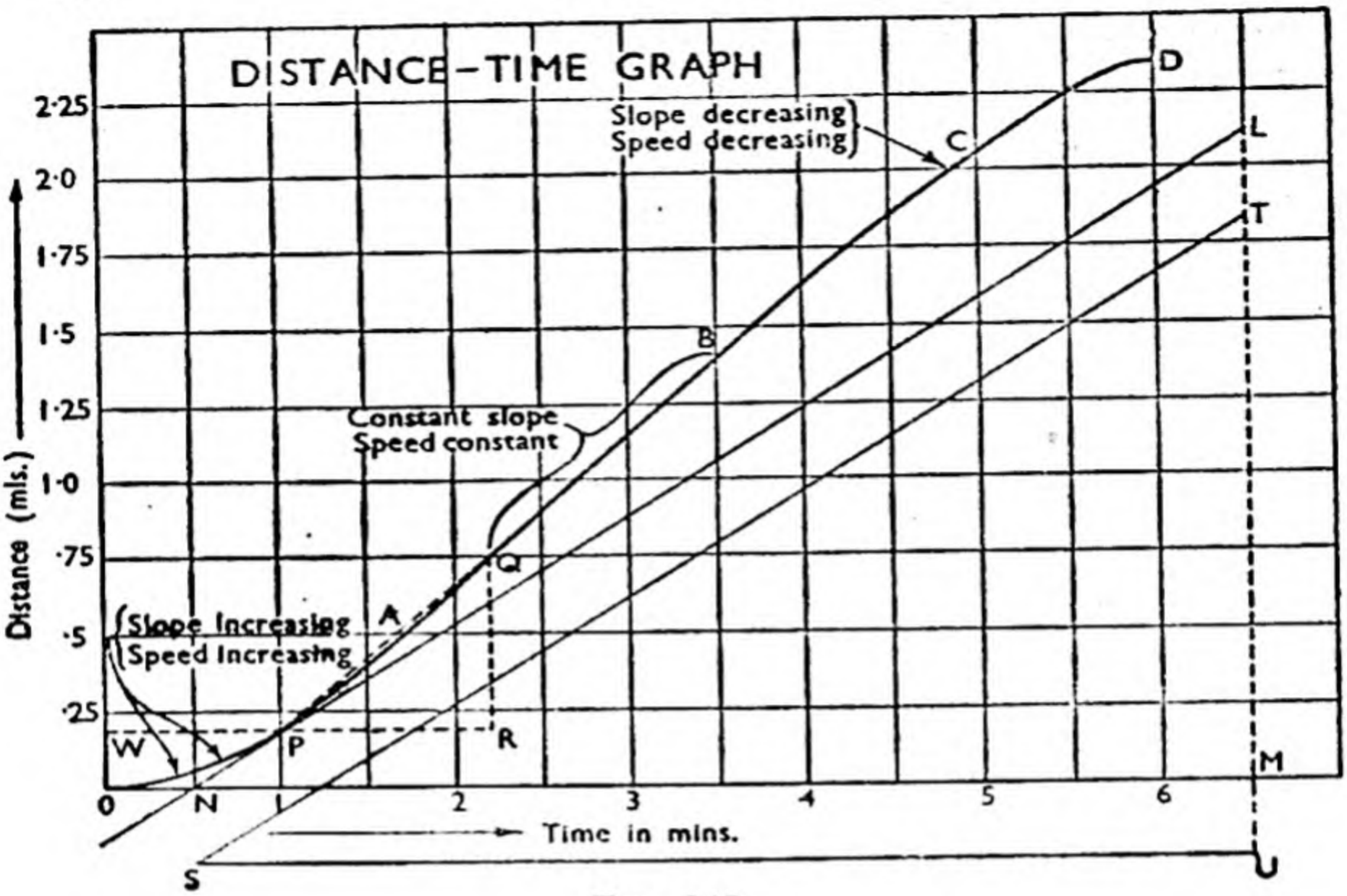

FIG. 147.

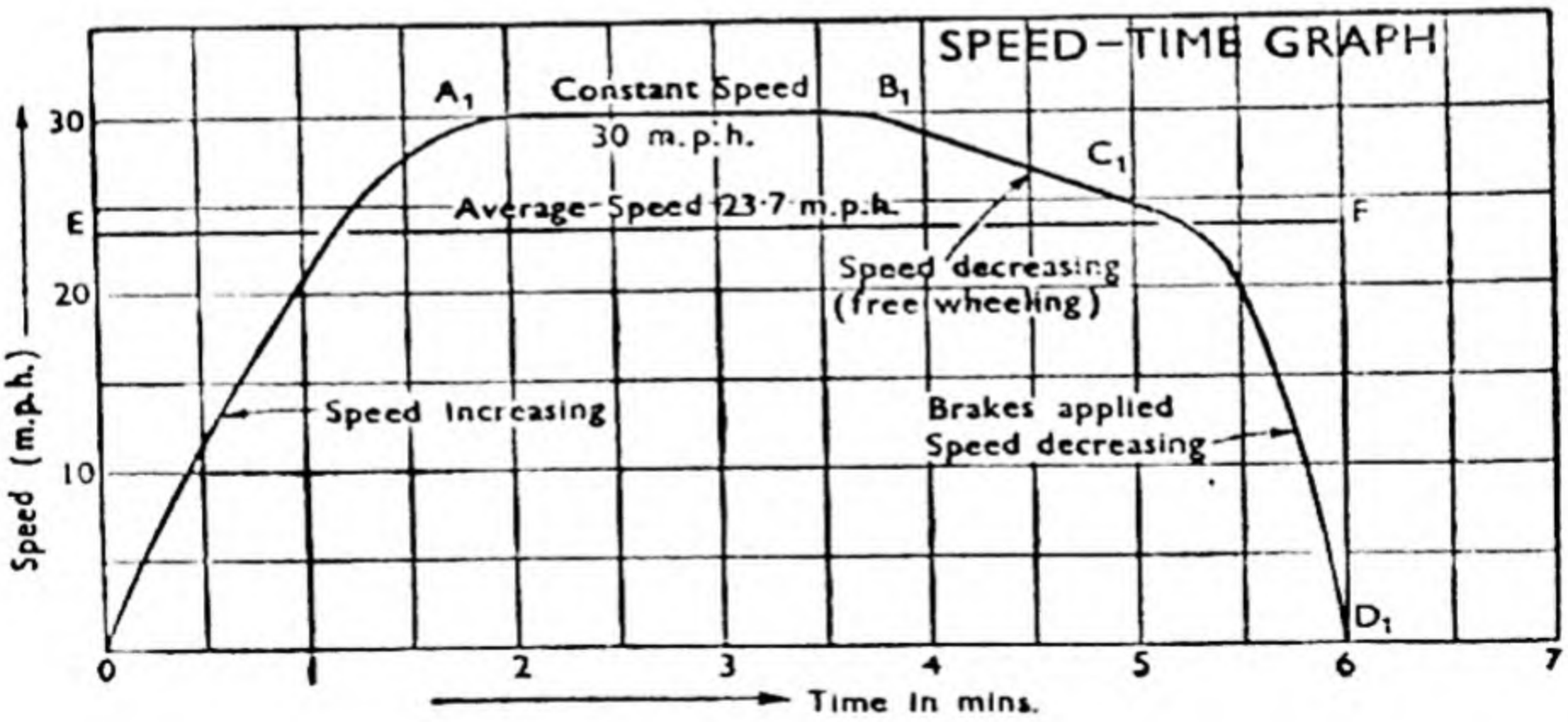

FIG. 148.

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$$\text{Instantaneous speed at P} = \frac{\text{LM}}{\text{MN}}$$

$$= \frac{2{\cdot}1}{6} \text{ miles per min.}$$

$$= 0{\cdot}35 \times 60 \text{ m.p.h.}$$

$$= 21 \text{ m.p.h.}$$

This result agrees with the speed as given by the speed-time graph.

3. *Average Speed.*—Consider the distance represented by the line QR on the graph and the corresponding time represented by the line PR. In the time PR the train travels the distance RQ, therefore the average speed during the time PR is given by

$$\text{Average speed} = \frac{\text{RQ (miles)}}{\text{PR (mins.)}} \text{ miles per min.}$$

$$= \frac{0{\cdot}56}{1{\cdot}2} \text{ miles per min.}$$

$$= 28 \text{ m.p.h.}$$

As the point Q approaches P, Q always being on the curve, the ratio of RQ to PR becomes more nearly equal to the ratio of LM to MN, and PQ becomes more nearly parallel to LN. Hence the smaller the time interval PR the more nearly is the average speed for this interval equal to the *instantaneous speed* at P.

Let the distance travelled by the train at P be s miles,

the time taken to travel this distance be t hrs.,

the increase in distance RQ be Δs miles,

and the increase in the time PR be Δt hrs.

Then the value of the ratio $\frac{\Delta s}{\Delta t}$ when Δt is made smaller and smaller is the *instantaneous speed at the time* t.

4. *Average Speed for the Whole Journey.*—(*a*) The average speed for the whole journey is given by

$$\frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{2{\cdot}35}{6} \text{ miles per min.}$$

$$= \frac{2{\cdot}35}{\frac{1}{10}} \times 60 \text{ m.p.h.}$$

$$= 23{\cdot}5 \text{ m.p.h.}$$

A uniform speed of 23·5 m.p.h. is shown on the speed-time graph by the straight line EF.

(*b*) Since the speed of the train is represented by the " height " of the speed-time graph, the average speed for the whole interval of time is equal to the *average height* of the graph.

First divide the area into a number of strips, of equal width, parallel to the speed axis. Write down the heights of these strips (on the speed scale) at the mean time. Thus if the strip is bounded by lines parallel to the speed axis corresponding to $\frac{1}{2}$ and 1 min., the " height " at $\frac{3}{4}$ min. is written down. We now add up these heights and divide the result by the number of heights taken (which is also equal to the number of strips).

Taking strips of " width " $\frac{1}{2}$ min.,

$$\text{Average speed} = \frac{6 + 16 + 24{\cdot}8 + 28{\cdot}2 + 3 \times 30 + 29{\cdot}8 + 28 + 25{\cdot}7 + 23{\cdot}5 + 12}{12}$$

$$= \frac{284{\cdot}0}{12}$$

$$= 23{\cdot}7 \text{ m.p.h.}$$

5. *To Find the Time at which the Speed has a given Value.*—Suppose that it is required to find the time at which the speed is 21 m.p.h. The line ST is drawn with a slope given by the ratio

$$\frac{\text{TU (miles)}}{\text{SU (min.)}} = \frac{2{\cdot}1}{6} = \frac{21}{60} \text{ of a mile in 1 min.} = 21 \text{ m.p.h.}$$

Parallel to ST a line is drawn to touch the curve, as shown in the figure. The point of contact P must be determined with great care, since considerable error is possible in estimating its position. The time at which the speed is 21 m.p.h. is represented by PW and is 1 min. from the start. There is a second position at which the speed has this same value, but this position has not been shown on the graph.

6. *To Draw the Speed-Time Graph given the Distance-Time Graph.*—It should now be clear that, as in (2) above, the speed at any time may be found from the distance-time graph. The

speed-time graph may be drawn by plotting the speeds so found on a time base and connecting the points by means of a continuous curve upon the assumption that the motion is not jerky.

Another method which may be used to obtain the speed-time curve from the distance-time curve is as follows :—

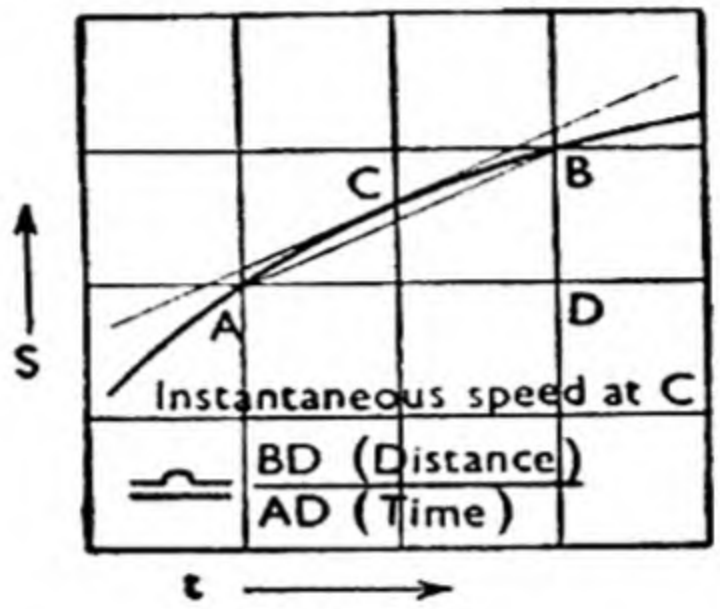

FIG. 149.

For small intervals of time the average speed of a moving body is nearly equal to the instantaneous speed at the midpoint of this interval of time. Thus by dividing the increase in distance in a given interval of time, by this interval of time the result is very nearly equal to the speed at the midpoint of the interval. Referring to the graph shown above, this means that the chord AB is nearly parallel to the tangent to the curve at C. If a graph is drawn in which the values of the speed so calculated are plotted against the times at the midpoints of the time intervals, a speed-time graph is obtained by connecting up these points by a curved line.

Consider the distance-time graph on p. 152, for which the following table gives corresponding values of distance and time. The table is arranged downwards so as to enable the subtractions to be carried out with the greatest ease.

Distance.	Time.	Increase in Distance.	Increase in Time.	Average Speed (Miles per Hour) = Speed at Mid-interval.	Time at Mid-interval.
Miles. 0·00	Min. 0·0	Miles.	Min.		
		0·05	0·5	6·0	0·25
0·05	0·5				
		0·14	0·5	16·4	0·75
0·19	1·0				
		0·21	0·5	25·2	1·25
0·40	1·5				
		0·24	0·5	28·8	1·75
0·64	2·0				
		0·26	0·5	31·2	2·25
0·90	2·5				
		0·25	0·5	30·0	2·75
1·15	3·0				
		0·24	0·5	28·8	3·25
1·39	3·5				
		0·24	0·5	28·8	3·75
1·63	4·0				

The speeds found by the above method may be compared with those shown on the speed-time graph.

EXAMPLES XXIV

1. From the distance-time graph shown, find the speed represented by OA and by BC in miles per hour. Explain the shape of the graph AB and find the average speed for the whole journey, occupying 75 min.

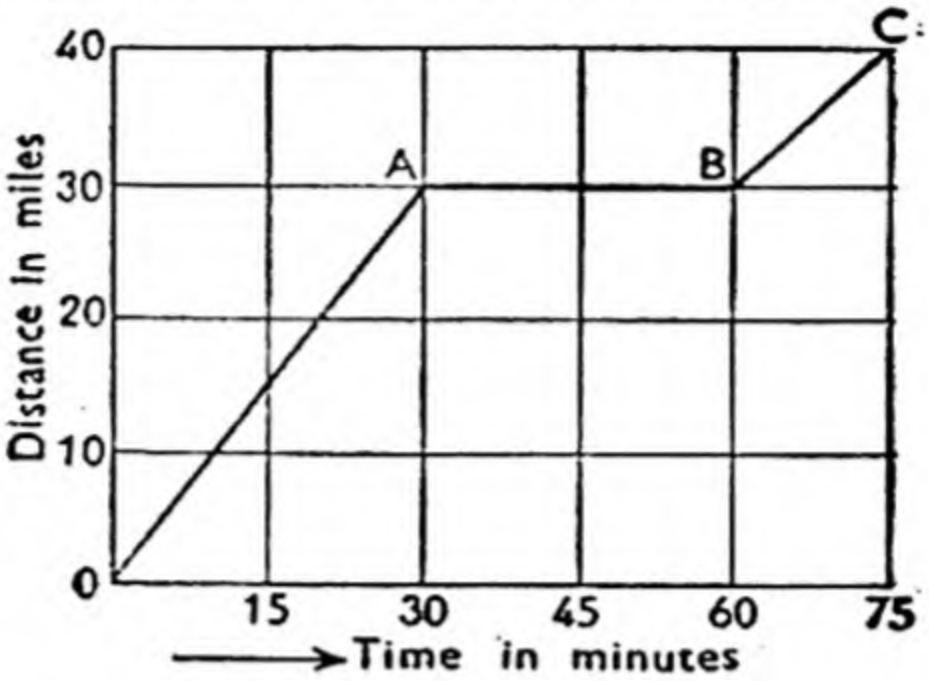

FIG. 150.

2. The distance-time graphs OABCDEFG and OG are those of a cyclist and a pedestrian who complete the same journey of 20 miles in 7 hrs. From the graph find the times at which the pedestrian passes

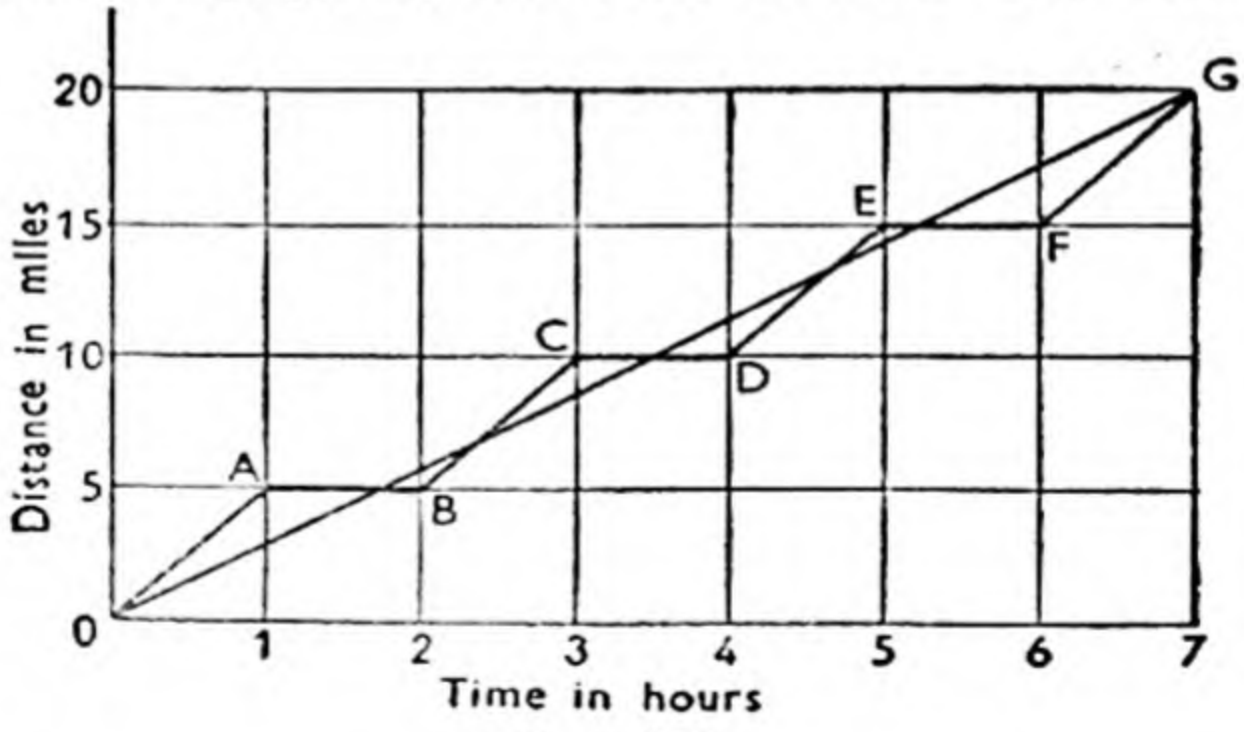

FIG. 151.

the cyclist and the times at which the cyclist passes the pedestrian. What is the average speed of the cyclist? When are they the farthest distance apart, and what is this distance?

3. A pedestrian walking at 4 m.p.h. passes a certain point in a road. Ten minutes later a motor car travelling at 20 m.p.h. in the same direction passes the same point. Draw a distance-time graph, showing the distances travelled by the pedestrian and the car during the 15 min. following the instant the pedestrian passes the certain point. From your graph find

when the car passes the pedestrian. How far are they apart (i) 5 min. before the motorist overtakes the pedestrian; (ii) at the end of the interval of 15 min. ?

4. A cyclist and a pedestrian start together along the same route. The cyclist rides at 8 m.p.h. for half an hour and rests for 1 hr. alternately. The pedestrian maintains a steady pace of 3 m.p.h. Draw the speed-time graph, showing the distances travelled by the cyclist and the pedestrian during 8 hrs. Find the times at which the pedestrian passes the cyclist and at which the cyclist passes the pedestrian. After what time will there be no more meetings ?

5. The distances travelled by an electric train between two stations corresponding to times measured from the beginning of the journey are given in the following table :—

Distance.	Time.	Distance.	Time.
Miles.	Min.	Miles.	Min.
0·00	0·0	1·15	3·0
0·05	0·5	1·63	4·0
0·18	1·0	1·80	4·5
0·40	1·5	1·88	5·0
0·65	2·0	1·90	5·5

Draw the distance-time graph, and find (i) the speed of the train when $t = 1$ min., (ii) the speed when $t = 3$ min., (iii) over what interval of time the speed of the train is approximately uniform.

6. In Example (5) find the speeds of the train at intervals of half a minute and draw the speed-time graph. From this graph find when the speed of the train is 25 m.p.h.

7. The distances of a moving body from a fixed point were determined at intervals of 1 sec., and are shown in the tables :—

Time in seconds . .	0	1	2	3	4	5
Distance in feet . .	0	3	12	27	48	75

Draw the distance-time graph for the body, and from your graph find (i) the average speed in the interval $t = 1{\cdot}5$ to $t = 4{\cdot}5$, (ii) the speed of the body at the end of 2·8 sec.

8. The distances fallen by a parachutist in certain times is given in the following table :—

Distance.	Time.	Distance.	Time.
Ft.	Sec.	Ft.	Sec.
0	0·0	1,240	15
100	2·5	1,360	20
240	4·0	1,440	25
400	5·0	1,520	30
680	7·0	1,600	35
960	10·0		

Draw the distance-time graph, and from it find (i) the greatest speed acquired by the parachutist, (ii) the constant " terminal speed " of the parachutist.

9. The distance s fallen by a body in time t in a vacuum at a certain place are given by the formula $s = 16 \cdot 1t^2$. Calculate the values of s corresponding to $t = 0$, 1, 2, 3, 4, 5, and draw the distance-time graph. From your graph find the speeds of the body at $t = 1 \cdot 5$, $2 \cdot 5$, $3 \cdot 5$, $4 \cdot 5$, and draw the speed-time graph for these values. Do you notice anything special about this graph ?

10. The following table gives corresponding values of distance s and time t for a piece of sliding mechanism :—

s.	t.	s.	t.
In.	Sec.	In.	Sec.
0·00	0·0	3·62	0·7
0·05	0·1	4·90	0·8
0·19	0·2	6·14	0·9
0·46	0·3	7·12	1·0
0·90	0·4	7·78	1·1
1·56	0·5	8·00	1·2
2·47	0·6		

Draw the distance-time graph, and from it find the maximum speed of the sliding piece. Describe the motion as fully as you can.

11. During the launching of a cruiser the distance s travelled by the cruiser down the sliding ways is given, together with corresponding times in the following table :—

s.	t.	s.	t.
Ft.	Sec.	Ft.	Sec.
0·00	0	173	24
0·04	4	245	28
5·00	8	320	32
27·00	12	402	36
64·00	16	484	40
112·00	20		

From the distance-time graph find the speed after 12 sec. and the speed when the distance is 100 ft. Find also the time taken to acquire half the final speed.

12. In the table below s represents the displacement of a piston from one end of its stroke, and t represents the time measured from the beginning of the stroke :—

s.	*t.*	*s.*	*t.*
Ft.	Sec.	Ft.	Sec.
0·000	0·00	1·352	0·35
0·041	0·05	1·575	0·40
0·158	0·10	1·757	0·45
0·343	0·15	1·892	0·50
0·575	0·20	1·973	0·55
0·834	0·25	2·000	0·60
1·100	0·30		

Draw the distance-time graph and the speed-time graph, and use the graphs to help you to describe the motion of the piston. What is the greatest speed of the piston and at what time does it possess this speed ?

90. The Area Beneath the Speed-Time Graph Represents the Distance Travelled.

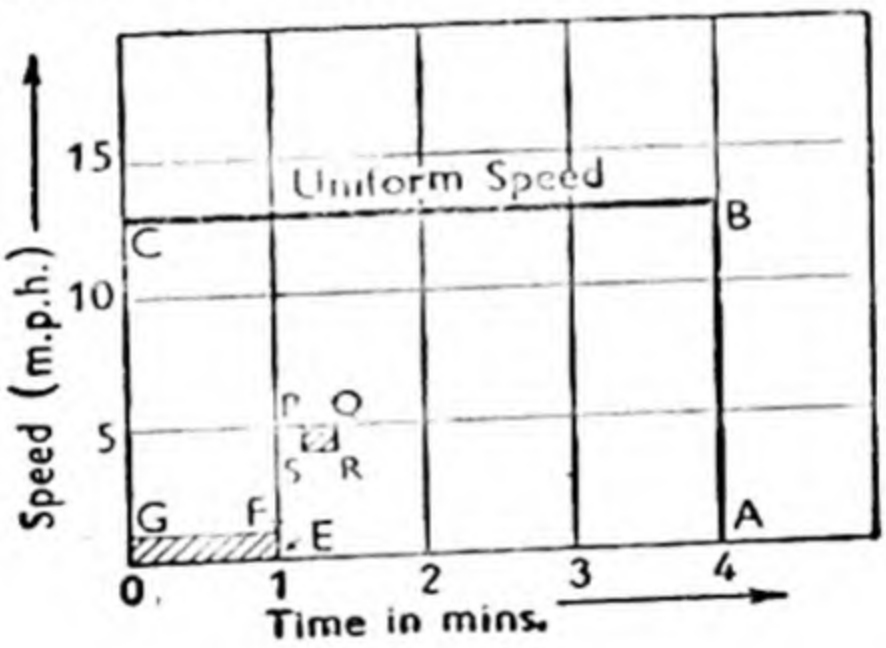

FIG. 152.

(*a*) *Uniform Speed.*—Consider the above speed-time graph. CB represents a uniform speed of 13 m.p.h. If CB is measured in the units of the time scale, viz., minutes, then CB divided by 60 represents the time for the whole journey in hours. OC measured in the units of the speed scale represents a speed of 13 m.p.h., which is uniform during the whole time of the journey.

Therefore $\frac{CB}{60} \times OC$ represents the total distance travelled in miles, for we have multiplied the uniform speed in miles per hour by the time taken in hours.

If OE is taken as a unit of length along the time base, and

OG the unit of length parallel to the speed axis, then OEFG represents a unit of area. If a body moves with the uniform speed represented by EF m.p.h. for a time represented by OE min., the distance which it travels is represented by the unit of area OEFG. Hence the area OEFG represents the distance travelled by a body which moves at 1 m.p.h. for a time of 1 min.

That is, area OEFG represents 1/60 mile.

Thus the area OABC measured in units equal to the area OEFG represents a total distance of $\frac{\text{OABC}}{60}$ miles.

That is, $\frac{\text{CB (min.)} \times \text{OC (m.p.h.)}}{60}$ is the total distance travelled.

The area in units equal to OEFG multiplied by 1/60 is equal to the distance travelled. Again, suppose that it is more convenient to estimate the area OABC in units equal to one small square. SR, the side of a small square parallel to the time base, represents $\frac{1}{5}$ min., and the adjacent side SP represents a speed of 1 m.p.h. The area of a small square is now taken as the unit of area, which therefore represents a distance of $1 \times \frac{1}{60 \times 5}$ miles. Hence if the number of small squares in the required area is found, and the result is multiplied by 1/300, the total distance travelled (in miles) is obtained.

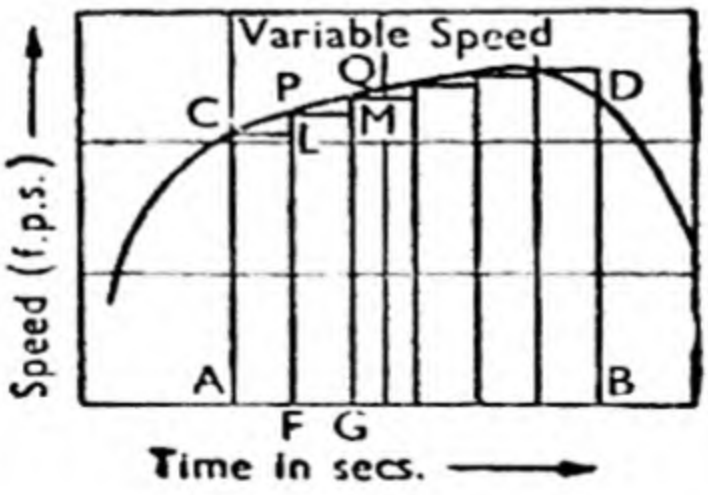

FIG. 153.

(*b*) *Variable Speed.*—Suppose that a body is travelling with variable speed. The speed-time graph for such a case is shown above. It is required to find the distance travelled during a time

represented by AB. Through A and B, AC and BD are drawn parallel to the speed axis to meet the curve at C and D respectively. AB is divided at F, G . . . into a number of equal parts and through F, G . . ., parallels to AC which meet the curve at P, Q . . . are drawn.

CLFA is a rectangle whose area has now been shown to represent the distance travelled by a body which moves at a uniform speed AC ft./sec. during a time AF sec. Similarly, the area of the rectangle PMGF represents the distance travelled by a body which moves during FG sec. at a uniform speed of FP ft./sec. In this manner the sum of the areas of all the rectangular strips represents the distance travelled by a body which at the time OF suddenly increases its speed from FL ft./sec. to FP ft./sec. and at the time AG suddenly increases its speed from GM ft./sec. to GQ ft./sec. and so on. If the number of rectangles is made larger the sudden increases in speed become smaller, and it is readily seen that the greater the number of rectangles taken the more nearly does the jerky motion become the continuous motion shown by the curve of the graph. Now the greater the number of rectangles becomes, the more nearly is the sum of their areas equal to the area beneath the curve. We therefore see that the area ACDB beneath the curve represents the total distance travelled by the body in the time AB sec.

(*c*) *The Total Distance Travelled is Equal to the Average Speed Multiplied by the Total Time.*—Since the average speed has been found by dividing the total distance travelled by the total time taken, it follows that if we are able to find the average speed during a journey we are also able to find the total distance travelled, for

Distance travelled = Average speed × Time taken.

91. Methods of Finding the Area Beneath the Speed-Time Graph.

1. Determine the average height of the graph (this corresponds to the average speed) and multiply by the total time. This method is used in the second worked example, and reference to the figure shows how this method is really the application of (3) below when the strips are of equal width.

2. Divide the area to be found into rectangles, triangles and trapeziums, the sum of the areas of these parts is the required area.

3. Use rectangular strips alone and balance the parts of the strips which fall outside the required area against the parts of the area not included in any of the rectangles. This method is capable of giving very satisfactory results. The figure used in the second worked example explains the method.

4. By the counting of small unit areas, each of which represents a known distance.

Example (1).—*The speed-time graph below is made up entirely of straight lines.*

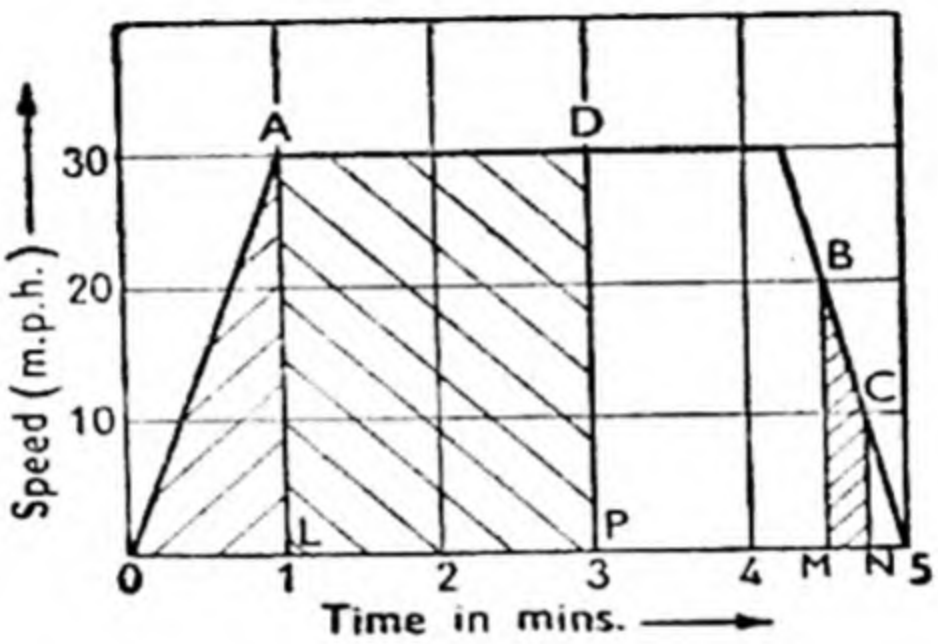

FIG. 154.

To find the distance travelled during the interval 1 *to* 3 *min.*

The distance travelled is represented by the area of the rectangle ADPL

$$= AD \,.\, DP = 2 \times 30 = 60 \text{ units.}$$

1 unit represents the distance travelled in 1 min at 1 m.p.h. $= 1 \times \frac{1}{60}$ miles.

Therefore $$60 \text{ units} = 60 \times \frac{1}{60} = 1 \text{ mile.}$$

The distance travelled by the train in the interval 1 to 3 min. is 1 mile.

To find the distance travelled during the first minute.

The distance travelled is represented by the area beneath the speed-time graph.

The distance travelled in the first minute is represented by the area of the triangle ALO.

$$\text{Area of triangle ALO} = \tfrac{1}{2} \,.\, OL \,.\, AL = \tfrac{1}{2} \times 1 \times 30 = 15 \text{ units.}$$

$$15 \text{ units} = 15 \times \frac{1}{60} = \tfrac{1}{4} \text{ mile.}$$

The distance travelled by the train in the first minute is $\frac{1}{4}$ mile.

To find the distance travelled in the interval $4\frac{1}{2}$ *to* $4\frac{3}{4}$ *min.*

The distance travelled is represented by the area of the trapezium BCMN.

Area of trapezium BCMN $= \frac{1}{2}(20 + 10) \times \frac{1}{4} = 3\frac{3}{4}$ units $= \frac{15}{4} \times \frac{1}{60}$ miles.

Therefore the distance travelled is $\frac{1}{16}$ miles.

Average speed in the interval $4\frac{1}{2}$ *to* $4\frac{3}{4}$ *min.*

$$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time taken}} = \frac{1}{16} \div \frac{1}{240} = \frac{1}{16} \times 240 = 15 \text{ m.p.h.}$$

Example (2).—*From the speed-time graph below, find the average speed and the total distance travelled.*

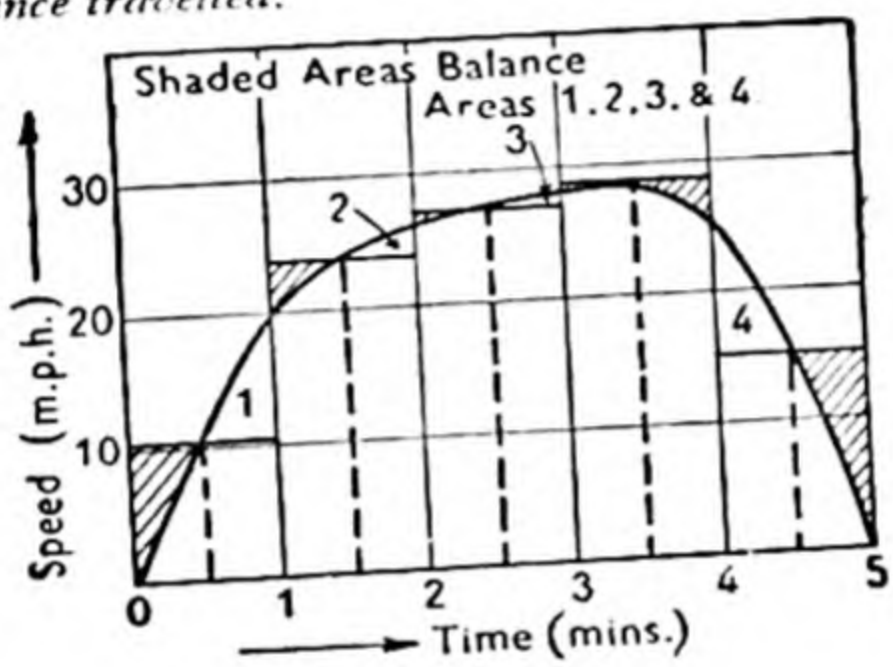

FIG. 155.

Dividing the area into five strips, the sum of the heights at the mean times is equal to the sum of the *dotted lines.*

$$\text{Average speed} = \frac{10{\cdot}5 + 24 + 27 + 28{\cdot}3 + 15}{5} = \frac{104{\cdot}8}{5} = 20{\cdot}96 = 21 \text{ m.p.h.}$$

Correct to the nearest 1 per cent., which may be taken as a satisfactory standard of accuracy for the present kind of graphical work.

Total distance travelled has been shown to be equal to the area beneath the speed-time curve, and this in its turn is equal to

$$\text{Average speed} \times \text{Time} = \frac{104{\cdot}8}{5} \times \frac{5}{60} = 1{\cdot}75 \text{ miles.}$$

(Correct to second decimal place.)

This method of finding the total distance amounts to the assumption that the area of a strip, whose total length is the sum of the dotted lines and width that of one strip, is equal to the required area. This can be tested by drawing the rectangles, whose heights are the dotted lines, and noticing how the area of the parts of these rectangles outside the required area balances that of the parts of the area not covered by any rectangle.

EXAMPLES XXV

1. The speed of a body at certain times is given in the table below. Draw the speed-time graph and find the distance travelled by the body during the first 5 sec. of its motion.

Time.	Speed.	Time.	Speed.
Sec.	Ft./sec.	Sec.	Ft./sec.
0	0·0	4	18·0
1	4·5	5	22·5
2	9·0	6	27·0
3	13·5		

2. The speed-time graph for a moving body is given below. From the graph find the distance travelled by the body during the journey.

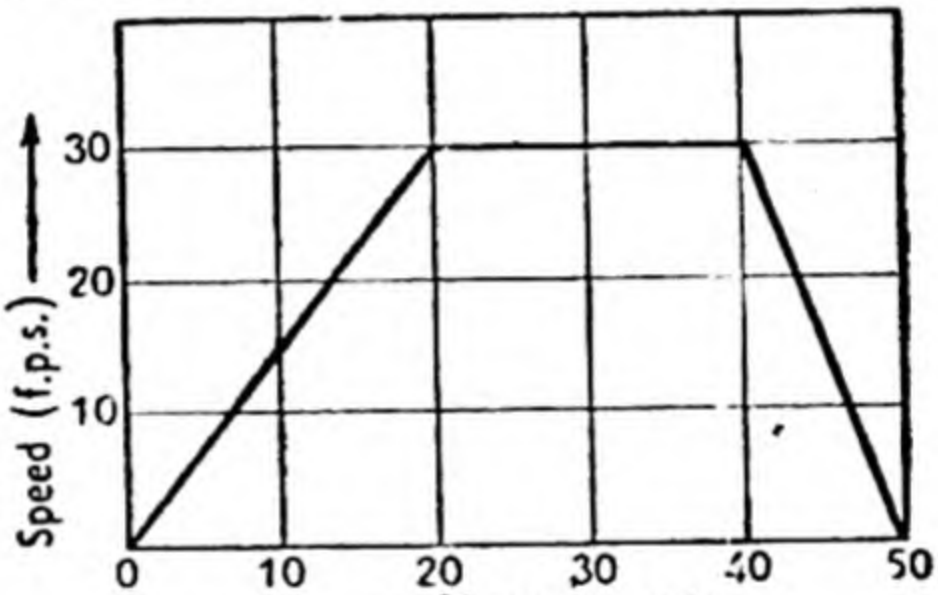

FIG. 156.

3. The speed-time graph for an underground tube train is shown in the figure below. Find the distance represented by one small square and the total distance travelled by the train.

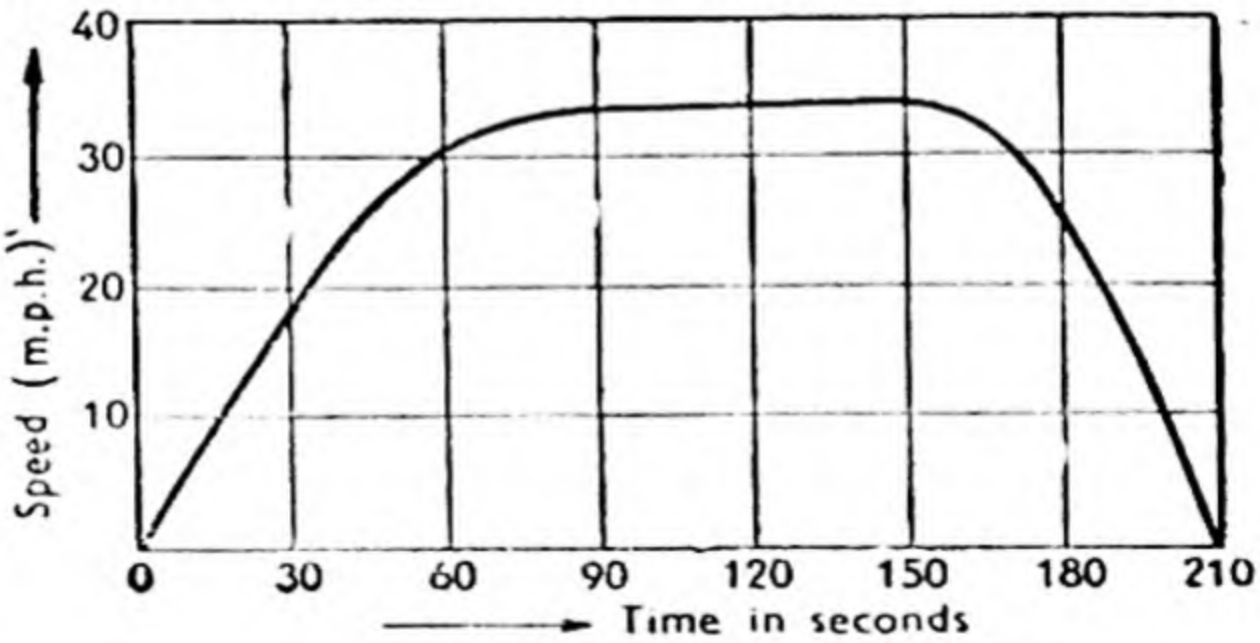

FIG. 157.

4. A train is brought to rest from a speed of 34 m.p.h., and the speeds corresponding to various times are given in the table below. Draw the speed-time graph, and from it find the distance travelled during the whole interval of 1 min.

Speed.	Time.	Speed.	Time.
M.p.h.	Sec.	M.p.h.	Sec.
34·0	0	25·0	30
33·5	6	16·5	42
33·0	12	7·0	54
31·0	18	0·0	60
29·5	24		

5. The angular speed A of the bob of a pendulum in radians per second is given by the formula $A = \cos 6t$, where t is the time measured from the instant the pendulum passes through the vertical position. Plot the angular speed against the time from $t = 0$ to $t = \frac{\pi}{12}$ at intervals of $\frac{\pi}{72}$ sec. From the graph find the average angular speed of the pendulum during the $\frac{\pi}{12}$ sec.

6. The speeds of a piston at times measured from the beginning of a stroke is given in the following table. Draw the speed-time graph, and from it find the total distance travelled and the average speed of the piston.

v.	*t.*	*v.*	*t.*
Ft./sec.	Sec.	Ft./sec.	Sec.
0·00	0·00	4·76	0·35
1·60	0·05	4·08	0·40
3·00	0·10	3·20	0·45
4·10	0·15	2·20	0·50
4·90	0·20	1·06	0·55
5·34	0·25	0·00	0·60
5·20	0·30		

7. A steam train travels according to the following table, in which times, distances and speeds are recorded :—

GOVT. GIRLS COLLEGE SRINAGAR ACC. NO.....

Time.	Distance.	Speed.	Time.	Distance.	Speed.
Min.	Miles	M.p.h.	Min.	Miles	M.p.h.
0·0	0·00	0·0	3·5	1·32	35·0
0·5	0·04	8·5	4·0	1·61	35·5
1·0	0·13	15·7	4·5	1·91	35·8
1·5	0·28	22·2	5·0	2·21	36·0
2·0	0·50	28·0	5·5	2·51	35·7
2·5	0·75	32·0	6·0	2·78	26·0
3·0	1·03	34·0	6·5	2·91	0·0

Plot the distance-time and the speed-time curves. (*a*) Describe the motion of the train by referring to the graphs. (*b*) From the distance-time graph find the instantaneous speed of the train 1½ min. after the start, and check your answer by means of the speed-time graph. (*c*) From the speed-time graph find the distance travelled in 2 min. (*d*) On the speed-time graph draw the line which represents a uniform speed equal to the average speed for the whole journey.

92. Velocity.

The velocity of a body is equal in MAGNITUDE to the speed of the body, but to completely describe the velocity it is necessary to state the DIRECTION in which the body is travelling. Thus the speed of a ship is 15 knots ; the VELOCITY of a ship is 15 knots in a direction 30° south of east.

Note.—1 SEA MILE = 6,080 ft.

If a ship travels along a line of latitude until its own latitude has changed 1°, it travels a distance equal to 1/360th of a meridian line which, passing through each of the poles, completely encircles the earth. When the ship changes its latitude by 1′ it travels 1/60th of the latter distance, and this is the distance which is equal to 6,080 ft.

1 KNOT = 1 SEA MILE PER HOUR.

When the motion of a body is in one straight line the definitions of UNIFORM SPEED, VARIABLE SPEED, SPEED and AVERAGE SPEED are equally true if we replace the word speed by the word velocity.

It is very important to notice, however, that whereas speed can change in magnitude only, VELOCITY changes (1) if its magnitude changes, or (2) if its direction changes.

It is, of course, quite usual for both the magnitude and the

direction to change, as, for example, in the case of a football which is kicked into the air, or any body which describes a curved path with varying speed.

When a gramophone record, of the circular disc type, is being played, a point on the rim of the record is moving with uniform speed, but since its direction is continually changing it is therefore travelling with VARIABLE VELOCITY.

93. Uniform Velocity.

A body travels with uniform velocity if it describes equal distances in equal consecutive times, however small these times may be IN A FIXED DIRECTION, *i.e.*, a body travelling with uniform velocity MOVES IN ONE STRAIGHT LINE.

94. Angular Velocity.

Most of us are familiar with the idea of a body rotating about an axis. The earth rotates about an axis once in 24 hrs.; a gramophone turntable rotates 78 times in a minute (this would be satisfactory for most records) about an axis through the centre of the disc and perpendicular to the plane of the disc; most machines make use of rotating parts, such as toothed wheels, pulley wheels and flywheels.

The rate at which a body turns about its axis of rotation is called the angular velocity of the body.

Consider the rotation of the earth, which is shown in

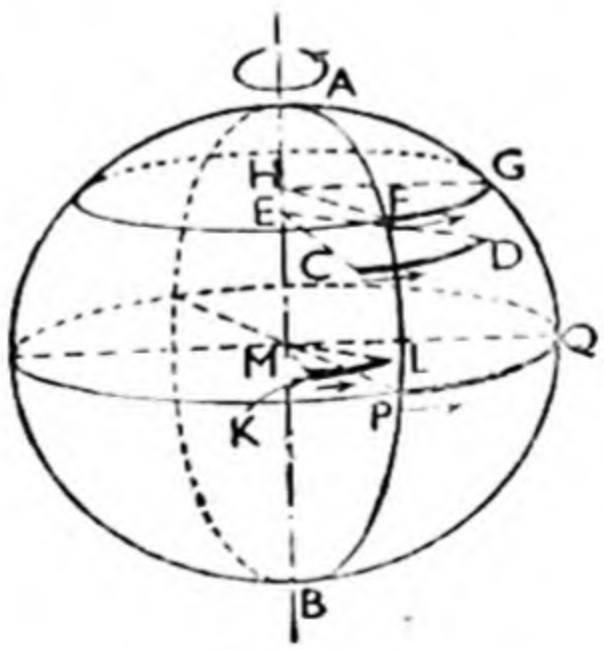

FIG. 168.

Fig. 158, with its axis vertical. APB is a great circle of longitude passing each of the poles A and B. Let us suppose that when the sphere turns about the axis AB, the great circle APB rotates into the position AQB. The points P, C and F on the surface of the sphere rotate into the positions Q, D and G respectively, and K, a point inside the sphere, turns into the position L. These points all describe circular arcs, and it is readily seen that the arcs subtend angles PMQ, CED, FHG and KML at points along the axis of the sphere. Moreover, all of these angles are equal. Hence in any rotation of a rigid body about a fixed axis, *all points of the body turn through equal angles.* If we now refer to Fig. 159 we see that the angle turned through by the point P as it moves from P to Q is equal to the angle through which the radius MP turns, in fact the angle through which a point such as P turns is *defined* as the angle turned through by the radius drawn to the point, and the angular velocity of P about M is equal to the rate at which the angle PMQ is changing.

Suppose the body is turning steadily so that all the points P, C, F and K have constant speeds ; speeds which are not as a rule equal to each other, for it is easy to see that the lengths of the arcs described by C, P, F and K in the same time are not equal to each other. Since these points turn through equal angles in equal times, they have the same angular velocity.

95. The Measurement of Angular Velocity.

Consider the angular velocity of the point P about M. We shall take the simplest case in which the radius MP turns through equal angles in equal times, that is, the case of *uniform angular velocity*.

$$\text{Angular velocity of P about M in degrees per second} = \frac{\text{Angle turned through by MP (in degrees)}}{\text{Time taken (in seconds)}} = \frac{\text{Angle PMQ (in degrees)}}{\text{Time taken (in seconds)}}.$$

Again, it is reasonable to measure the angular velocity of P by giving the number of complete rotations which it makes about

M in 1 sec. The usual method of measuring angular velocity makes use of a unit of turning called *the radian*. An angle of 1 radian is the angle which an arc of a circle equal to its radius subtends at the centre of the circle.

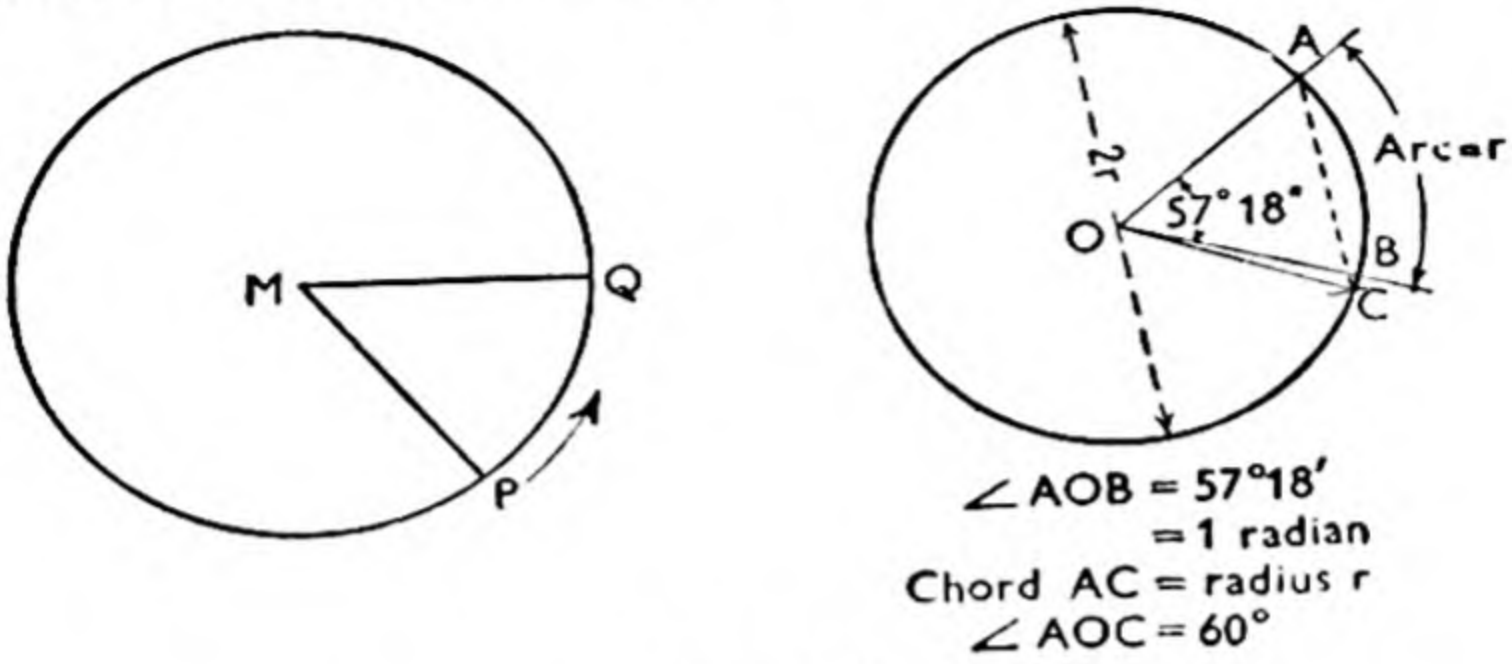

FIG. 159.

Referring to the figure above, it is readily appreciated that an angle of 1 radian is a little less than 60°. In fact,

$$1 \text{ radian} = 57^\circ\ 18'.$$

The measure of the angle which an arc of a circle subtends at its centre is

$$\frac{\text{Arc}}{\text{Radius}} \text{ radians.}$$

In particular,

$$1 \text{ complete turn} = 360^\circ = \frac{\text{Circumference}}{\text{Radius}} = \frac{2\pi r}{r} \text{ radians.}$$

$$1 \text{ complete turn} = 360^\circ = 2\pi \text{ radians.}$$

Thus if a point describes an arc of a circle with uniform angular velocity about the centre of the circle,

$$\begin{aligned}\text{Angle in radians turned through per second} &= \text{Angular velocity} \\ &= \frac{\text{Arc described per second}}{\text{Radius}} \\ &= \frac{\text{Speed of the point}}{\text{Radius}}.\end{aligned}$$

If ω radians per second is the angular velocity,
v ft./sec. is the speed of the point along the circumference,
and r ft. the radius of the circle, the result may be expressed,

$$\omega = \frac{v}{r}$$

or

$$v = r\omega.$$

Example.—*A pulley rotates* 100 *times per min. If the pulley has a diameter of* 3 *ft., and it is used to drive a belt :*

(a) *What is the angular velocity of the pulley :*
(1) *In turns per second,*
(2) *In degrees per second,*
(3) *In radians per second ?*

(b) *What is the speed of the belt ?*
(c) *What is the speed of a point* 1 *ft. from the centre of the pulley ?*

(*a*) (1) The pulley turns 100 times in 60 sec.

Hence it turns $\frac{100}{60}$ times in 1 sec. $= 1\frac{2}{3}$ times per sec.

Angular speed $= 1\frac{2}{3}$ turns per sec.

(2) The pulley turns through $\frac{5}{3} \times 360°$ per sec. $= 600°$ per sec.

(3) 1 turn $= 360° = 2\pi$ radians.

$1\frac{2}{3}$ turns per sec. $= \frac{5}{3} \times 2\pi$ radians per sec.

$= 10{\cdot}47$ radians per sec.

(*b*) Angular velocity of the pulley $= \frac{10\pi}{3}$ radians per sec.

Speed of a point on the rim of the pulley is equal to the speed of the belt = Angular velocity of the belt × Radius

$$= \frac{10\pi}{3} \times \frac{3}{2}$$
$$= 15{\cdot}7 \text{ ft./sec.}$$

(*c*) Speed of a point 1 ft. from the centre of the pulley

= Angular velocity × Distance from axis of pulley

$$= \frac{10\pi}{3} \times 1$$
$$= 10{\cdot}5 \text{ ft./sec.}$$

EXAMPLES XXVI

1. Express in radians (*a*) 600°, (*b*) 12·25 complete turns.
2. Express in degrees (*a*) 20π radians, (*b*) 20 radians, (*c*) 40 complete turns.

3. Express the following angular velocities in radians per second :—

(*a*) 100 turns per min.
(*b*) 480° in 12 sec.

4. Calculate the angular velocities of the hour-hand and the minute-hand of a clock in radians per second. (Assume that the clock keeps correct time.)

5. A flywheel rotates 100 times per min. Calculate its angular velocity in radians per second. Find also the speeds of points on the flywheel, 3 in., 5 in. and 1 ft. from its axis of rotation.

6. Assuming that the earth turns once about its axis in 24 hrs., find its angular velocity (*a*) in degrees per second, (*b*) in radians per hour. Find also the linear speed of a point at the equator which describes a circle of diameter 8,000 miles in miles per hour.

7. Assuming that the earth describes a circle of radius 90,000,000 miles about the sun as centre once in 365 days, calculate the angular velocity of the earth about the sun in radians per day and the speed of the earth in miles per minute. (Correct to three significant figures.)

8. A conveyor belt is used for moving packages from one end of a workshop to another. The speed of the belt is 30 ft./min. If it runs on rollers of diameter 4 in. and is driven by a pulley of diameter 1 ft. 6 in. calculate the angular speeds of the rollers and of the pulley in radians per second.

CHAPTER IX

ACCELERATION

96. Acceleration.

If we describe the motion of a car by saying that it is " speeding up " or " slowing down," we have stated in ordinary language that the car is travelling with an *acceleration* or with a *retardation.*

When the velocity of a body is changing it is said to travel with an acceleration.

The acceleration of a body is the rate at which its velocity is changing.

Uniform Acceleration.

When the velocity of a body increases by equal amounts in equal times, however small these times may be, it is said to travel with uniform acceleration.

If the velocity of a body travelling in a straight line increases from 10 to 25 ft./sec. in 5 sec. and its acceleration is uniform, then we may describe this acceleration by stating that the increase in velocity in 5 sec. is 15 ft./sec.

It is usual to express uniform acceleration as the *increase in velocity in unit time.*

In the above instance, therefore,

$$\text{Acceleration} = 15 \text{ ft./sec. in } 5 \text{ sec.} = 3 \text{ ft./sec. in } 1 \text{ sec.}$$
$$= 3 \text{ ft./sec./sec.}$$

The notation feet per second per second is usually written more briefly, ft./sec.^{2}, but it should be read as ft./sec./sec.

Again,

$$\text{Acceleration} = \frac{15 \times 15}{22} \text{ m.p.h. in } 5 \text{ sec.}$$

$$= \frac{15 \times 15}{22} \times 12 \times 60 \text{ m.p.h. } \textit{per hour}$$

$$= 7363\tfrac{7}{11} \text{ m.p.h./hr., or } 7363\tfrac{7}{11} \text{ m.p.h.}^2$$

It is readily appreciated from the two results obtained above that for accelerations of motor cars and trains, and, in fact, of

most bodies with which we have to deal in everyday life, it is more convenient to measure acceleration in feet per second per second than in miles per hour per hour, since the latter units require such large numbers to be used.

Suppose that we require to find the velocity of the body whose motion has been considered above after the elapse of 7 sec.

Velocity at the end of the first second $= 10 + 3$ ft./sec.
" " " second " $= 10 + 2 \times 3$ ft./sec.
" " " third " $= 10 + 3 \times 3$ ft./sec.
" " " seventh " $= 10 + 7 \times 3$ ft./sec.
" " " $= 31$ ft./sec.

Note.—At present we shall only consider changes in the *magnitude* of the velocity, *i.e.*, the acceleration when the body is moving in a straight line.

97. Variable Acceleration.

When the velocity of a body in a straight line does not increase by equal amounts in equal times it is said to move with variable acceleration.

Instantaneous Acceleration.

Suppose that the velocity of a train at the ends of successive minutes is 10, 30, 40, 43 ft./sec. . . . The increases in velocity in successive minutes are not equal. It would therefore be wrong to state that the acceleration at any instant for this train is equal to the increase in velocity per unit time following this instant.

The acceleration of a body which is travelling with variable acceleration is equal to the increase in velocity in unit time following the instant at which the acceleration is required, upon the assumption that during this unit time the acceleration remains uniform.

Average Acceleration.

However the velocity of a body which is travelling in a straight line is changing during any interval of time the *average acceleration* during this time is given by

$$\textit{Average acceleration} = \frac{\text{Total increase of velocity}}{\text{Time taken}}.$$

Thus, in the instance of the train given in the last paragraph,

Average acceleration for the first minute $= \frac{20}{60} = \frac{1}{3}$ ft./sec.^{2}

" " " " 2 min. $= \frac{30}{120} = \frac{1}{4}$ ft./sec.^{2}

" " " " 3 " $= \frac{23}{180} = 0{\cdot}128$ ft./sec.^{2}

Example (1).—*A train begins to ascend an incline with a speed of 40 m.p.h., and during the ascent, which takes 10 min., it has a retardation of 0·02 ft./sec.^{2} It then proceeds down an incline, and upon reaching a level stretch has acquired a speed of 50 m.p.h. in 4 min. With what acceleration does the train descend the incline?*

Since the retardation of the train up the incline is 0·02 ft./sec.^{2}, the decrease in speed in 10 min. is

$$0{\cdot}02 \times 600 = 12 \text{ ft./sec.} = 12 \times \frac{15}{22} = 8\tfrac{2}{11} \text{ m.p.h.}$$

The speed of the train at the top of the incline $= 31\tfrac{9}{11}$ m.p.h.
Down the second incline the increase of speed in 4 min. is

$$(50 - 31\tfrac{9}{11}) = 18\tfrac{2}{11} \text{ m.p.h.}$$

$$= \frac{200}{11} \times \frac{22}{15} \text{ ft./sec.}$$

The increase of speed in 240 sec. $= \frac{80}{3}$ ft./sec.

The acceleration is therefore

$$= \frac{80}{3} \div 240 = \frac{80}{3} \times \frac{1}{240} = 0{\cdot}111 \text{ ft./sec.}^2$$

Example (2).—*A body moving with uniform acceleration 6·4 ft./sec.^{2} reaches a speed of 48 ft./sec. after 4 sec. What is its initial speed and the time after starting with this speed which it takes to reach a speed of 100 ft./sec.?*

Increase in speed in 4 sec.
= Acceleration in ft./sec.^{2} × Time (sec.) = 6·4 × 4 = 25·6 ft./sec.

Initial speed + Increase in speed = Final speed.

Initial speed = 48 − 25·6 = 14·4 ft./sec.

Total increase in speed when the final speed is 100 ft./sec.
= 100 − 14·4 = 85·6 ft./sec.

Increase in speed in 1 sec. = 6·4 ft./sec.

Time for the speed to increase 85·6 ft./sec.

$$= 85{\cdot}6 \div 6{\cdot}4 = \frac{85{\cdot}6}{6{\cdot}4} = 13\tfrac{3}{8} \text{ sec.}$$

EXAMPLES XXVII

1. A body is travelling with uniform acceleration, 1·5 ft./sec.^{2} If it starts from rest, what is its speed after 20 sec. ?

2. A body moving with uniform acceleration increases its speed from 20 to 60 ft./sec. in $\frac{1}{2}$ min. Calculate the acceleration of the body.

3. A train is accelerating uniformly at 0·025 ft./sec.^{2} How long does it take for its speed to increase from 10 to 50 m.p.h. ?

4. Two trains are travelling in the same direction on parallel lines with speeds of 10 and 15 m.p.h. They are together at a certain instant. If their accelerations are 0·12 and 0·08 ft./sec.^{2} respectively, after what time are they travelling at the same speed, and what is this speed ?

5. A train travels with a uniform acceleration of 1 ft./sec.^{2} for $\frac{1}{4}$ min. and then moves with retardation of 2 ft./sec.^{2} for 22 sec., after which it is at rest. Find the initial speed of the train and the greatest speed at which it travels during the motion.

6. Express the following uniform accelerations in feet per second per second :—

(*a*) 10 m.p.h. in 20 sec.
(*b*) 300 ft./min. in 3 sec.
(*c*) 825 m.p.h. in 5 sec.

7. The speeds of a body at times 0, 5, 15 and 22 sec. are 5, 17, 41 and 52·8 ft./sec. respectively. Are these speeds and times consistent with uniform acceleration ?

8. (*a*) Express an acceleration of 9,000 m.p.h.^{2} in feet per second per second.
(*b*) Express an acceleration of 2,100 m.p.h.^{2} in centimetres per second per second.

9. Express an acceleration of 30 cm./sec.^{2} in yards per minute per minute. (1 ft. = 30·48 cm.)

98. Suppose that a body is travelling with uniform acceleration f ft./sec.^{2}, and that when first observed its speed is u ft./sec. The speed of the body after t sec. may be found as follows :—

The speed at the beginning of the first second $= u$ ft./sec.

The speed at the end of 1 sec. $= (u + f)$ ft./sec.
" " " " 2 " $= (u + 2f)$ ft./sec.
" " " " t " $= (u + tf)$ ft./sec.

If v ft./sec. is the velocity at the end of t sec.,

$$\mathbf{v = u + ft.}$$

From this expression,

$$v - u = ft,$$

$$t = \frac{(v-u)}{f},$$

and

$$f = \frac{(v-u)}{t}.$$

Example.—*A train moving with uniform acceleration increases its speed from* 10 *to* 25 *m.p.h. in* 9 *sec.*

(a) *Find the acceleration of the train in feet per second per second.*
(b) *If this acceleration is kept up for another* 6 *sec., calculate the final speed of the train.*

Let u ft./sec. be the initial speed,
and v ft./sec. the speed after t sec.
If f ft./sec.^{2} is the acceleration of the train,

(*a*)
$$f = \frac{(v-u)}{t}.$$

$$v - u = \frac{22}{15}(25 - 10) = 22.$$

$$t = 9.$$

$$f = \frac{22}{9} = 2{\cdot}44.$$

The acceleration of the train is 2·44 ft./sec.^{2}

(*b*) Again,

$$v = u + ft.$$

$$u = \frac{22}{15} \times 10.$$

$$f = \frac{22}{9}.$$

$$t = 15.$$

$$v = \frac{22}{15} \times 10 + \frac{22}{9} \times 15 = \frac{154}{3}.$$

The speed of the train is $51\frac{1}{3}$ ft./sec. $= \frac{154}{3} \times \frac{15}{22} = 35$ m.p.h.

99. The Determination of Acceleration from the Speed-Time Graph.

The acceleration of a body along its path may be obtained from the speed-time graph by much the same method which

has already been used to obtain the speed of a body from its distance-time graph.

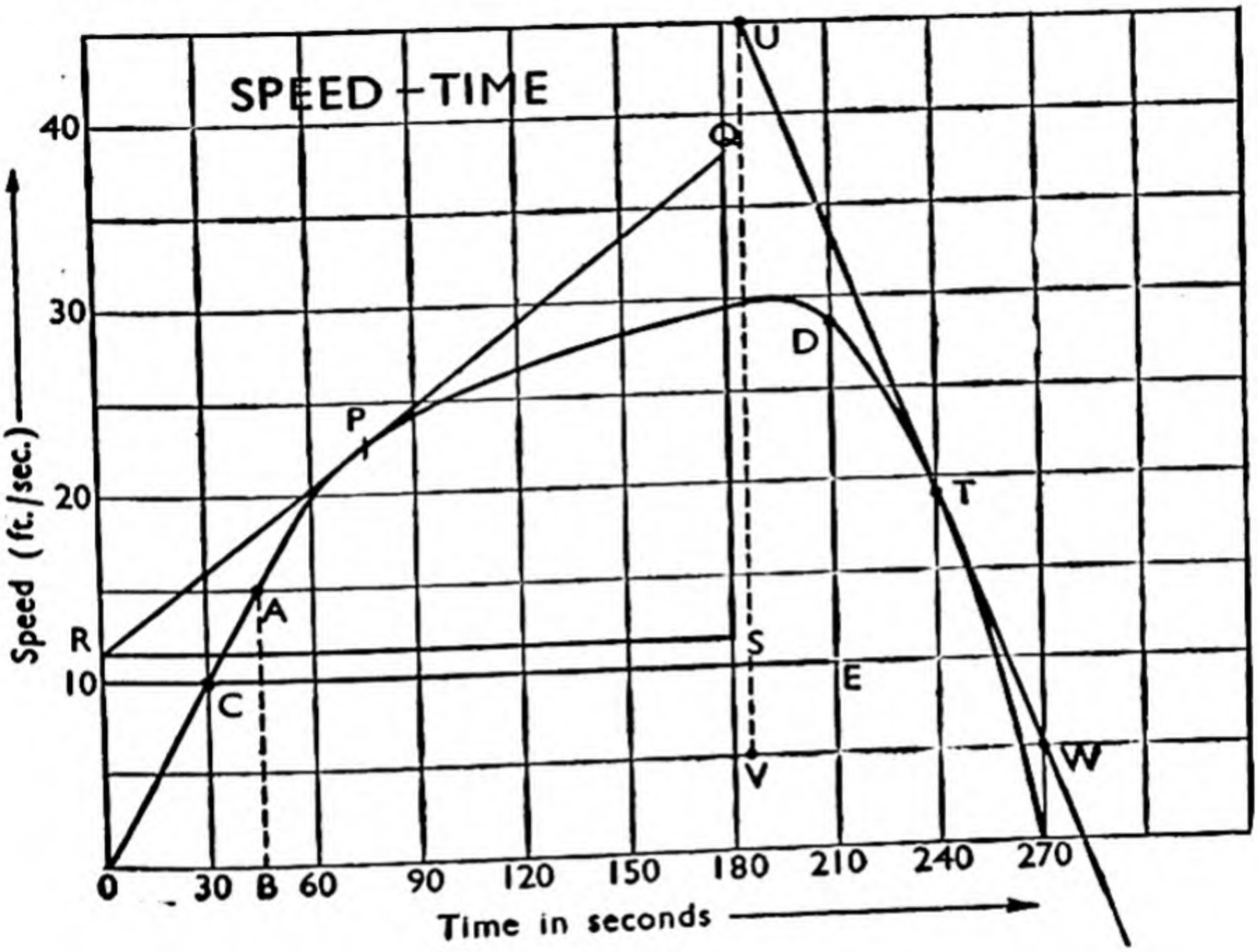

FIG. 160.

(*a*) *Uniform Acceleration.*—On the speed-time graph shown above, equal increases in speed in equal times are shown to occur during the time OB sec. by the "steps" drawn on the graph. *When the slope of the speed-time graph is constant and the graph is therefore a straight line, the corresponding motion of the body is that of uniform acceleration.*

Acceleration during the time OB sec.

$$= \frac{\text{BA (ft./sec.)}}{\text{OB (sec.)}} = \frac{15}{45} = \tfrac{1}{3} \text{ ft./sec.}^2$$

(*b*) The acceleration is greatest where the slope of the graph is greatest.

(*c*) Zero acceleration, *i.e.*, uniform speed is shown on the graph by a straight line parallel to the time axis.

(*d*) *Instantaneous Acceleration.*—The instantaneous acceleration at a certain time is found by drawing the tangent to the curve at the point on the graph corresponding to this time. The instantaneous acceleration is measured by the increase in speed in unit time, following the instant at which the acceleration is required, *upon the assumption that the acceleration is uniform during this unit time.* Hence the uniform acceleration which is found from the slope of the tangent is the required acceleration.

The instantaneous acceleration after 75 sec. is found by drawing the tangent at the point P on the graph and is equal to

$$\frac{\text{SQ (ft./sec.)}}{\text{RS (sec.)}} = \frac{26{\cdot}5}{180} = 0{\cdot}147 \text{ ft./sec.}^2$$

The instantaneous retardation after 4 min. is

$$\frac{\text{UV (ft./sec.)}}{\text{VW (sec.)}} = \frac{40}{84} = 0{\cdot}476 \text{ ft./sec.}^2$$

(*e*) *To Find the Time at which the Acceleration has a given Value.*—Suppose that it is required to find the time at which the acceleration has the value 0·147 ft./sec.^{2}

A line is drawn on the graph to represent an acceleration (uniform) of 0·147 ft./sec.^{2}, and parallel to this line a tangent is drawn to the curve. The point of contact of this tangent with the curve (in the example given this would be the point P) gives the instant at which the acceleration has the required value.

(*f*) *To Find the Average Acceleration during any Interval of Time.*—Since the average acceleration during any interval of the time is given by

$$\frac{\text{Total change in speed}}{\text{Time taken}},$$

the average speed in the interval from 15 sec. to 3 min. 15 sec. is

$$\frac{\text{ED (ft./sec.)}}{\text{CE (sec.)}} = \frac{18{\cdot}5}{180} = 0{\cdot}103 \text{ ft./sec.}^2$$

100. The Acceleration-Time Graph.

If the accelerations of a body are known at times during its motion, the acceleration-time graph for the body may be drawn.

Since it has now been shown that the instantaneous acceleration of a body may be found from the speed-time graph, it is readily seen that the acceleration-time graph may be drawn if the speed-time graph is given.

In particular, if the acceleration of a body is uniform, the acceleration-time graph is a straight line parallel to the time axis.

It is left as an exercise for the reader to show that the area beneath the acceleration-time graph is equal to the increase in speed of the body in the case of uniform acceleration, and when the acceleration is variable, this result applied to area beneath a curve enables us to show that however the acceleration changes *the area beneath the acceleration-time graph is equal to the change in speed.* (See p. 163 where it is shown that the area beneath the speed-time graph is equal to the distance travelled.)

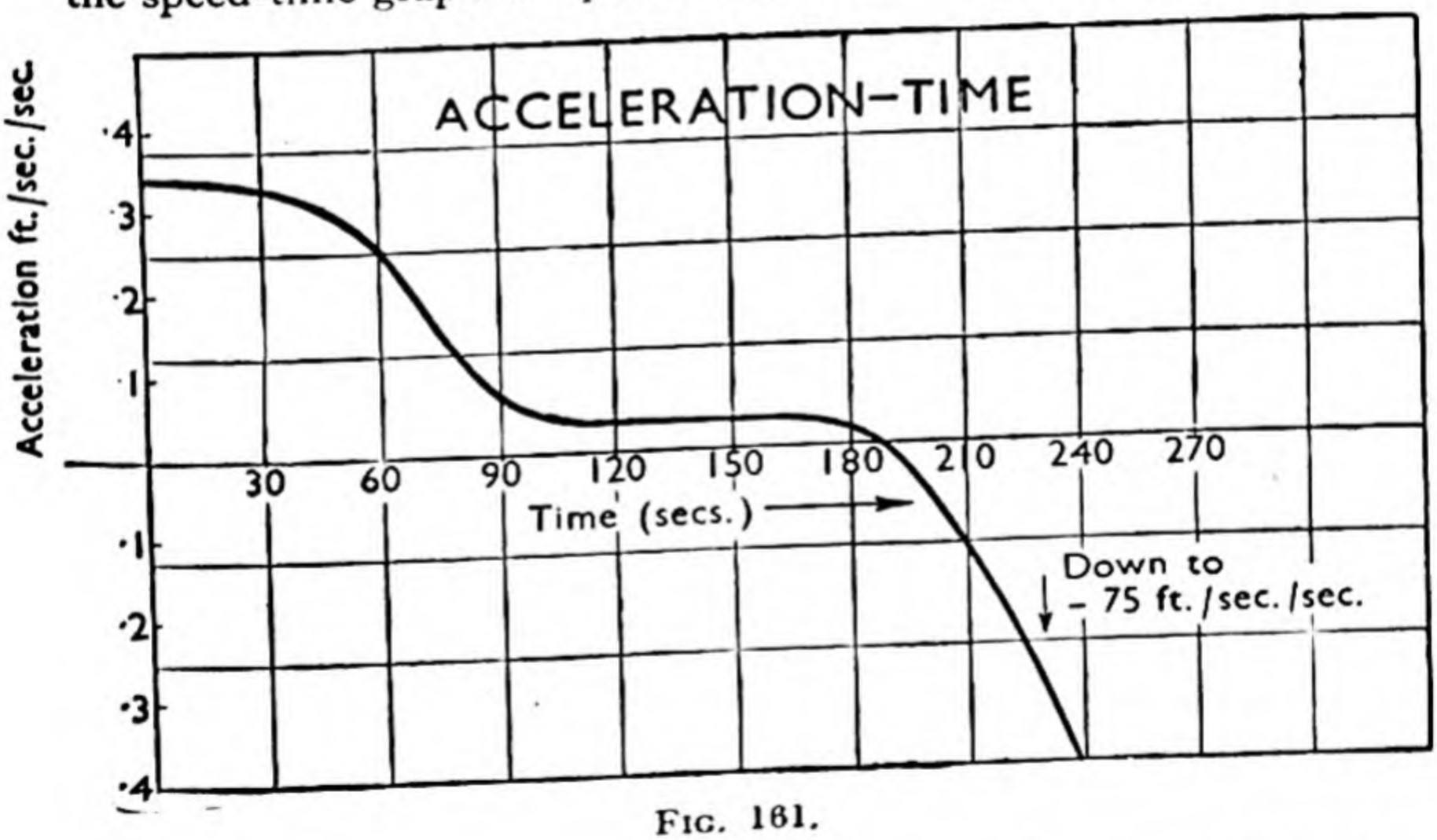

FIG. 181.

The accompanying acceleration-time graph has been obtained from the speed-time graph on p. 181, and it should be examined

together with the speed-time graph. Notice that a *retardation is represented on the acceleration-time graph below the time axis.* This is readily understood from the fact that a retardation is decrease in speed in unit time, whereas acceleration is increase in speed in unit time, and to represent these on the same graph a retardation must be regarded as a *negative acceleration.*

Summary of the Properties of the Graphs

	Slope or Gradient.	Area Represents
Distance-time graph .	Speed	...
Speed-time graph .	Acceleration	Distance travelled.
Acceleration-time graph	...	Change in speed.

Example.—*A body starts from rest and accelerates uniformly for* 200 *ft. It then travels at the speed acquired,* 15 *m.p.h. for a further* 600 *ft., and is finally brought to rest with uniform retardation in* 150 *ft. Find the time taken and draw the speed-time graph for the complete journey.*

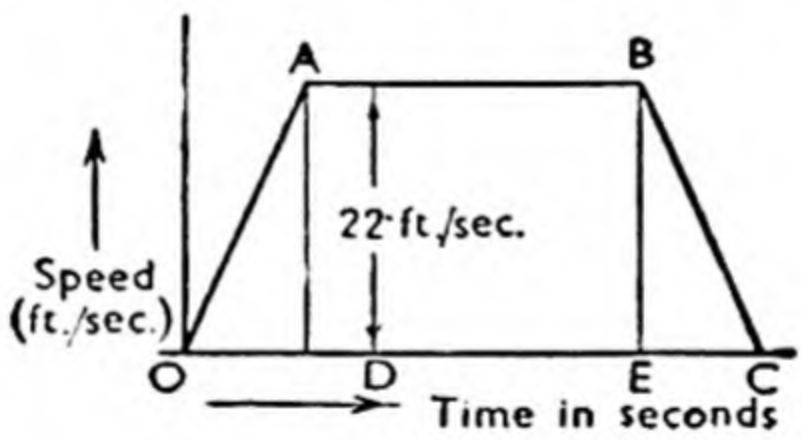

FIG. 162.

The speed-time graph consists of the three straight lines OA, AB and BC.

OA represents the part of the journey when the body is accelerating uniformly.

AB represents the part of the journey when the body is travelling with uniform speed.

BC represents the part of the journey when the train is retarding uniformly.

Referring to the speed-time diagram which is at first drawn, to show the type of motion, and is not intended to represent accurately the motion of the body.

The area of the triangle OAD represents the distance (200 ft.) travelled while the body is accelerating.

$$AD \equiv 15 \text{ m.p.h., or } 22 \text{ ft./sec.}$$

If OD represents t_1 sec.

$$\frac{t_1 \times 22}{2} = 200.$$

Therefore
$$t_1 = \frac{200}{11} \text{ sec.}$$

Again, ABED represents the distance travelled while the body is moving with uniform speed.

Let DE represent t_2 sec., then

$$t_2 \times 22 = 600$$

$$t_2 = \frac{300}{11} \text{ sec.}$$

Finally, if t_3 sec. is the time during which the body travels with uniform retardation

$$\frac{t_3 \times 22}{2} = 150.$$

$$t_3 = \frac{150}{11} \text{ sec.}$$

The total time is therefore

$$\left(\frac{200}{11} + \frac{300}{11} + \frac{150}{11}\right) = 59\frac{1}{11} \text{ sec.}$$

The speed-time graph may now be drawn accurately to scale.

EXAMPLES XXVIII

1. A train starting from rest is timed over successive distances of 220 yds., with the following result :—

Distance.	Time.	Distance.	Time.
Yds.	Sec.	Yds.	Sec.
220	45·0	880	90·0
440	63·6	1,100	100·6
660	78·0	1,320	110·2

Plot a space-time graph and a velocity-time graph. Is the acceleration uniform ? What is its average value ? At what point was the velocity 30 m.p.h. ?

2. Simultaneous values of speed and time for a train are given in the table below. Draw the speed-time curve and find the acceleration in feet per second per second at the end of the second minute and the distance in miles passed over in attaining a speed of 50 m.p.h.

Time.	Speed.	Time.	Speed.
Sec.	M.p.h.	Sec.	M.p.h.
0	0·0	200	48·3
50	24·6	250	51·3
100	36·3	300	53·5
150	43·7		

3. A body starting from rest moves in a straight line, and the following observations are taken :—

Time.	Distance.	Time.	Distance.
Sec.	Cm.	Sec.	Cm.
0·0	0·00	1·0	13·2
0·2	0·55	1·2	18·5
0·4	2·10	1·4	23·7
0·6	4·80	1·6	29·0
0·8	8·50		

Plot a graph showing distances attained on a time base. Determine as nearly as you can the velocities of the body in centimetres per second at the middle of each interval of time. What conclusions do you draw as to the nature of the motion ? (*O.C.*)

4. Derive an expression for the distance moved through in t sec. by a body possessing uniform acceleration f ft./sec.^{2} and an initial velocity of u ft./sec.

A train starts from rest and moves with a uniform acceleration of 6 in./sec.^{2} for 1½ min. It then travels at a uniform speed for another 4 min., and is brought to rest again with a uniform retardation in 450 yds. Draw a velocity-time graph of the train and use it to find the total distance the train travels. Verify the result you obtain by calculation. (*L.M.*)

5. If a curve is drawn showing the velocity of a body at each instant, explain how the distance traversed in any interval may be obtained from it. An electric train accelerates uniformly from rest and attains its maximum speed of 35 m.p.h. in 30 sec. It travels at this speed until the brakes are applied, when it is brought to rest in 25 sec. If the time allowed between two stations is 85 sec., what is their distance apart ? (*G.S.*)

6. Explain what information can be obtained from a graph in which speed is plotted against time.

A train starting from rest accelerates uniformly until it has traversed 1¼ miles ; its speed remains constant for the next 2 miles, and the brakes then produce a uniform retardation, bringing it to rest after a farther ¾ mile. If the whole journey takes 7½ min., find the maximum speed in miles per hour. (*G.S.*)

7. A body starts from rest and is uniformly accelerated for 250 ft. It then travels at the speed acquired, 7·5 m.p.h., for a further 500 ft., and is finally brought to rest with uniform retardation in 200 ft. Find

the time taken, and draw the velocity-time graph for the complete journey. (*L.M.*)

8. A train starts from rest at a station A, with an acceleration of 1 ft./sec.^{2}; it acquires a maximum speed of 45 m.p.h., and when approaching a second station B, it retards at a constant retardation of 9 in./sec.^{2} and stops at B. If A and B are 10 miles apart, calculate the time for the journey. (*L.M.*)

9. A particle starts from rest and travels in a straight line. The following table shows the distances s travelled in times t :—

t (sec.)	0	1	2	3	4	5	6
s (in.)	0	2	8	18	32	48	64

(*a*) Draw a graph of the motion, plotting distances vertically and times horizontally.

(*b*) Briefly describe the motion.

(*c*) Find from the graph, approximately, the velocity at the end of $1\frac{1}{2}$ sec. (*J.*)

10. The speedometer readings of a motor car, taken at $\frac{1}{2}$-min. intervals, are recorded in the following table :—

Time.	Speed.	Time.	Speed.
Min.	M.p.h.	Min.	M.p.h.
0	5	4	25
$\frac{1}{2}$	5	$4\frac{1}{2}$	$27\frac{1}{2}$
1	5	5	30
$1\frac{1}{2}$	10	$5\frac{1}{2}$	30
2	15	6	30
$2\frac{1}{2}$	$17\frac{1}{2}$	$6\frac{1}{2}$	15
3	20	7	0
$3\frac{1}{2}$	$22\frac{1}{2}$		

Plot the velocity-time graph for the car.

From the graph determine (*a*) the acceleration at time 4 min., (*b*) the total distance travelled in 7 min. Explain clearly in each case how the value is got from the graph. (*O.C.*)

11. A body starting from rest is found to have travelled in a straight line, according to the following table :—

Time.	Distance from Start.	Time.	Distance from Start.
Sec.	Cm.	Sec.	Cm.
0·0	0·0	1·0	14·7
0·2	0·7	1·2	18·2
0·4	2·8	1·4	20·3
0·6	6·3	1·6	21·0
0·8	10·5		

Plot a graph showing time horizontally and distance vertically, and determine from the graph the maximum velocity and the velocity after 1·3 sec. What do you infer from the graph as to the nature of the acceleration ?

12. Explain the terms *velocity* and *acceleration*. A ball rolls down an inclined plane, and the total distance travelled at the end of each second is given in the following table :—

t (sec.) . . .	1	3	5	7	9
s (ft.) . . .	0·2	1·8	5·0	9·8	16·2

From the information given, construct a graph which shows that the ball rolls with uniform acceleration, and find the value of the acceleration. (*L.M.*)

101. Uniform Acceleration.

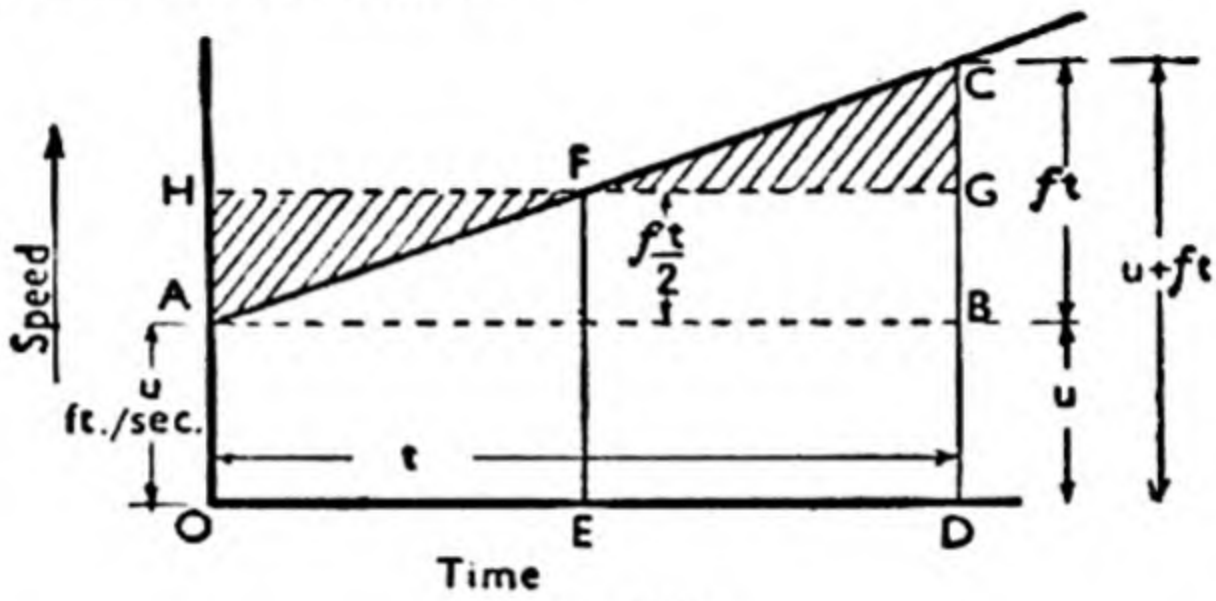

FIG. 163.

The speed-time graph for a body which travels with uniform acceleration is a straight line of constant slope. The speed-time graph shown in the above figure is that of a body which has a speed of u ft./sec. at the instant it comes under observation (*i.e.*, at zero time), and whose speed at the end of t sec. is v ft./sec.

Suppose that f ft./sec.^{2} is the acceleration of the body, then, since this is uniform,

$$\text{Acceleration} = \frac{\text{Total change in speed}}{\text{Time taken}}.$$

Referring to the graph,

$$\text{Acceleration} = \frac{\text{BC (ft./sec.)}}{\text{AB (sec.)}} \text{ ft./sec.}^2$$

Therefore
$$f = \frac{(v-u)}{t}$$
$$ft = v - u$$
$$\mathbf{v = u + ft.}$$

From the graph OA represents u ft./sec.

If AB is drawn parallel to OD to meet CD at B, then BD = OA.

Since DC = DB + BC and DC represents v ft./sec., it follows that BC represents ft ft./sec.

OD is bisected at E, and EF is drawn parallel to OA to meet AC at F.

It has been shown that the area beneath the speed-time graph represents the distance travelled in the corresponding time interval. The distance travelled in t sec. is therefore represented by the area of the trapezium ODCA expressed in the proper units.

$$\text{Area of trapezium} = \tfrac{1}{2}(\text{OA} + \text{DC}) \,.\, \text{OD}.$$

Now, OA represents u ft./sec.

DC ,, v ft./sec.

OD ,, t sec.

The distance travelled $= \tfrac{1}{2}(u+v)t$.

If the distance travelled is represented by s, as is usual, the last result may be written

$$s = \tfrac{1}{2}(u+v)t.$$

If through F a straight line HFG is drawn parallel to OD to meet the speed axis at H and CD at G, it is readily seen that the triangles AHF and CGF are equal in all respects, and in particular CG = HA. Thus EF $= \tfrac{1}{2}$(OA + CD), and

rectangle ODGH = trapezium ODCA.

Now, Distance travelled = Average velocity × Time.

Therefore EF represents the average velocity during the time t sec. *Hence if a body is travelling with uniform acceleration the average speed during any time interval is equal to the speed of the body at the mean time, and this is also equal to the average of the original and final speeds.*

Again, $$v = u + ft,$$

therefore $$\tfrac{1}{2}(u+v) = \tfrac{1}{2}(u+u+ft) = \tfrac{1}{2}(2u+ft) = u + \tfrac{1}{2}ft.$$

Thus, since $$s = \tfrac{1}{2}(u+v)t,$$

$$s = (u + \tfrac{1}{2}ft)t = ut + \tfrac{1}{2}ft^2,$$

$$\mathbf{s = ut + \tfrac{1}{2}\,ft^2}.$$

It should be noticed that $\frac{1}{2}ft^2$ represents the extra distance travelled by the body due to the fact that it is accelerating, ut representing the distance that the body would travel in time t with uniform speed u.

Since $$s=\tfrac{1}{2}(u+v)t$$

$$2s=(u+v)t,$$

and $$2fs=(u+v)ft \quad . \quad . \quad . \quad . \quad (1)$$

But $$ft=(v-u).$$

Hence putting $(v-u)$ for ft in equation (1),

$$2fs=(u+v)(v-u)=v^2-u^2$$

$$v^2-u^2=2fs, \text{ or } \mathbf{v^2=u^2+2fs.}$$

If the initial speed is zero, *i.e.*, $u=0$, the above formulæ become,

$$v=ft$$

$$s=\tfrac{1}{2}vt$$

$$s=\tfrac{1}{2}ft^2$$

$$v^2=2fs.$$

Formulæ for uniformly retarded motion.

When a body is retarding, *i.e.*, is losing speed at the rate of f ft./sec.^{2}, it is easy to see that the velocity after time t with the same notation as above is given by

$$v=u-ft,$$

the distance travelled by

$$s=ut-\tfrac{1}{2}ft^2,$$

and that $$v^2=u^2-2fs.$$

Thus these three formulæ differ from the formulæ for uniform acceleration in that f is replaced by $-f$.

Example (1).—*A train moves from rest with uniform acceleration 1·5 ft./sec.^{2} for 30 sec., after which it moves with uniform speed for 2 min., and finally is reduced to rest in 20 sec. What is the total distance travelled during the motion and the average speed for the whole journey in miles per hour (assuming that the retardation is uniform while the train is being brought to rest)?*

The motion of the train may be divided into three parts :—

The period of uniform acceleration, 30 sec.
The period of uniform speed, 2 min.
The period of uniform retardation.

During the first 30 sec. the train travels with uniform acceleration, 1·5 ft./sec.², therefore, since

$$\text{Increase in speed} = \text{Acceleration} \times \text{Time},$$

$$\text{Speed at the end of 30 sec.} = 1{\cdot}5 \times 30 = 45 \text{ ft./sec.}$$

$$\text{Distance travelled} = \text{Average speed} \times \text{Time} = \frac{(0+45)}{2} \times 30 = 675 \text{ ft.}$$

During the next 2 min. the speed is uniform, and

$$\text{Distance travelled} = \text{Uniform speed} \times \text{Time} = 45 \times 120 = 5{,}400 \text{ ft.}$$

During the last 20 sec. the speed is reduced from 45 ft./sec. to zero, and since the retardation is uniform,

$$\text{Distance travelled} = \frac{45+0}{2} \times 20 = 450 \text{ ft.}$$

Hence the total distance travelled

$$= (675 + 5400 + 450) = 6525 \text{ ft.} = 1 \text{ mile } 415 \text{ yds.}$$

Average speed for the journey

$$= \frac{\text{Total distance travelled}}{\text{Time taken}} = \frac{6525}{170} \text{ ft./sec.} = \frac{6525}{170} \times \frac{15}{22} = 26{\cdot}2 \text{ m.p.h.}$$

Example (2).—*A tram car accelerates uniformly for 25 sec. During the next 20 sec. it moves with uniform retardation through a distance of 220 yds., and its final speed is 15 m.p.h. Find the total distance through which the tram car has moved.*

Consider the motion during the last 20 sec.

Let v ft./sec. be the initial speed.

$$\text{Final speed} = 15 \times \frac{22}{15} = 22 \text{ ft./sec.}$$

$$\text{Average speed} = \tfrac{1}{2}(v + 22) \text{ ft./sec.}$$

$$\text{Distance travelled} = \text{Average speed} \times \text{Time.}$$

$$660 = \tfrac{1}{2}(v + 22) \times 20.$$

$$66 = v + 22.$$

$$v = 44.$$

Consider the motion during the first 25 sec.

The initial speed is zero.

The final speed is 44 ft./sec.

$$\text{Distance travelled} = \tfrac{1}{2}(0 + 44) \times 25 = 550 \text{ ft.}$$

$$\text{The total distance travelled} = (660 + 550) = 1210 \text{ ft.} = 403 \text{ yds. } 1 \text{ ft.}$$

Example (3).—*A body has an initial speed of 20 ft./sec. and moves with uniform acceleration 3 ft./sec.² Find the distance which it travels*

during the 5th sec. of its motion. Find also the distance travelled by the body in the t-th sec. of its motion.

Let s ft. be the distance travelled in t sec.,

u ft./sec. be the initial speed,

f ft./sec.^{2} be the acceleration.

Using the formula $s = ut + \frac{1}{2}ft^2$,

$$s = 20t + \frac{3}{2}t^2.$$

In the first 5 sec. the distance travelled $= 20 \times 5 + \frac{3}{2} \times 5^2 = 137\frac{1}{2}$ ft.

In the first 4 sec. the distance travelled $= 20 \times 4 + \frac{3}{2} \times 4^2 = 104$ ft.

The distance travelled in the 5th sec. is $(137\frac{1}{2} - 104) = 33{\cdot}5$ ft.

The distance travelled in t sec. $= 20t + \frac{3}{2}t^2$ ft.

The distance travelled in $(t-1)$ sec.

$$= 20(t-1) + \frac{3}{2}(t-1)^2$$

$$= 20t - 20 + \frac{3}{2}t^2 - 3t + \frac{3}{2}$$

$$= \frac{3}{2}t^2 + 17t - 18\frac{1}{2}.$$

Therefore, $20t + \frac{3}{2}t^2 - \left(\frac{3}{2}t^2 + 17t - 18\frac{1}{2}\right) = 3t + 18\frac{1}{2}$ ft.

is the distance travelled in the t-th sec.

If $t = 5$ we see that

$$3t + 18{\cdot}5 = 3 \times 5 + 18{\cdot}5 = 33{\cdot}5,$$

and this checks the previous result.

EXAMPLES XXIX

1. A train starts from rest at a station, and for 30 sec. it travels with uniform acceleration 1·2 ft./sec.^{2} What is the speed of the train at the end of 30 sec., and how far has it travelled ?

2. A body moves from rest with uniform acceleration, and in 10 sec. acquires a speed of 45 ft./sec. What is its acceleration, and how far has the body travelled in the 10 sec. ?

3. Two racing cars start from rest side by side and both accelerate uniformly. The acceleration of one car is twice that of the other, and the cars are 484 yds. apart in 44 sec. Find their accelerations and their speeds at the end of the 44 sec.

4. A motor car travelling at 30 m.p.h. is brought to rest with uniform

retardation in $5\frac{1}{2}$ sec. What is its retardation, and what distance has it travelled in this time?

5. A train travelling at 40 m.p.h. is to be brought to rest in a distance of 200 yds. With what uniform retardation should it travel and how long will it take to travel the 200 yds.?

6. A train starts from rest, accelerates uniformly for 10 sec. at 3 ft./sec.^{2} and then continues with uniform speed for 4 min. It is finally brought to rest with retardation 2 ft./sec.^{2} Find the total distance travelled from rest to rest.

7. A car after travelling round a bend in a road starts on a straight course with a speed of 15 m.p.h. It accelerates uniformly for 15 sec. until its speed is 48 m.p.h. How far has it travelled in the 15 sec., and what is its acceleration?

8. A cyclist descends a slope with uniform acceleration $\frac{1}{3}$ ft./sec.^{2} in 45 sec. At the bottom of the slope he free-wheels 200 yds. along a level road with uniform retardation for 1 min. before he is brought to rest. Find his speed at the bottom of the hill, at the top of the hill and the total distance travelled.

9. A tram car accelerates uniformly for 30 sec. from rest. During the next 25 sec. it moves with uniform retardation through a distance of 300 yds., and its final speed is 10 m.p.h. Find the total distance the car has travelled.

10. Two cars A and B travelling in opposite directions pass each other with speeds of 10 and 15 m.p.h. respectively. They are both moving with uniform accelerations after passing each other for 20 sec., during which time A has travelled 200 yds. and B has acquired a speed of 30 m.p.h. How far are the cars apart at the end of 20 sec., and at what speed are they separating?

11. Draw the speed-time curve over an interval of 10 sec. for a body which has an initial speed of 15 cm./sec. and an acceleration of 2 cm./sec.^{2} From the graph find the speed of the body after 5·5 sec. and the time at which the speed is 20 cm./sec. Find from the graph the distance travelled on the time interval 4·5 to 8·5 sec.

12. By using the formula $s = ut + \frac{1}{2}ft^2$, find how far a body travels in the 6th sec. of its motion if its initial speed is 10 ft./sec. and its acceleration is 1·3 ft./sec.^{2}

13. Prove that the distance travelled by a body where initial speed is u cm./sec. and acceleration f cm./sec.^{2} in the t-th sec. of its motion is $u + f(t - \frac{1}{2})$ (1) by using the formula $s = ut + \frac{1}{2}ft^2$, (2) by using the formulæ

$$v = u + ft \text{ and } s = \frac{(u + v)}{2} \cdot t.$$

14. A body travelling with uniform acceleration 4 ft./sec.^{2} travels 38 ft. in the 5th sec. of its motion. Show that the initial speed of the body is 20 ft./sec.

15. Use the algebraic formulæ on pp. 188-90 to solve the following problems:—

(*a*) Given $u = 3$, $f = 4$, $t = 5$, find v and s.
(*b*) Given $u = 12$, $v = 3$, $s = 12$, find f and t.

The units are centimetres and seconds. In each case state the problem which you are solving in words.

16. Two cars A and B are racing abreast round a track at 70 m.p.h. when B decides to stop and change a wheel. B comes to rest with a uniform retardation of $3\frac{2}{3}$ ft./sec.^{2}, takes 37 sec. to change the wheel and then accelerates at $4\frac{2}{5}$ ft./sec.^{2} to attain a uniform speed of 75 m.p.h. What time elapses before B is again abreast of A ? (*L.M.*)

17. Two trains start from rest at a station A at 12 noon and proceed in opposite directions to stations B and C respectively, which are 13·2 miles apart.

The first train just acquires a speed of 60 m.p.h. as it passes station B, having accelerated uniformly at 2 in./sec.^{2}

The second train accelerates at half the rate of the first. What is the speed of the second train as it passes station C ?

At what time do the trains pass through their respective stations ? (*L.M.*)

18. What is meant by the statement that " a body is moving with a uniform acceleration of 5 m.p.h. per sec." ? What is this in feet per second per second ?

A body has an initial velocity of u ft./sec. and is subjected to an acceleration of f ft./sec.^{2} Find an expression for the displacement after t sec.

A body has an initial velocity of 15 m.p.h., and is uniformly accelerated at the rate of 1 yd./sec.^{2} for $\frac{1}{4}$ mile. Find, in feet per second, the velocity it acquires. (*L.M.*)

19. A motor car travelling along a main road at 40 m.p.h. approaches a village which is $\frac{1}{4}$ mile long. If the driver retards in order to pass through the village at 10 m.p.h., and then accelerates to 40 m.p.h., how much extra time does his act of courtesy put on the journey if the acceleration and retardation is 4 ft./sec.^{2} ? (*L.M.*)

20. Two particles moving in the same straight line pass a point O at the same time ; one is uniformly accelerated with an acceleration of 5 ft./sec.^{2}, and passes over a certain distance in 12 sec. The second particle is travelling with a uniform velocity of 24 ft./sec. and travels over the same distance in 19 sec. Determine the velocity of the first particle in passing the point O. Also determine when and where the first particle will overtake the second. (*L.M.*)

21. A car is timed to take 15 sec. over 220 yds., and 10 sec. over the next 220 yds. Assuming constant acceleration, what is the speed of the car at the end of the observed motion ? (*O.C.*)

22. The driver of an express train travelling at 60 m.p.h. sees on the same track 200 yds. in front of him a slow train travelling in the same direction at 20 m.p.h. If the slow train cannot increase its speed, what is the least retardation that must be applied to the express train so as to avoid a collision ? (*O.C.*)

23. The brakes on a train reduce its speed from 60 to 20 m.p.h. while it runs 220 yds. Assuming that they exert a constant retarding force, find how much farther the train will run before coming to rest and how long it will take. Justify any formulæ you employ. (*G.S.*)

24. What is meant by " speed," " velocity," " acceleration " ? If the speed of a lift is not allowed to exceed 10 ft./sec. and it is accelerated and retarded at 4 ft./sec.^{2}, find the least time required to ascend 60 ft. from rest to rest. (*G.S.*)

25. A train capable of a maximum speed of 40 m.p.h. accelerates and

retards at a uniform rate of 1·5 ft./sec.², what is the least time required to travel between two stations 1,000 yds. apart ? (*G.S.*)

26. A car travelling at 40 m.p.h. has its speed reduced in 2·5 sec. to 10 m.p.h. Assuming the retardation to be uniform, what distance was covered in the time ? (*C.W.B.*)

27. Obtain the expression for the distance travelled by a uniformly accelerated body in terms of the initial and final velocities and its acceleration.

A train moves from rest with a uniform acceleration of 1·5 ft./sec.² After what time and distance will its speed be 30 m.p.h. ? (*C.W.B.*)

CHAPTER X

FALLING BODIES

Aristotle (384-322 B.C.) is described as the most famous of the Greek thinkers. He was a disciple of Plato and was tutor to Alexander, who afterwards became Alexander the Great. He compiled a work in which he recorded the knowledge of his times, and his greatness may be judged from the respect with which scholars have regarded this work. We are here concerned only with Aristotle's ideas related to mechanics. We are able to understand the meaning of his statement that " heavy bodies tend to their proper place in the universe more strongly than light bodies," but are surprised to find that he believed that heavy bodies fall to the ground more quickly than light bodies ; that, in fact, a 10-lb. weight falls ten times more quickly to the earth than a 1-lb. weight, the distance fallen through being the same in each case. It is still more surprising to find that this belief was not proved false until 2,000 years later when *Galileo Galilei* (A.D. 1564-1642), in about A.D. 1590, showed in his famous experiment at Pisa that heavy and light bodies fall to the earth with equal rapidity (allowing for the error caused by the different effects of the resistance of the air upon the bodies). In this experiment Galileo allowed a 100-lb. shot and a 1-lb. shot to fall simultaneously from a height of 100 ft. It was observed that the bodies reached the ground practically together.* The building used for the experiment was the Leaning Tower of Pisa.

Galileo argued, moreover, that if Aristotle's views were correct a heavy body and a light body tied together would fall to the ground less rapidly than the heavy body, since the light body would act as a drag upon its motion, but Aristotle's belief required that the two bodies together being heavier than either body separately should fall to the ground more quickly than either.

* The 100-lb. shot reached the ground 2 in. ahead of the 1-lb. shot.

Thus Aristotle's view led to an obvious contradiction. We are familiar to-day with the idea that scientific progress is largely dependent upon faithfully carried out experiment, and it is important to notice that great progress in science took place in and after Galileo's time chiefly because of the fact that this idea was put into practice by such men as Galileo, Kepler, Stevinus and Tycho Brahé. These men paved the way for the greatest genius of all, *Sir Isaac Newton* (A.D. 1642-1727).

In the present connection we may refer to Newton's *Guinea and Feather* experiment. We all know that if a coin and a feather are allowed to fall to the ground from the same height the feather takes considerably the longer time to reach the ground. In order to show that this difference in the behaviour of falling bodies could be entirely accounted for by the effect of the resistance of the air, Newton enclosed the feather and the guinea in a vertical cylindrical glass tube from which he exhausted as much air as possible. The two objects were then allowed to fall from the same height to the bottom of the tube, and it was observed that they reached the bottom practically together. We may therefore conclude that *in vacuo* bodies fall to the earth through the same height in equal times.

EXPERIMENT.—Allow the following pairs of objects to fall simultaneously from the same height to the ground :—

(*a*) A tennis ball and a cricket ball.
(*b*) A small piece of paper and a thimble.
(*c*) Place the small piece of paper inside the thimble and drop the thimble open end upwards.

Write down the results of these experiments and give your reasons for the differences in behaviour of the objects in the three cases.

102. Having shown to his own satisfaction * that Aristotle's ideas concerning the motion of falling bodies were not entirely correct, Galileo proceeded to attempt to describe this motion more accurately. He first *assumed* that the speed acquired by a

* There were many, some even among those who had seen the experiment at Pisa, who were not convinced that Galileo was correct. Galileo is said to have had an unfortunate habit of ridiculing his opponents, and on this occasion he wrote of those who still supported Aristotle's view that they sought to find an error of 99 ft. in the 2 in. In 1591 he was forced to resign his position as Professor of Mathematics at Pisa, having given considerable cause for offence by his behaviour.

falling body was proportional to the distance through which it had fallen from rest. He finally rejected this assumption, a fact which will be readily understood when we consider how he showed that his next assumption could actually be verified by experiment. This latter *assumption* was that the speed of a body which falls freely from rest is proportional to the time ; in fact, that a body which falls during 2 sec. acquires twice the speed which it acquires during 1 sec. This is the motion which we have now described as uniform acceleration. We are therefore in a position to compare Galileo's argument with those which have been used in describing the same kind of motion.

1. *Let it be assumed that the speed of a body A which falls from rest is proportional to the time ; then the time taken by A to fall through a given distance is equal to the time taken by a body B, which moves with uniform speed equal to half the final speed of A, to travel the same distance.*

Suppose that A is travelling with half its final speed at a certain instant. We consider the motions of A and B before and after this instant. At a given interval of time before this instant the speed of A is as much less than that of B as it is greater than that of B at the same interval of time after the given instant. Thus Galileo argued the way in which B is gaining distance over A in the first instance is exactly balanced out by the way in which it is losing distance in the second instance. This is true during the whole of the motion and therefore the above proposition is proved.

2. *The distance travelled by the body A is proportional to the square of the time which it takes to fall freely through this distance.*

If the speed of A increases g ft./sec. in each second and t sec. is the total time, then by (1) the total distance travelled by A is equal to that travelled by B, which has a uniform speed of $gt/2$ ft./sec.

Therefore the distance travelled is

$$\frac{gt}{2} \times t = \tfrac{1}{2}gt^2 \text{ ft.}$$

Since g is constant the distance is proportional to the square of the time.

103. Galileo's Experiment.

The time taken by a body to fall to the earth is very short for distances which can be conveniently used in a laboratory experiment, and Galileo had great difficulty in measuring such times accurately. In order to make his experiments cover longer intervals of time he assumed that if a marble is allowed to roll down a straight groove cut in a piece of wood which is inclined to the horizontal, the effect of the attraction of the earth would be lessened, but that the laws connecting distance, speed and time which he was seeking to test would remain unaffected. His method of measuring times was to allow water to flow from a small hole in a vessel of very large cross-sectional area. The rate of flow of the water was kept almost constant, for the amounts of outflow were very small in comparison with the contents of the reservoir, the surface of which remained practically unchanged. The times of outflow were thus proportional to the outflow. In fact the weight of water collected in a certain time could be taken as a measure of that time.

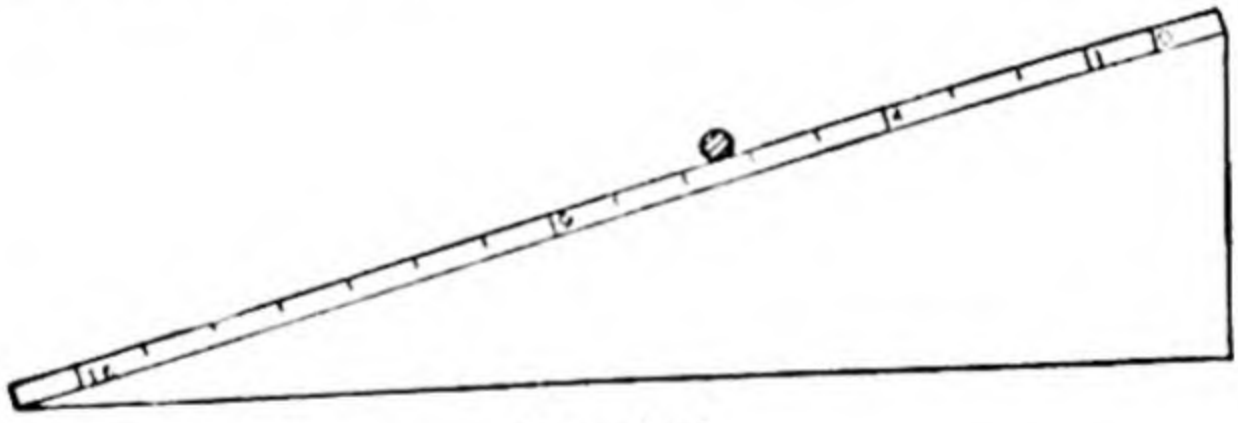

FIG. 164.

It has been shown that Galileo proved that if the speed of a body is proportional to the time, then the distance which it travels from rest is proportional to the square of the time. Now, he argued, if the distance travelled is proportional to the square of the time, then the speed will be proportional to the time, and will therefore increase equal amounts in equal times. He consequently marked the positions of points along the groove from the top point at distances in the proportion of $1 : 4 : 9$, *i.e.*, $1^2 : 2^2 : 3^2$, and observed the times taken by the marble to roll from the top point to these positions. He achieved this by

releasing the marble and taking his finger from the hole in the reservoir simultaneously, and replaced his finger as the marble rolled past the mark on the groove. The weights of water collected in this way and the distances travelled are tabulated below :—

Distances Travelled. (Proportion.)	Weights. (Proportion.)	Squares of Times. (Proportion.)
1	1	1
4	2	4
9	3	9
16	4	16

In this way Galileo tested his assumption, and showed that his experimental results supported his *working hypothesis*.

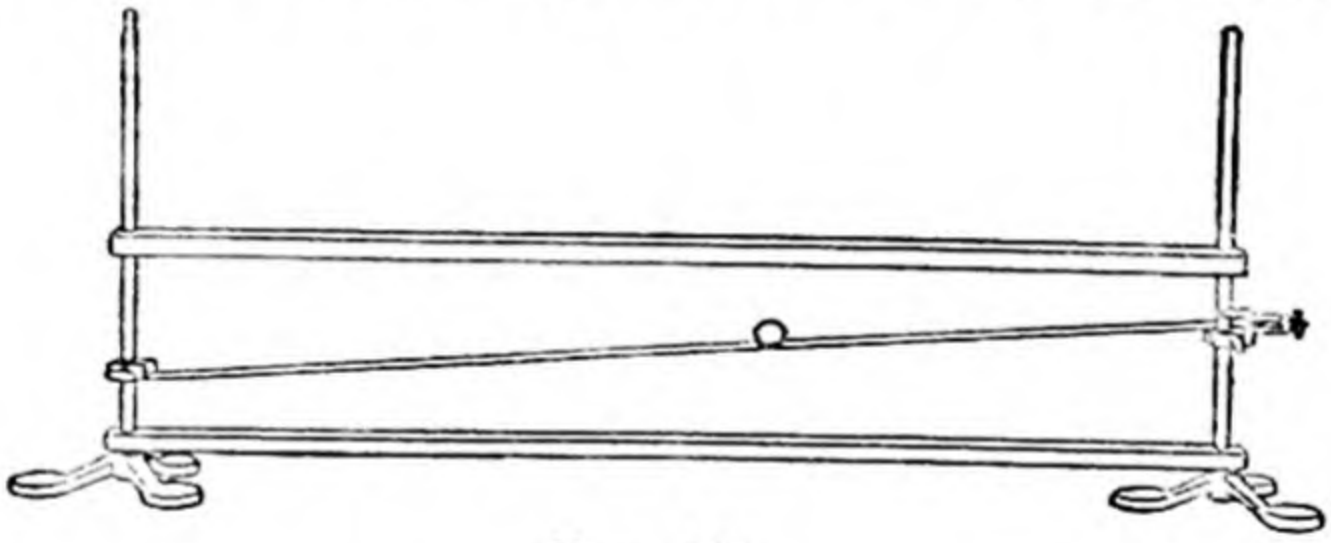

FIG. 165.

104. EXPERIMENT.—Repeat Galileo's experiment, using a stop-watch which measures times to one-fifth of a second. The inclined plane, with a groove in it, may be replaced by a glass tube secured to an inclined plane. Again, two stiff wires stretched parallel to each other may be used in place of the inclined plane and the groove, provided that a fairly light ball is used.

If the distances travelled by the ball in 1, 2, 3 . . . sec. are observed, then the ratios of these distances to the squares of the corresponding times should be equal if the motion of the ball is one of uniform acceleration.

The values of s, t and s/t^2 are given in the table below for an experiment similar to the above. From the table the values of s/t^2 agree satisfactorily. We may also show that s varies as the square of t by drawing a graph in which s is plotted against t^2.

The points are found to lie on a straight line, and we deduce that s varies directly at t^2 and that the motion of the ball is one of uniform acceleration.

s (cm.)	20	40	60	80	100	120
t	2·8	4·0	4·8	5·6	6·2	6·8
$\frac{s}{t^2}$	2·55	2·50	2·60	2·55	2·60	2·59

Suppose that f ft./sec.^{2} is the acceleration, then

$$s = \tfrac{1}{2} ft^2, \text{ and } \frac{s}{t^2} = \tfrac{1}{2} f.$$

From the above results taking the average value of $\frac{s}{t^2}$ as 2·57,

$$\tfrac{1}{2} f = 2{\cdot}57, \text{ and } f = 5{\cdot}14.$$

The acceleration of the ball is 5·14 ft./sec.^{2}

105. To show that the Acceleration due to Gravity is Constant and to find its Value.

Since we are able to measure times more accurately than Galileo, we are able to show that a body which falls freely under gravity moves with constant acceleration, and, moreover, we are able to find its value with fair accuracy.

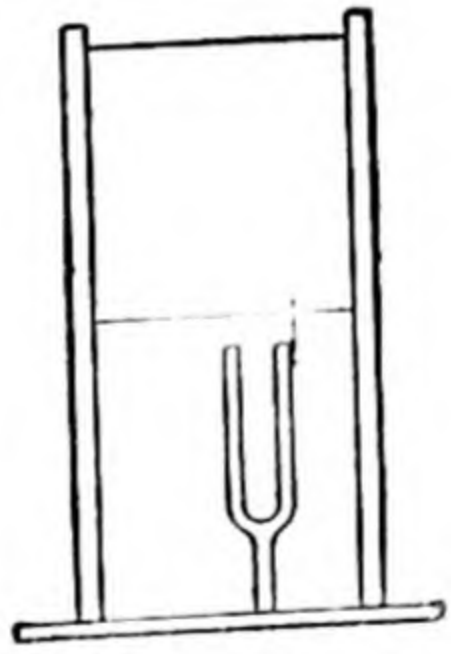

FIG. 166.

EXPERIMENT.—A suitable piece of apparatus is shown above. A tuning-fork is arranged so that a bristle, which is attached to one of its prongs by means of a light piece of sealing-wax, is in contact with a smoked-glass plate attached to the vertical face of a heavy board, which is free to fall between vertical guides without friction. If the fork is made to vibrate, and the board is allowed to fall freely from rest, the bristle describes a wavy line on the surface of the smoked glass without impeding the motion of the board to an extent which affects the accuracy required

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from the experiment. If the fork makes a large number of vibrations per second, then the times taken by the board to fall through different distances can be found by counting the number of complete waves described by the fork. In the experiment, for which the results are given below, the fork made 384 vibrations per second. The time for one complete wave was therefore 1/384th sec. It is found that the waves at the beginning of the motion of the plate are so closely packed that counting them cannot be satisfactorily carried out. Moreover, when the plate is released, whatever method of releasing the plate is used, such as burning through the string supporting the board, the initial motion of the plate is described freely under gravity. For these reasons we shall not attempt to make use of the first waves registered on the plate.

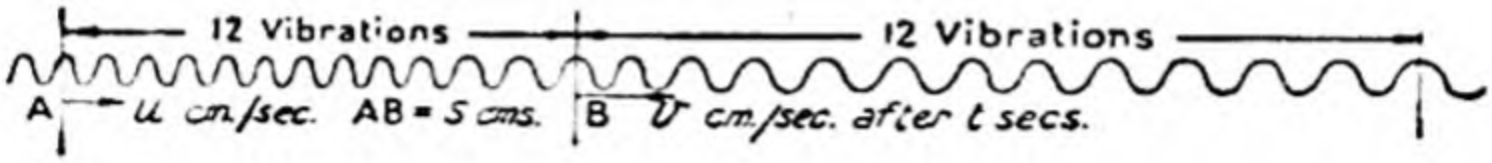

FIG. 167.

Consider the above record of the path of the bristle on the smoked glass.

Let A be a point at which the waves become normal in shape, *i.e.*, of the continuous wavy-line shape.

u cm./sec. is the speed of the plate at A.

v cm./sec. is the speed of the plate at B t sec. later.

If the acceleration of the plate is uniform and equal to f cm./sec.^{2},

$$v = u + ft.$$

Since, $$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time taken}},$$

$$\frac{v+u}{2} = \frac{AB}{t},$$

$$v + u = \frac{2AB}{t},$$

i.e., $$2u + ft = \frac{2AB}{t}. \qquad (v = u + ft.)$$

Now u is an unknown constant, therefore the graph of the right-hand side of the last equation will be a straight line if the acceleration of the board is constant and *its gradient will be the value of* g. The wavy line is therefore marked off in equal numbers of complete waves (groups of twelve were used in the experiment) and the values of $(v + u)$ are calculated from the above expression. These results are plotted on a time base, as shown on the graph below, and it is seen that the points are in one straight line, thus showing that the board falls with constant accelera-

tion. In the graph the unit on the time base is 1/456th sec., since the tuning-fork made 456 complete vibrations per second.

s = AB.	$2s$.	t.	$\frac{2s}{t}$.
Cm.		Sec.	
2·19	4·38	$\frac{12}{456}$	$4{\cdot}38 \times \left(\frac{456}{12}\right)$
5·06	10·12	$\frac{24}{456}$	$5{\cdot}06 \times \left(\frac{456}{12}\right)$
8·64	17·28	$\frac{36}{456}$	$5{\cdot}76 \times \left(\frac{456}{12}\right)$
12·93	25·86	$\frac{48}{456}$	$6{\cdot}46 \times \left(\frac{456}{12}\right)$

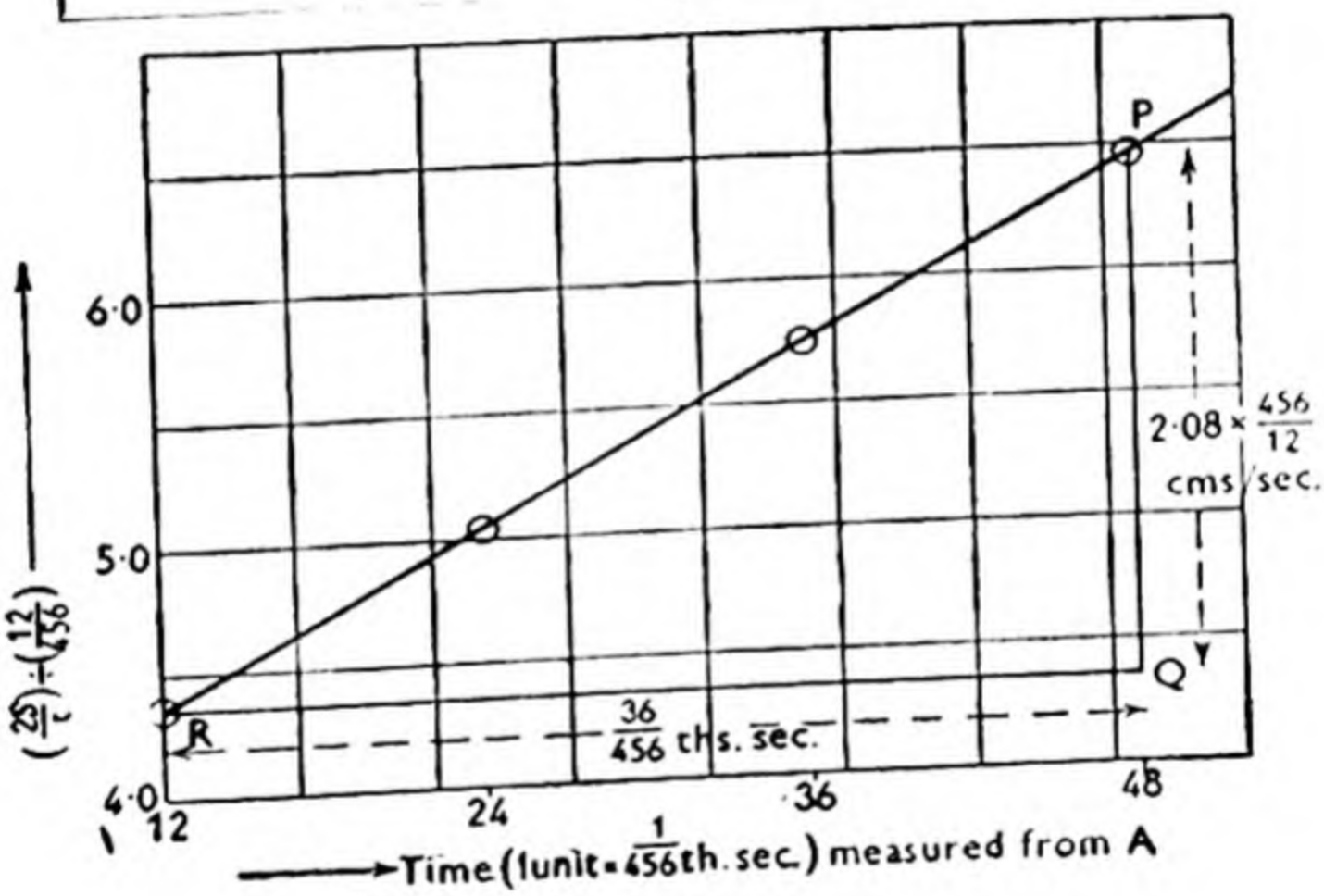

FIG. 168.

$$\text{Gradient of graph} = \frac{PQ}{QR}.$$

$$QR = \frac{36}{456} \text{ sec.}$$

$$PQ = 2{\cdot}08 \times \left(\frac{456}{12}\right).$$

$$g = \frac{2{\cdot}08}{3} \times \left(\frac{456}{12}\right)^2 = 1000 \text{ cm./sec.}^2$$

106. While he was still a student at the University of Pisa, where he was afterwards Professor of Mathematics, Galileo noticed that the times of successive swings of a swinging candelabrum in the cathedral were equal whether the candelabrum described large or small arcs. He timed the changing swings by his pulse. It is appropriate that the following experiment for determining the acceleration due to gravity makes use of the pendulum and the fact referred to above, originally discovered by Galileo.

FIG. 169.

EXPERIMENT.—The apparatus consists essentially of a pendulum, which for small swings takes exactly equal times. The time of one complete oscillation (*i.e.*, from one extreme position to the other *and back* to the first position) may be altered by changing the position of a weight attached to the pendulum at a point in its length. The pendulum is pulled aside through a small angle, and is kept in a position displaced from the vertical through its point of suspension by a light thread, which after passing round a smooth peg at A and over another at B supports a small plummet, which just touches the straight-edge of the pendulum when the latter is in a vertical position.

When the string is burned through the pendulum and the plummet are released simultaneously, and the time taken by the pendulum to swing into a vertical position is so arranged that the plummet strikes it near its lower end. In order that the mark of contact may be clearly shown the edge of the pendulum is blackened with soot. The vertical distance through which the plummet falls is measured as accurately as possible. The pendulum is then made to oscillate, and the time for 50 or 100 complete swings is observed.

If the experiment is repeated three times for each new period of oscillation, and the average value of s/t^2 is calculated for each new period of oscillation, by comparing the average values of s/t^2 we may show that the acceleration due to gravity is constant.

The following is the result obtained in an experiment similar to that described.

Distance through which the plummet falls = 4 ft. 1·5 in.
Time required for 50 complete swings = 100·8 sec.
" " ½ complete swing = 0·504 sec.

Since $s = \frac{1}{2}gt^2$,

$$g = \frac{2s}{t^2} = \frac{4 \cdot 125 \times 2}{0 \cdot 504^2} = 32 \cdot 5.$$

One value of g is not sufficient, however, and the experiment should be repeated at least three times, and the average value of g may be taken as that found from these three results, *provided that they are in fairly close agreement with each other.*

107. Vertically Downward Motion under Gravity.

It has now been shown that when a body falls freely to the earth the motion is one of uniform acceleration if the resistance of the air is neglected.

The increase in velocity (v ft./sec.) of a body which falls freely from rest in t sec. is given by

$$v = gt,$$

and the distance through which it falls by

$$s = \tfrac{1}{2}gt^2.$$

The value of g is different at different places on the earth's surface, since its value depends upon the force of attraction of the earth.

In foot-second units the value of g *in vacuo* at Greenwich is 32·19 ft./sec.^{2}

In centimetre-second units the value of g *in vacuo* at Greenwich is 981·2 cm./sec.^{2}

(g is 32·09 at the equator and 32·25 at the poles.)

Since for uniform acceleration

$$v^2 = u^2 + 2fs,$$

the velocity acquired by a body which falls from rest through h ft. is given by

$$v^2 = 2gh.$$

$$v = \sqrt{2gh},$$

If a body is thrown vertically downwards with initial velocity u ft./sec., then in t sec.

$$\mathbf{v = u + gt},$$

$$\mathbf{s = ut + \tfrac{1}{2}gt^2},$$

$$\mathbf{v^2 = u^2 + 2gh}.$$

h being the height through which the body falls.

Example (1).—*A stone is dropped from a bridge into a river, and it is observed by means of a stop-watch that after 2·4 sec. the stone strikes the water. What is the height through which the stone falls and the velocity with which it strikes the water?*

The height, h ft., through which the stone falls is obtained from the formula

$$h = \tfrac{1}{2}gt^2.$$

Taking g as 32, $\quad h = 16 \times 2{\cdot}4^2 = 92{\cdot}16.$

The required height is therefore 92 ft.

(Since g is only correct to a little less accuracy than $\frac{1}{2}$ per cent., the nearest foot is a reasonable accuracy for the height.)

Let v ft./sec. be the velocity with which the stone strikes the water,

$$v = gt = 32 \times 2{\cdot}4 = 76{\cdot}8.$$

The velocity is 77 ft./sec.

Example (2).—*A stone is dropped from the top of a cliff 262 ft. high. After what time does it reach the foot of the cliff? What is its velocity when it has fallen 100 ft.?*

Let t sec. be the time taken by the stone to reach the foot of the cliff

Using

$$s = \tfrac{1}{2}gt^2,$$

$$262 = 16t^2,$$

$$t^2 = \frac{262}{16},$$

$$t = \sqrt{\frac{262}{16}} = 4{\cdot}05.$$

The stone reaches the foot of the cliff after 4·05 sec.
If v ft./sec. is the velocity acquired after falling 100 ft.,

$$v^2 = 2gh = 64 \times 100,$$

$$v = \sqrt{6400} = 80.$$

The velocity acquired is 80 ft./sec.

Example (3).—*A stone is thrown vertically down a well 110 ft. deep, with initial speed 40 ft./sec. If the speed of sound is 1,100 ft./sec., find*

the time after projection which elapses before the sound of the stone striking the water is heard at the top of the well, and the velocity with which the stone strikes the water.

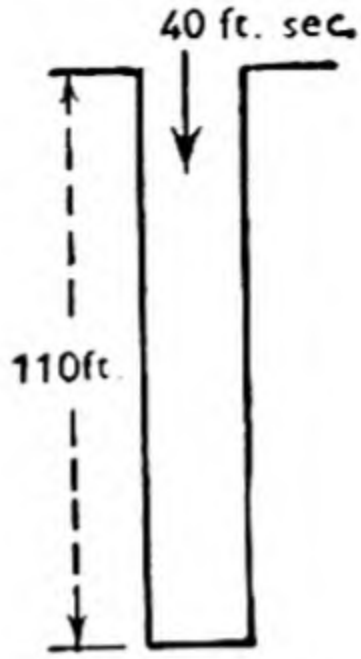

FIG. 170.

The total time may be divided into two parts, that required for the stone to reach the bottom of the well, and that required for the sound of the splash to travel from the bottom to the top of the well.

Let t sec. be the time for the stone to reach the bottom of the well.

Using
$$s = ut + \tfrac{1}{2}gt^2,$$
$$110 = 40t^2 + 16t^2. \qquad (g = 32),$$
$$8t^2 + 20t - 55 = 0.$$

Solving this quadratic equation,
$$t = 1{\cdot}655 \text{ sec.}$$

The second value of t, which is a solution of the equation, is negative and is not the solution required in the problem.

The time taken by the sound of the splash to reach the top of the well is the time taken by sound to travel 110 ft. at 1,100 ft./sec., *i.e.*, 1/10th sec.

The total time required is therefore 1·76 sec.

Let v ft./sec. be the velocity with which the stone strikes the water, then using
$$v^2 = u^2 + 2gh,$$
$$u = 40 \text{ and } h = 110,$$
$$v^2 = 40^2 + 2 \times 32 \times 110 = 1600 + 7040 = 8640,$$
$$v = \sqrt{8640} = 93{\cdot}0.$$

Otherwise, since the time for the stone to fall 110 ft. is 1·655 sec.,
$$v = u + gt = 40 + 32 \times 1{\cdot}655 = 93.$$

The stone strikes the water with velocity 93 ft./sec.

EXAMPLES XXX

[$g = 32$ ft./sec.^{2} or 981 cm./sec.^{2}]

1. A stone is dropped from a window and strikes the ground after 1·5 sec. Through what distance does the stone fall ?

2. A brick is let fall from the top of a building and reaches the ground in 3 sec. How high is the building and with what speed does the brick strike the ground ?

3. A stone is projected vertically downwards with a speed of 12 ft./sec. How long does it take to acquire a speed of 44 ft./sec., and through what distance has it then fallen ?

4. Ballast, thrown overboard from a balloon which is at rest, reaches the ground in 10 sec. With what speed does the ballast reach the ground, and what is the height of the balloon ?

5. Calculate the distances through which a body falls from rest in the 1st, 2nd, 3rd, 4th and 5th sec. of its motion. How far does it fall in the nth sec. of its motion ?

6. A stone is dropped from a height of 100 ft. How long does it take to reach the ground ?

7. A stone is dropped from a height of 144 ft. Find the time which it takes to reach the ground and the velocity of the stone after 2 sec.

8. A stone is dropped from a height of 196 ft. Calculate the distances through which it falls in the 1st, 2nd and 3rd sec. of its motion. With what velocity will it strike the ground ?

9. How long does it take a body, which is allowed to fall freely, to acquire a speed of 80 ft./sec. ? How far has it fallen, and how much farther will it fall before it acquires a speed of 120 ft./sec. ?

10. A body is released from a balloon which is descending with a velocity of 20 ft./sec. and it reaches the ground in 10 sec. With what velocity does it strike the ground ? How high was the balloon when the body was released ?

11. A parachutist releases a pen-knife when he is descending vertically with a speed of 80 ft./sec. and at a height of 2,400 ft. Show that the knife takes 10 sec. to reach the ground.

12. A cliff is 30 m. in height, calculate the time taken by a stone to fall from the top to the bottom of the cliff.

13. A body is projected vertically downwards over the edge of a cliff with a velocity of 30 ft./sec. If it reaches the base of the cliff in 1·2 sec., with what speed does it strike the ground at the foot of the cliff, and what is the height of the cliff ?

14. A stone is dropped down a well 64 ft. deep. How long does it take to reach the bottom ? If the speed of sound is 1,120 ft./sec., how long after the stone is first released does the sound of its striking the water reach the top of the well ?

15. A stone is projected down a well shaft 62½ ft. deep with a velocity 30 ft./sec. Assuming that the speed of sound is 1,100 ft./sec., find the time which elapses from the instant of projection to the instant that the sound of the stone striking the bottom of the well is heard at the top of the well.

16. A stone is let fall and a second stone is thrown after it 1 sec. later with an initial velocity of 60 ft./sec. Find where and when the second stone passes the first.

17. A stone is dropped from the top of a tall scaffolding, and 1 sec. later another stone falls from a platform 112 ft. vertically below. If the two stones reach the ground simultaneously, calculate the height of the scaffolding and the velocity with which each stone reaches the ground. (*G.S.*)

18. Draw the velocity-time curve for a body projected vertically upward from the ground with a velocity of 60 ft./sec. From the graph find the time at which the body strikes the ground again, and using the value of this time calculate the maximum height reached by the body. (*O.C.*) ($g = 32$ ft./sec.^{2})

19. Describe a laboratory method of measuring the acceleration due to gravity. A flat steel strip, fitted with an inked brush at its upper end, is clamped vertically in a vice, and a flat board carrying a paper strip falls freely past the spring so that the brush makes a wavy trace on the paper. From a particular wave-crest the following distances were measured for successive crests : 3·7, 9·5, 17·3, 27·3, 39·4, 53·7. Discuss to what extent these readings indicate a uniform acceleration. (*J.*)

20. Describe the motion of a freely falling body, assuming $g = 32$ ft./sec.^{2} A tape-measure is hung vertically and a bullet dropped from the zero mark. A snapshot of the falling bullet is taken, and shows that while the shutter is open the bullet falls from the 16-ft. mark to 16 ft. 4 in. How long is the shutter open ? (*O.C.*)

21. Show that if a body is thrown vertically downwards with an initial velocity u ft./sec. and the acceleration due to gravity is g, it will fall $u + g(n - \frac{1}{2})$ ft. in the nth sec. of its motion.

A body falls through 90 and 122 ft. in the 3rd and 4th sec. of its descent. Find its initial velocity and the value of g.

22. A stone is dropped into a well and reaches the bottom with a velocity of 96 ft./sec. The sound of the splash on the water reaches the top of the well $3\frac{2}{3}$ sec. from the time the stone starts. Find the velocity of sound. ($g = 32$.)

108. Bodies Projected Vertically Upwards.

When a body is projected vertically upwards its speed decreases at the rate of g ft./sec.^{2} Hence the time taken for the initial speed to be reduced to zero is u/g sec.

The time to reach maximum height is therefore $\frac{u}{g}$ sec.

Otherwise, using

$$v = u + ft,$$

$$0 = u - gt. \qquad (f = -g),$$

$$t = \frac{u}{g}.$$

It is important to notice that at the top point of the path the height reached has a maximum value, the speed of the body is

instantaneously zero, and that during the whole of the motion the speed is decreasing at the constant rate of g ft./sec.^{2}

t sec. after projection the velocity of the body is given by

$$v = u - gt,$$

and is vertically upwards. Thus when t is greater than u/g, v is negative, *i.e.*, the stone is moving vertically downwards. When the stone again reaches the point of projection its velocity is equal to $-u$ ft./sec. measured in the vertically upward direction and it passes through every point of its downward path with the same speed with which it passed upward through the same point.

Maximum Height Reached.—Let h be the maximum height reached.

Since Distance travelled = Average velocity × Time,

$$h = \frac{(u+0)}{2}t = \frac{(u+0)}{2} \cdot \frac{u}{g} = \frac{u^2}{2g}.$$

Otherwise, using the formula,

$$v^2 = u^2 + 2fs,$$
$$v^2 = u^2 - 2gh,$$
$$0^2 = u^2 - 2gh,$$
$$h = \frac{u^2}{2g}.$$

General Formulæ.—

$$\mathbf{v = u - gt},$$
$$\mathbf{h = ut - \tfrac{1}{2}gt^2},$$
$$\mathbf{v^2 = u^2 - 2gh}.$$

It will be seen that when these formulæ are used, the direction in which velocity and distance are considered as positive is the vertically upward direction, and if negative answers are obtained for velocity or distance when these formulæ are applied to a problem the meaning of the sign *must on no account be overlooked.* Thus a negative answer for velocity means that the body is moving

downwards, and a negative answer for distance means that the body is below the point of projection.

Example.—*A stone is thrown vertically upwards at the edge of a cliff with a velocity* 100 *ft./sec.*

(a) *How long does it take to reach its greatest height?*

$$\text{Time to reach greatest height} = \frac{u}{g} = \frac{100}{32} = 3\tfrac{1}{8} \text{ sec.}$$

(b) *What is the greatest height reached?*

Using
$$v^2 = u^2 - 2gh,$$
$$0^2 = 100^2 - 64h,$$
$$h = \frac{10000}{64} = 156\tfrac{1}{4}.$$

The greatest height reached is $156\frac{1}{4}$ ft.

(c) *After what time is the stone* 150 *ft. above the edge of the cliff?*

Let t sec. be the time to reach a height of 150 ft.

Using
$$s = ut - \tfrac{1}{2}ft^2,$$
$$s = 150,\ f = -g,\ u = 100,$$
$$150 = 100t - 16t^2,$$
$$8t^2 - 50t + 150 = 0,$$
$$(4t - 15)(2t - 5) = 0.$$

Either
$$4t - 15 = 0,\ i.e.,\ t = 3\tfrac{3}{4},$$

or
$$2t - 5 = 0,\ i.e.,\ t = 2\tfrac{1}{2}.$$

Hence the stone is travelling vertically upwards after $2\frac{1}{2}$ sec., and vertically downwards after $3\frac{3}{4}$ sec. (Since the times from the top point to these positions are equal, the time for greatest height is their average, *i.e.*, $3\frac{1}{8}$ sec.)

(d) *What is the position and velocity of the stone after* 8 *sec.?*

Let h ft. be the height above the point of projection after 8 sec., then using
$$s = ut + \tfrac{1}{2}ft^2,$$
$$h = 100t - 16t^2 = 100 \times 8 - 16 \times 8^2 = 800 - 1024 = -224.$$

Supposing that the projectile passes over the edge of the cliff, it will be 224 ft. below the point of projection after 8 sec.

If v ft./sec. vertically upwards is the velocity of the stone after 8 sec.,
$$v = u + ft = 100 - 32 \times 8 = -156.$$

The velocity of the stone is 156 ft./sec. vertically downwards.

EXAMPLES XXXI

Motion under Gravity; Bodies projected Upwards, etc.

[$g = 32$ ft./sec.^{2}, or 981 cm./sec.^{2}]

1. A stone is thrown vertically upwards with a speed of 96 ft./sec. Calculate its speed after 2 sec. and after 4 sec., and the height of the stone above the ground in each case.

2. A stone is thrown vertically upwards and returns to the point of projection after 4·8 sec. Calculate the velocity of projection and the maximum height reached.

3. A stone is thrown vertically upwards with a velocity of 100 ft./sec. Calculate the velocity of the stone after 2 sec., and find the maximum height which the stone reaches.

4. A stone is thrown vertically upwards with a velocity of 90 ft./sec. Calculate the times at which the stone is 125 ft. above the ground.

5. A stone is thrown vertically upwards with a velocity of 60 ft./sec., and 1 sec. later a second stone is thrown after it with the same initial velocity. Find the velocity of each stone when they pass each other.

6. Draw the speed-time graph for a body which is allowed to fall freely under gravity, and from it deduce the distance travelled in 3 sec.

7. Derive an expression for the distance moved through in t sec. by a body possessing uniform acceleration f ft./sec.^{2} and an initial velocity of u ft./sec. A body is projected upwards with a velocity 88 ft./sec. Draw graphs showing the velocity and the position of the body at subsequent times. Use the graphs to find the velocity of the body when it returns to a position 72 ft. above the ground and the time that has elapsed before it arrives there. (*L.M.*)

8. A stone is projected upwards with a velocity of 120 ft./sec. from the end of a pier 64 ft. above the sea-level. How long must elapse before another stone is dropped over the side of the pier so that both stones reach the water at the same instant? (*L.M.*)

9. A stone is dropped into a well x ft. deep. Write down an expression for the time (t sec.) which elapses before the sound of the splash is heard by the observer at the top of the well. Calculate to the nearest 1/100th sec. the values of t corresponding to the following values of x: 36, 64, 100, 144, 196 ft. Plot a graph to demonstrate your results, and from it read off the depth of the well for the case in which t equals 2·31 sec. (You may neglect air resistance and assume that the velocity of sound in air is 1,100 ft./sec.) ($g = 32$ ft./sec.^{2}) (*G.S.*)

10. The gravitational acceleration at the surface of the moon is nearly 5·4 ft./sec.^{2} Calculate (*a*) the time taken by a projectile starting vertically upwards from the surface of the moon with a velocity of 120 ft./sec. to return to its starting point, and (*b*) the maximum height reached. How do these results compare with those upon the earth's surface? (*L.M.*)

11. A stone is thrown vertically upwards and returns to the ground after 5 sec. Determine its initial velocity and the greatest height reached. ($g = 981$ cm./sec.^{2}) (*L.M.*)

12. A balloon ascends from rest with uniform acceleration, and a stone is dropped from it after 6 sec. The stone reaches the ground 4 sec. after being released. Find the acceleration of the balloon and the height from which the stone fell. ($g = 32$ ft./sec.^{2}) (*L.M.*)

13. A balloon is ascending vertically, and at a height of 1,500 ft. a stone is released. If the stone reaches the ground in 10 sec., find the velocity of the stone at the instant of release and the height to which the stone rises immediately after release. (*G.S.*)

14. A stone is projected vertically upwards from a point, and at the same time a second stone is allowed to fall from rest vertically downwards from the same point. After 5 sec. the second stone is 480 ft. vertically below the first stone. Determine the velocity of projection of the first stone. (L.)

15. A stone is thrown vertically upwards at the edge of a cliff with a velocity of 88 ft./sec. If the height of the cliff is 75 ft., find the time taken by the stone to reach the foot of the cliff and the velocity with which it strikes the foot of the cliff.

CHAPTER XI

THE PARALLELOGRAM OF FORCES

109. The Graphical Representation of a Force.

When the forces acting upon a body have been considered in the previous pages, the body has often been drawn in elevation and the forces have been shown in position and direction. If we wish to describe fully the way in which a force acts upon a body, it is necessary to state :

(1) Its magnitude.
(2) Its direction.
(3) The point of the body at which it acts, *i.e.*, the position of the line of action of the force.

We may include these three pieces of information on the same diagram by drawing a line, which represents the force in position and direction *of length proportional to the magnitude of the force.* It is often convenient to represent a force in *magnitude and direction only*, in this case the first point of the line which represents the force may be taken as the point in the body at which the force acts.

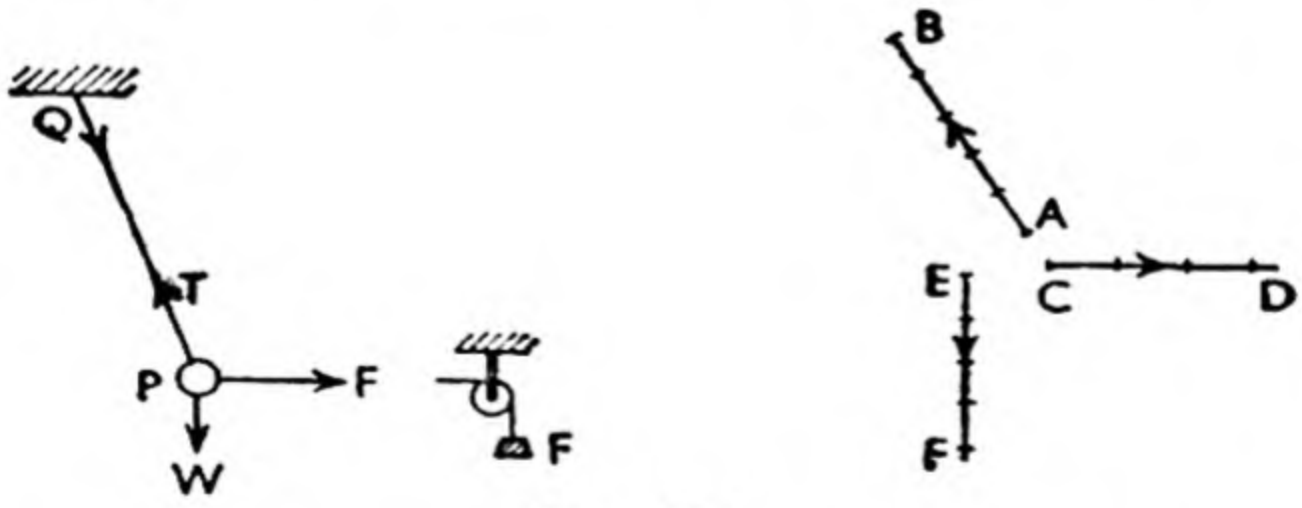

FIG. 171.

In the figure on the left above the forces acting on the weight P, which is suspended from Q by a light string QP, are :

(1) The tension T in the string.
(2) The horizontal force F.
(3) The weight of the body W.

On this figure the forces are shown as they act on the body in position and direction. If T = 5 lb. wt., F = 3 lb. wt. and W = 4 lb. wt., these forces may be represented in magnitude and direction but not in position by the three lines AB, CD and EF *drawn to the same scale*, such as 1 in. to represent 1 lb. wt.

It will be noticed that, following the usual procedure, an arrowhead on the line, which represents the force, indicates the direction in which the force acts. It is not enough to show the direction of a force by a line of given slope, thus, for example, the tension in the string PQ acts upon the weight in the direction PQ taken in the direction of the letters, *i.e.*, from the point P towards the point Q, whereas the tension in the string acts at Q in the direction QP, *i.e.*, from Q towards P. When a force is represented by a straight line, such as AB in the above figure, it is to be understood that the force acts from A towards B.

In a *force diagram* the arrowheads showing the directions of the forces represented on the diagram should *never* be omitted.

110. Three Forces in One Plane Acting at a Point.

It has been shown by experiment that the effect of a force upon the body on which it acts is the same at whichever point of the line of action of the force it is considered to act.

Two forces which act upon a body have equal and opposite effect upon the body and are therefore in equilibrium if the forces are equal in magnitude, opposite in direction and have the same line of action. Two forces which do not fulfil all these conditions are not in equilibrium, and if we consider a body at rest acted upon by two such forces, it is clear that the body cannot remain at rest. In this case a third force may be introduced to balance the combined effect of the first two. In Fig. 171, if the string is cut the weight, which is then under the action of two forces F and W only, will no longer remain at rest. The combined effect of these two latter forces is balanced

by the third force T. In fact, the combined effect of any two of the forces is balanced by that of the third remaining force. Suppose that the lines of action of two forces are produced and meet at a point, then the two forces may be considered to act at this point. If the line of action of a third force, which acts upon the same body and is in the same plane as the first two forces, does not pass through the point of intersection of the lines of action of the first two forces, it cannot balance the resultant of the other two. The three forces cannot therefore keep the body at rest, and it is not possible for them to be in equilibrium. *Three non-parallel forces acting in one plane on a body can be in equilibrium only if their lines of action pass through one point.* We shall show at a later stage, where the above fact is treated more fully, that this can be demonstrated by experiment. We proceed by examining experimentally the simplest case of three forces acting at a point, viz., that in which three forces act along three strings knotted together at a common point, as shown in the figure below. The three forces act upon the knot in the string. The second of the figures below shows the forces acting at the end of a jib in a jib-and-tie arrangement. If the three forces acting at the end of the jib are not in equilibrium the end of the jib cannot remain at rest.

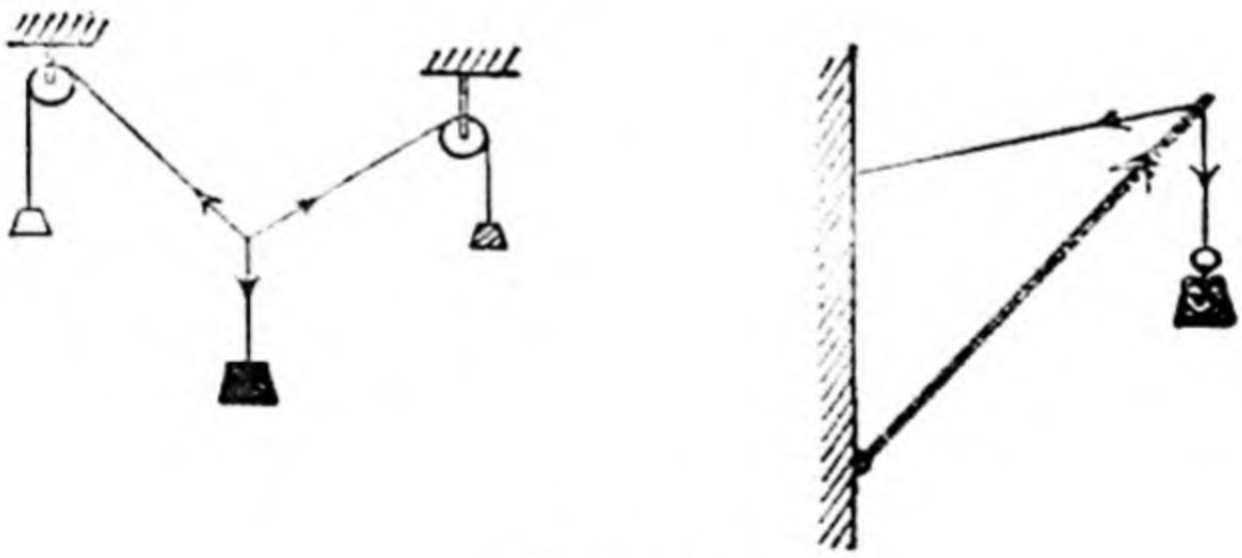

FIG. 172.

111. EXPERIMENT.—Three strings are knotted together at a common point O, and are attached to weights at their other ends. One string hangs vertically downwards, whilst the other two pass over smooth pulleys in the same plane. The common point O is kept at rest by the

tensions in the three strings, arranged as shown in the following figure. (An alternative arrangement using two spring balances is also shown.) When the knot is disturbed it should return to its original position, when it is released. Again, if one or both of the pulleys is altered in position, the weights being kept unaltered, it will be observed that the strings take up positions parallel to their original positions. It thus appears that when three forces act at a point and are in equilibrium the angles between their lines of action are uniquely determined by the magnitudes of the forces. (We may compare this with the fact that if three sides of a triangle are given in magnitude the angles of the triangle are uniquely determined. Is there any fact concerning the three forces which corresponds to the property of a triangle that any two sides are together greater than the third ?) A sheet of paper is arranged on a

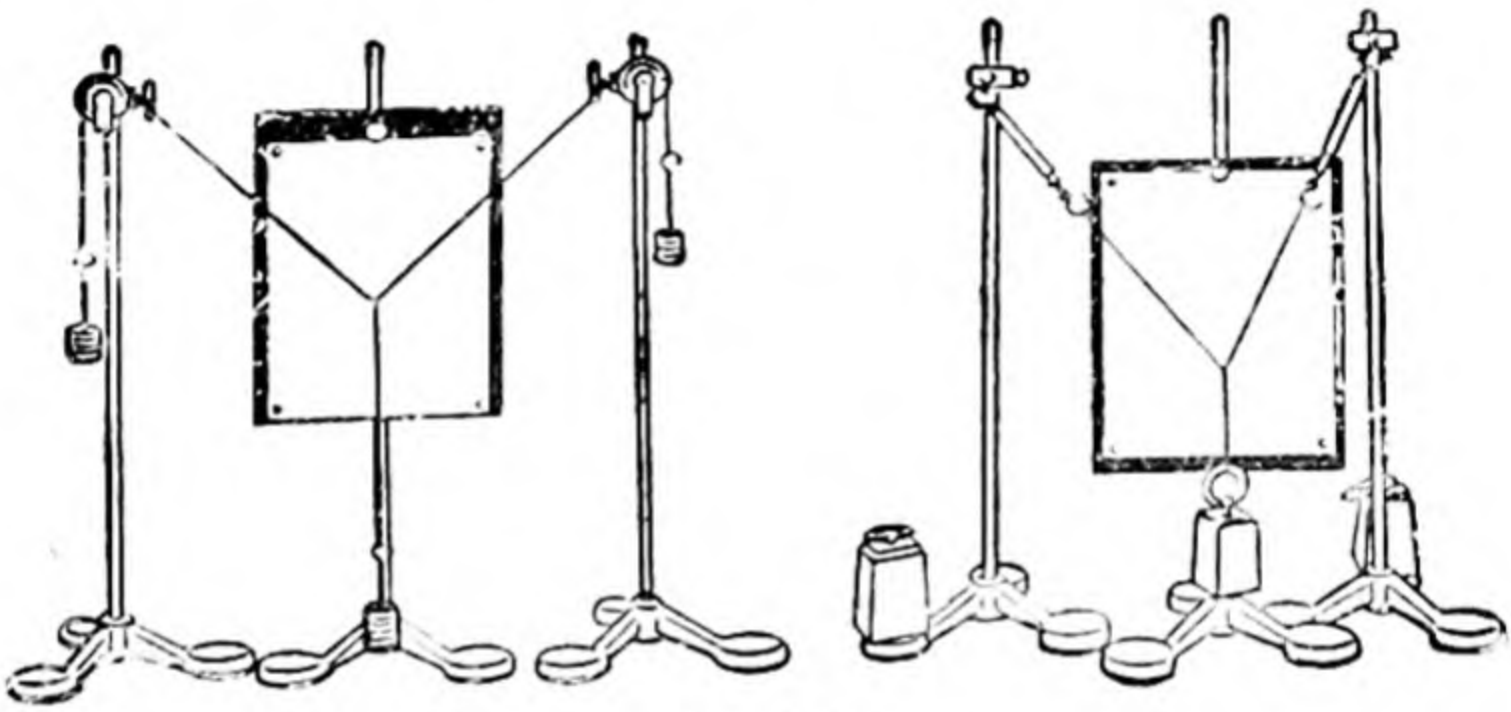

FIG. 173.

plane vertical board to be immediately behind the three strings as close to them as possible. Upon this sheet of paper the point O is marked as accurately as possible, and the lines of action of the three tensions are marked also by means of a number of pricked points immediately behind the strings. Before removing the board and paper the magnitudes of the three forces are marked upon the appropriate lines. The figure below is referred to in the discussion which follows.

Along the lines drawn on the sheet of paper set off distances OA, OB and OC to represent the three forces T_1, T_2 and T_3 to a suitable scale.

Complete the parallelogram OADB.

OA represents T_1.
OB ,, T_2.
OC ,, T_3.

Since the point O is at rest, T_1, T_2 and T_3 are in equilibrium.

The combined effect of T_1 and T_2 upon the knot at O is balanced by the effect of T_3. T_3 is therefore the equilibrant of T_1 and T_2.

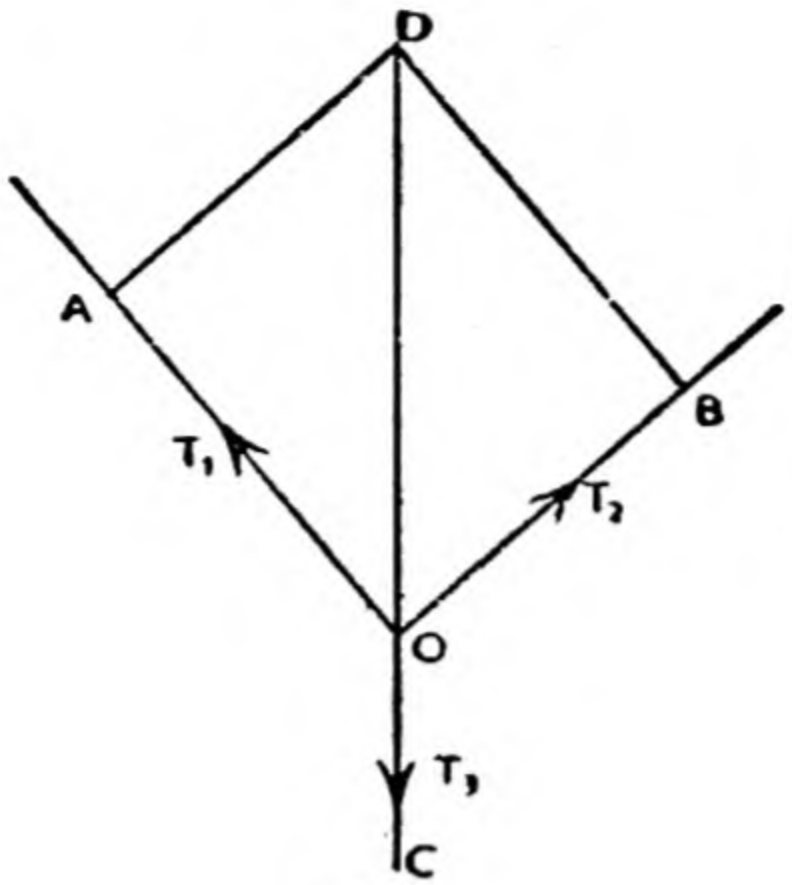

FIG. 174.

A force equal to T_3, but acting in the opposite direction at O, would be in equilibrium with T_3.

Thus a force equal to T_3 acting vertically upward at O has the same effect as the combined effect of T_1 and T_2.

When the diagonal OD of the parallelogram OADB is drawn, it is found that D, O and C are in one straight line, and that the diagonal OD represents, to the same scale as that used for T_1, T_2 and T_3, a force equal in magnitude to T_3, but acting in the opposite direction.

The diagonal OD *represents a force which has the same effect as the combined effect of* T_1 *and* T_2. *Hence* OD *represents the resultant of* T_1 *and* T_2 *in magnitude and direction and position if* O *represents the position of the knot.*

The experiment may be carried out for a number of different values of T_1, T_2 and T_3. Within the limits of experimental error we have demonstrated the *Parallelogram Law for Forces* :—

If two forces acting at a point O are represented in magnitude and direction by two adjacent sides OA and OB of a parallelogram OADB, their resultant is represented in magnitude and direction by the diagonal OD of the parallelogram.

Example (1).—*Two ropes* OA *and* OB *are attached to a point* O *of a barge, and the ropes are pulled by two horses which walk along opposite banks of a canal in the same direction. If the tensions in the ropes* OA *and* OB *are* 15 *and* 25 *lb. wt. respectively and* $\angle$AOB *is* 30°, *what is the magnitude of the resultant force on the barge and what angle does its direction make with* OA *?*

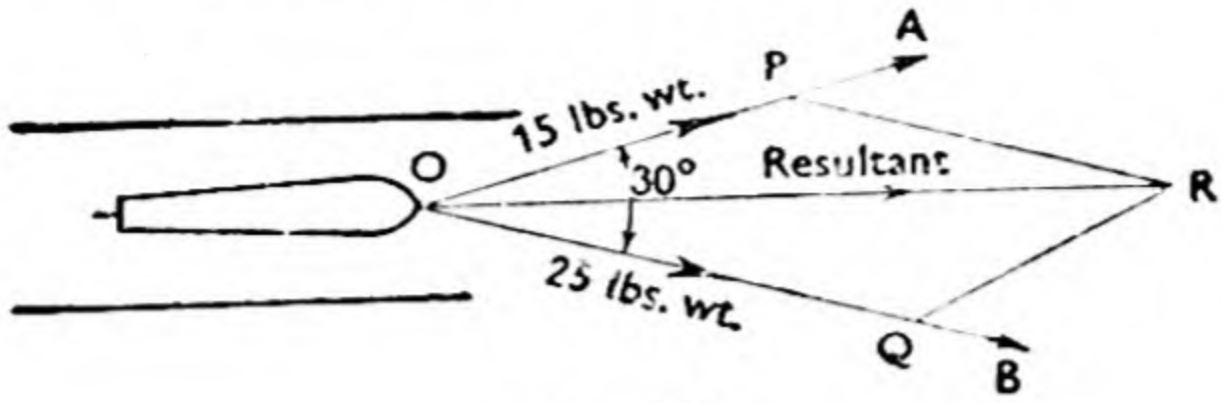

FIG. 175.

Draw OA, OB so that $\angle AOB = 30°$.

Along OA mark off OP equal to 7·5 cm. to represent 15 lb. wt.
„ OB „ OQ „ 12·5 „ „ 25 „

Complete the parallelogram OPRQ.

OR represents the resultant of the two forces in magnitude and direction.

From the drawing OR = 19·5 cm., and represents 39 lb. wt. Also

$$\angle POR = 16°.$$

Example (2).—ABC *is a jib-and-tie arrangement.* AC *is the jib and* BC *the tie. A load of* 10 *cwt. is suspended at* C. *It is required to find the force* T *exerted by the tension in* CB *at* C, *and the resultant of this force and the load of* 10 *cwt. which is exerted along the jib* CA.

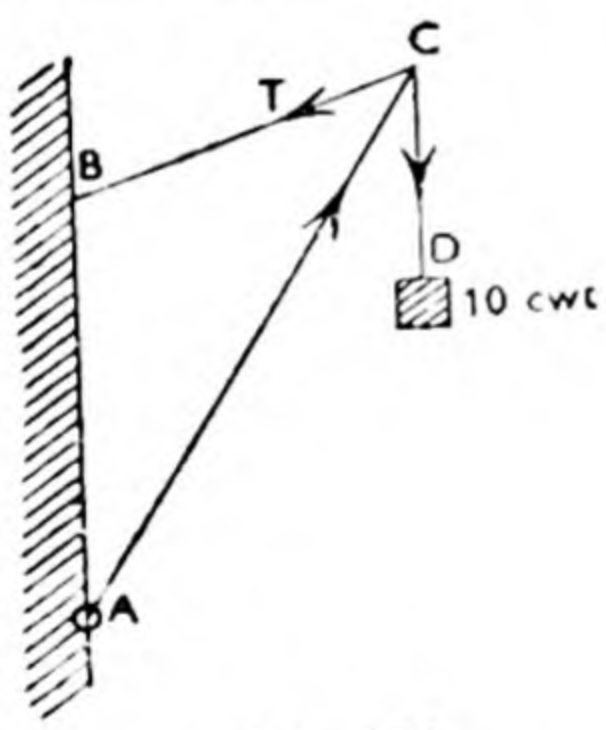

FIG. 176.

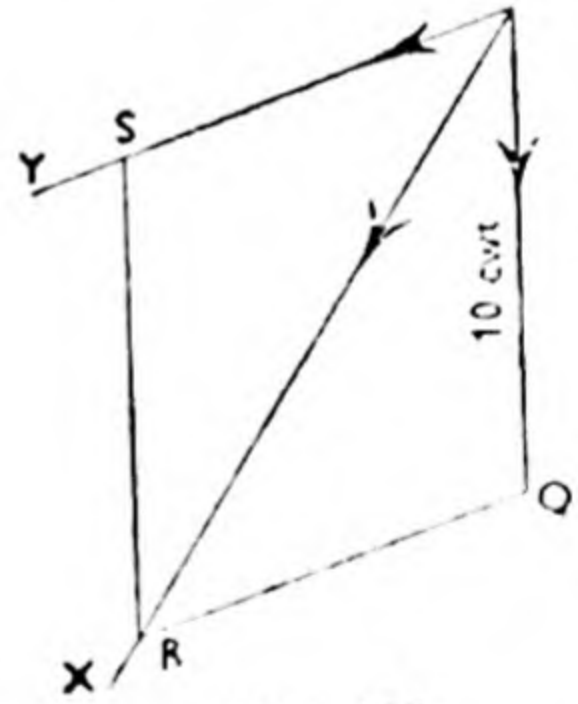

FIG. 177.

Draw PQ parallel to CD and equal to 10 cm. to represent 10 cwt.
Through P draw PX / / to CA and PY / / to CB.
Through Q draw QR / / to PY to meet PX at R.
Complete the parallelogram PQRS.
From the drawing, PS = 7·4 cm.
But PS represents T, therefore T = 7·4 cwt.
PR represents the resultant of T and the load of 10 cwt.

PR = 14·6 cm.

The resultant force along the jib CA is 14·6 cwt.

This force is balanced by an equal but opposite force exerted by the jib AC, as indicated in Fig. 176.

112. Resultant of Two Forces Acting at a Point by Calculation.

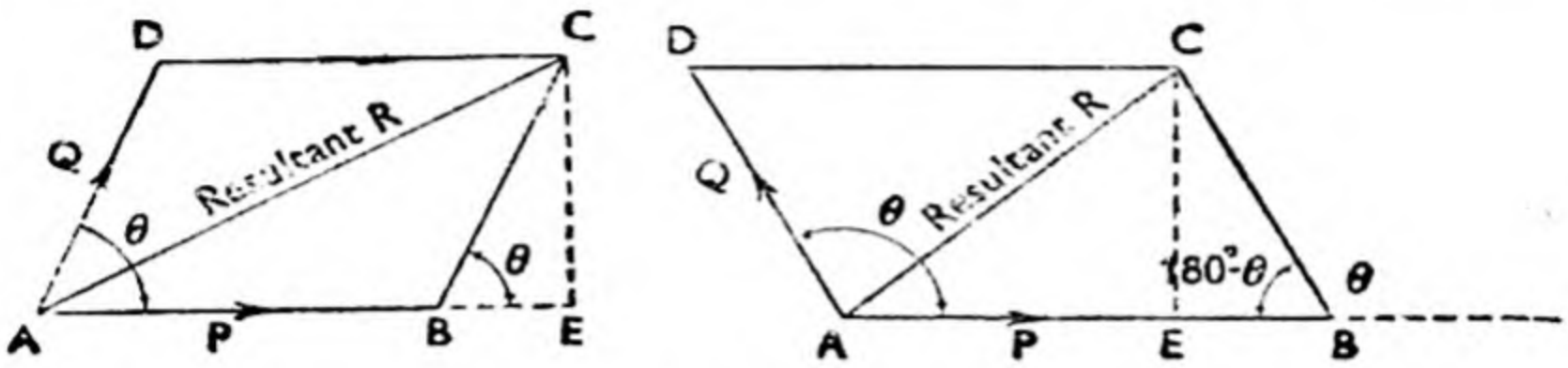

FIG. 178.

Consider the figure on the left above.

AB and AD represent two forces P and Q acting at a point in magnitude and direction. AC represents their resultant. θ is the angle between the lines of action of the forces.

The resultant of the forces P and Q is completely known when (1) its magnitude is known, (2) the angle which its line of action makes with that of a given force is known.

Let CE be the perpendicular from C on to AB produced.

$$\text{Tan CAE} = \frac{CE}{AE} = \frac{CE}{AB + BE}$$

$$= \frac{BC \sin \theta}{AB + BC \cos \theta} \qquad \left(\frac{CE}{BC} = \sin \theta\,;\ \frac{BE}{BC} = \cos \theta\right)$$

$$= \frac{Q \sin \theta}{P + Q \cos \theta}.$$

By the extension of Pythagoras's Theorem for the obtuse-angled triangle ABC,

$$AC^2 = AB^2 + BC^2 + 2\,.\,AB\,.\,BE = AB^2 + BC^2 + 2AB\,.\,BC \cos \theta.$$

Consider the figure on the right.

$$AE = AB - EB$$
$$= AB - BC \cos CBE$$
$$= AB + BC \cos \theta. \qquad (\text{Cos } CBE = -\cos \theta.)$$

By the extension of Pythagoras's Theorem for the acute-angled triangle ABC,

$$AC^2 = AB^2 + BC^2 - 2AB \,.\, EB = AB^2 + BC^2 + 2 \,.\, AB \,.\, BC \cos \theta.$$

Hence, for both cases,

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta,$$

and

$$\tan CAE = \frac{Q \sin \theta}{P + Q \cos \theta}.$$

Note.—These formulæ need not be memorised. The student should rather remember the method of obtaining the result until the formulæ become thoroughly familiar to him. At this stage the formulæ may be quoted and used.

When the lines of action of two forces acting at a point are perpendicular to each other the parallelogram of forces becomes a rectangle, and the resultant of the two forces is easy to calculate.

Suppose that two forces of 4 and 5 lb. wt. acting at a point are represented in magnitude and direction by the sides OA and OB of the rectangle OBCA; their resultant is represented in magnitude and direction by the diagonal OC of the rectangle.

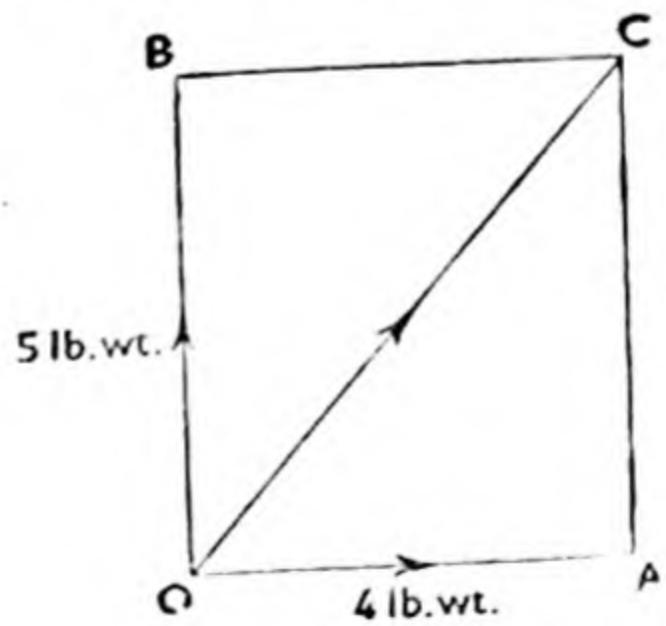

FIG. 179

From the right-angled triangle OBC,

$$OC^2 = OB^2 + BC^2 = 25 + 16,$$

$$OC = \sqrt{41}, \text{ or } R = 6{\cdot}40 \text{ lb. wt.}.$$

$$\text{Tan COB} = \frac{BC}{OB} = \frac{4}{5} = 0{\cdot}8,$$

$$\underline{/COB} = 38^\circ\ 40'.$$

The resultant is a force of 6·40 lb. wt. inclined at 38° 40′ to the 5 lb. wt. force.

113. Resultant and Equilibrant of a Number of Forces Acting in One Plane upon a Body.

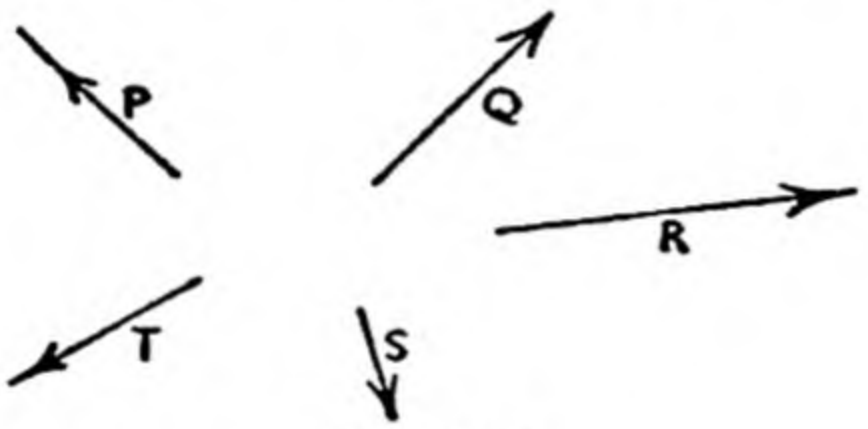

FIG. 180.

Suppose that a number of forces P, Q, R, S and T, acting in one plane upon a body, keep the body at rest. The forces are therefore in equilibrium. Of these forces consider any one force, such as P. The effect of the force P upon the body balances the combined effect of the forces Q, R, S and T. The force P is the *Equilibrant* of the forces Q, R, S and T.

If the combined effect of a number of forces acting upon a body is balanced by the effect of a single force, the single force is the equilibrant of the original number of forces.

Now the force P, referred to above, is in equilibrium, with an equal but opposite force having the same line of action, hence this latter force has the same effect upon the body as the combined effect of the forces Q, R, S and T, and is therefore the *Resultant* of the forces Q, R, S and T.

If the combined effect of a number of forces acting upon a body is the same as that of a single force, the single force is the resultant of the original number of forces.

The resultant and equilibrant of a number of forces are equal in magnitude, opposite in direction and have the same line of action.

114. The Resultant of a Number of Co-planar Forces.

Let P, Q, R, S . . . be a number of forces acting in one plane upon a body. To find their resultant we may proceed as follows. Find the resultant X of the forces P and Q by producing their lines of action to meet at a point and then using the parallelogram

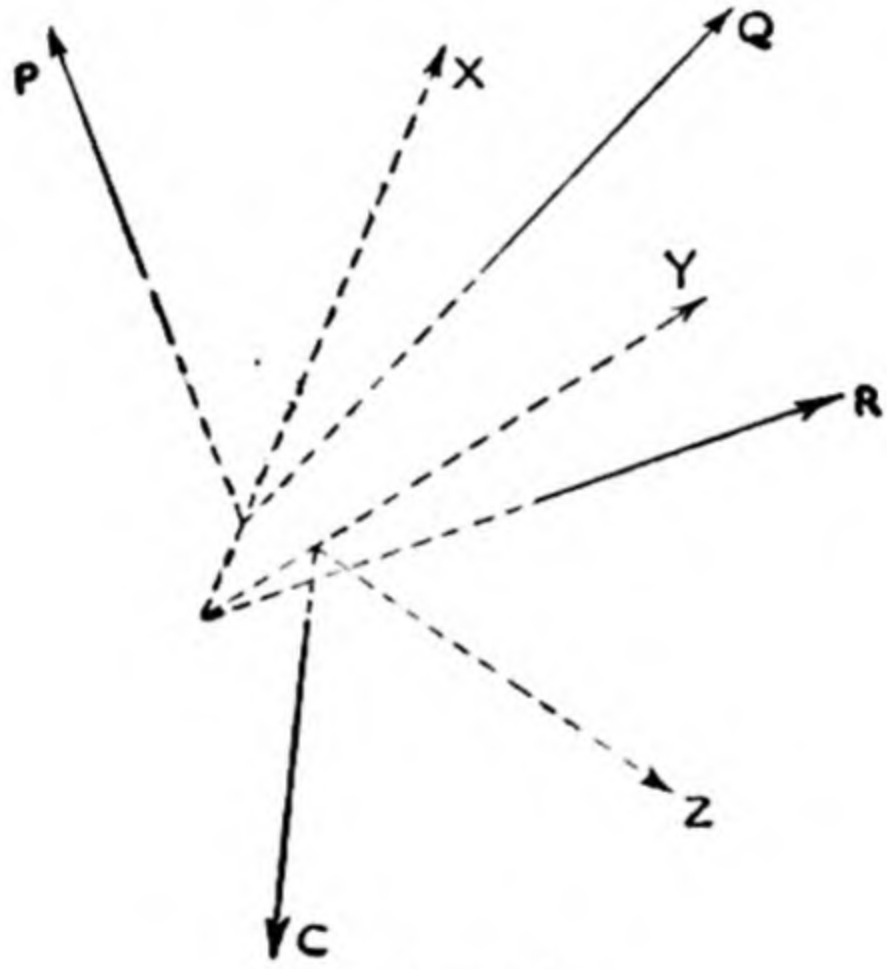

FIG. 181.

law. Find the resultant Y of X and R in a similar manner, and so on. In this way the resultant of any number of co-planar forces may be found in magnitude, direction and position. Since the the parallelogram law can only be applied if the lines of the forces meet, the above method cannot be used when the resultant of two parallel forces must be determined. This difficulty is dealt with later.

EXAMPLES XXXII

1. Find the magnitude of the resultant of two forces P and Q whose lines of action are inclined at angle θ. State also the angle α which the line of action of the resultant makes with the force P.

(*a*) P = 5 lb. wt.,	Q = 7 lb. wt.,	$\theta = 30°$.
(*b*) P = 100 lb. wt.,	Q = 150 lb. wt.,	$\theta = 45°$.
(*c*) P = 6 gm. wt.,	Q = 10 gm. wt.,	$\theta = 70°$.
(*d*) P = 5 cwt.,	Q = 3 cwt.,	$\theta = 110°$.
(*e*) P = 4 tons wt.,	Q = 5 tons wt.,	$\theta = 170°$.

2. Find the magnitude of the resultant of forces of 6 and 8 lb. wt. which act on a particle when the angle between their directions is (*a*) 0°, (*b*) 45°, (*c*) 90°, (*d*) 135°, (*e*) 180°. How does the magnitude of the resultant change with the angle between the forces ?

3. A weight of 4 lb. is suspended by a string AB from a point A. The weight at B is drawn aside by a horizontal force of 2 lb. wt. Find the force exerted on the support at A in magnitude, and the angle which the string makes with the vertical.

4. The resultant of a force of 10 gm. wt. and another force P gm. wt. acting at a point is a force of 15 gm. wt. acting at an angle of 30° to the force of 10 gm. wt. Find the magnitude of P and the direction its line makes with the force of 10 gm. wt.

5. A load P lb. wt. is supported by a wire rope and pulley arrangement. If the forces exerted on the pulley towards A and B are each

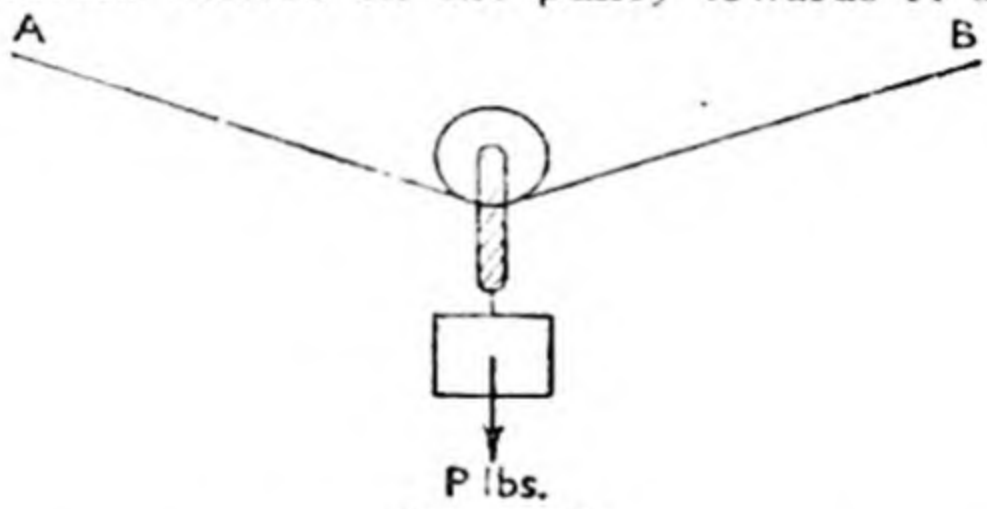

FIG. 182.

equal to 1,000 lb. wt. and the angle between the inclined parts of the rope is 160°, calculate resultant force which supports the load P.

6. A barge is drawn by two horses attached to the ends of a rope which passes round a smooth pulley on the barge. If the force exerted by each horse is equal to 30 lb. wt. and the angle between the two parts of the rope at the pulley is 30°, find the resultant pull on the barge.

7. Two forces of 5 and 7 lb. wt. act at a point. At what angle to each other should they be arranged to act so that their resultant may (*a*) have its least possible value, (*b*) have its greatest possible value, (*c*) act at right angles to the 5 lb. wt. force. In each case find the resultant of the two forces.

8. A load of 100 lb. wt. is raised by means of a rope which passes over a smooth fixed pulley. If the angle between the parts of the rope is 60°, calculate the magnitude and direction of the resultant force exerted on the pulley.

9. Find by calculation the resultant of the forces 5 and 12 lb. wt.

whose lines of action are at right angles to each other. Test your answers by an accurate drawing.

10. Forces of 10 and 20 lb. respectively act at an angle θ with each other. Obtain graphically or otherwise their resultant (R) for values of $\theta = 0°$, 45°, 90°, 135°, and 180°. Tabulate your results, and plot a graph to demonstrate them. From the graph find the value of θ when R = 20 lb. and confirm your result. (*G.S.*)

11. State the theorem known as the parallelogram of forces, and describe an experiment in verification.

If two perpendicular forces A and B act at a point, show how the resultant force may be determined in magnitude and direction. (*G.S.*)

12. Find graphically or otherwise the magnitude and position of the resultant of the three forces given in the figure. (*O.C.*)

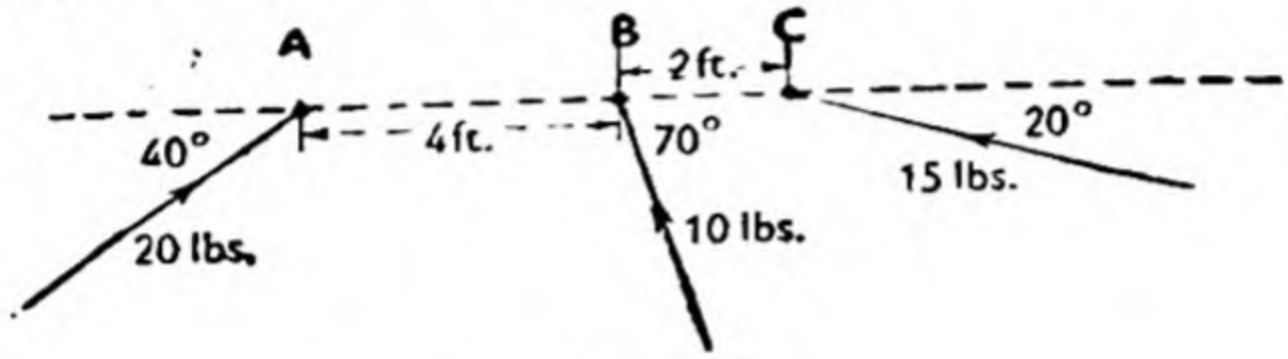

FIG. 183.

115. Components of a Force.

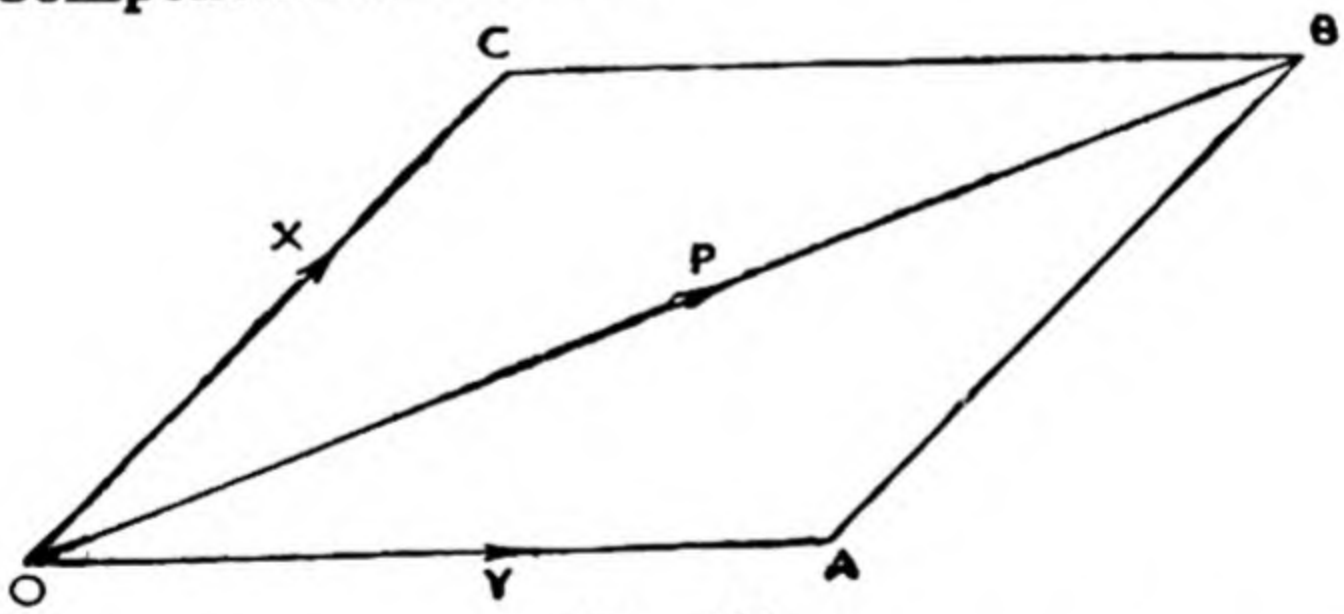

FIG. 184.

In the figure OB represents a force P, and *any* parallelogram is drawn, having OB as its diagonal.

By the parallelogram of forces OB represents the resultant of forces represented by OA and OC. Thus any force such as P, represented by OB, may be replaced by forces X and Y, represented by the adjacent sides OA, OC of a parallelogram, of which OB is the diagonal. The forces X and Y are called the components of P in the directions OC and OA.

116. It is important to notice that the components of a force may either or both be greater than the force itself. If, for example, a vertical load is carried by a pair of shear legs, the greater the angle between the legs the greater will be the components of the load taken by the legs, as shown in the figures below.

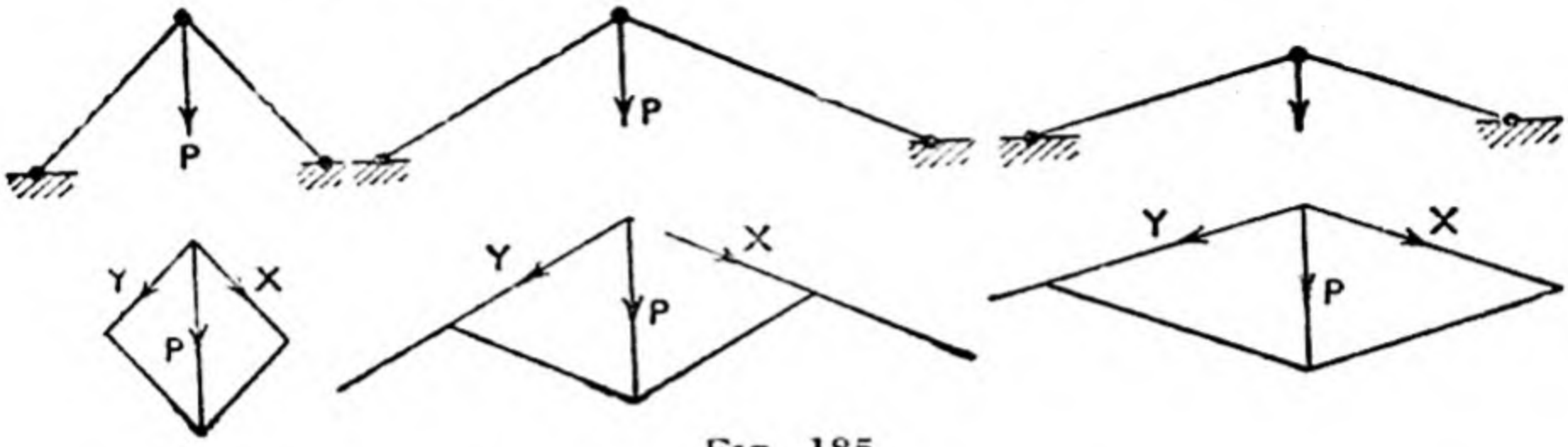

FIG. 185.

Example (1).—*A vertical load of* 15 *cwt. is taken by two shear legs* AB *and* AC. *Calculate the forces exerted by the shear legs along* AB *and* AC *on their supports* B *and* C *respectively.*

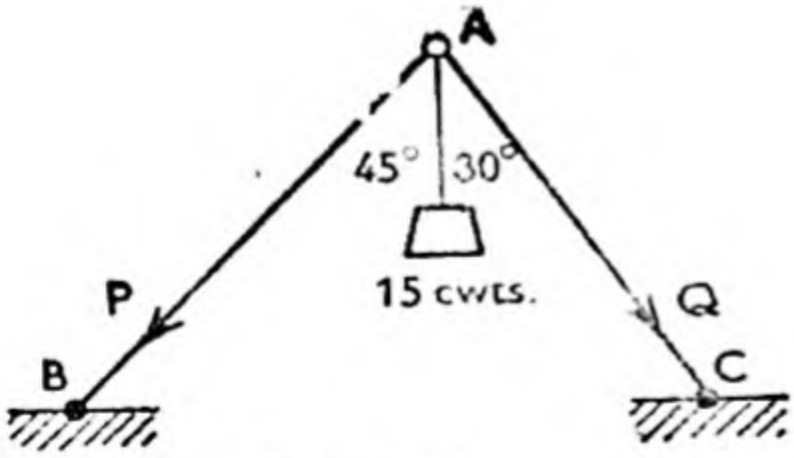

FIG. 186.

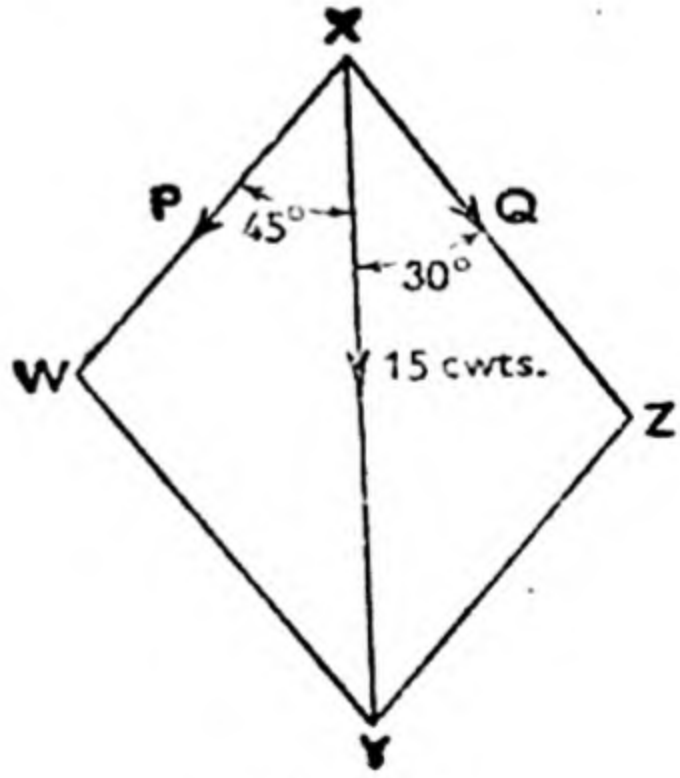

FIG. 187.

P and Q are the components of the load of 15 cwt. along AB and AC respectively.

Let the forces exerted by the shear legs AB and AC on their supports B and C be P and Q cwt. respectively.

Draw XY to represent 15 cwt.

Complete the parallelogram XZYW in which XZ is parallel to AC and XW is parallel to AB.

Then XZ represents Q and XW represents P.

$$P = 7{\cdot}8 \text{ cwt.}, \quad Q = 11{\cdot}0 \text{ cwt.}$$

Example (2).—*A force of 50 lb. wt. acting along a rod* AB *produces a thrust* P *perpendicular to the guide bars* CD *and a force* Q *parallel to the bars. If* $\angle ABC = 30°$, *calculate the forces* P *and* Q.

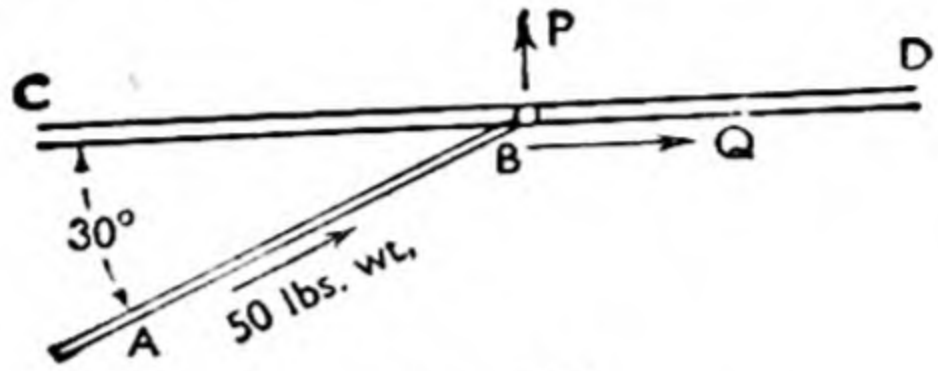

FIG. 188.

P and Q are the components of the force of 50 lb. wt., or, conversely, the force of 50 lb. wt. may be regarded as the resultant of the two forces P and Q.

Draw XY to represent the force of 50 lb. wt.

Complete the parallelogram XWYZ in which XW is parallel to the line of P and XZ is parallel to the line of Q.

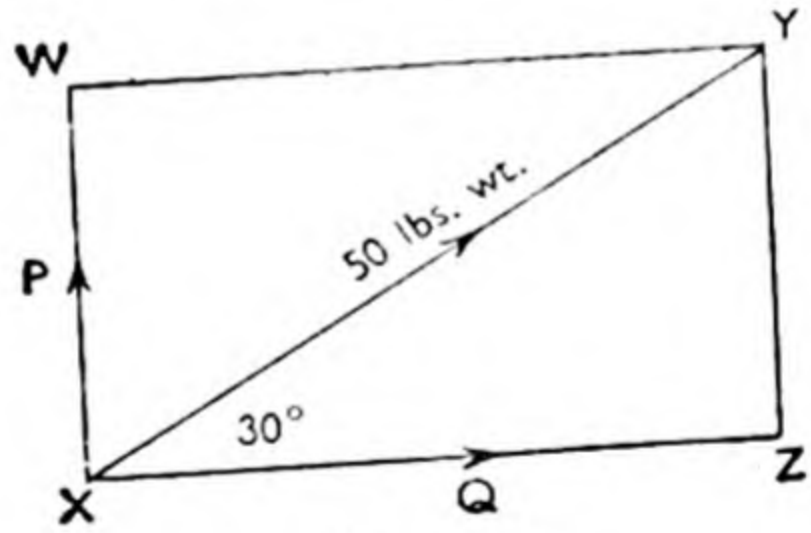

FIG. 189.

XW represents the force P and XZ represents the force Q.

By calculation,

$$\frac{XZ}{XY} = \cos 30°. \quad \text{(Right-angled triangle ZY.)}$$

i.e.,
$$\frac{Q}{50} = \cos 30° \text{ and } Q = 50 \times 0{\cdot}866 = 43{\cdot}3 \text{ lb. wt.}$$

Also
$$\frac{ZY}{XY} = \sin 30°, \text{ i.e., } \frac{P}{50} = \sin 30, \text{ or } P = 50 \times 0{\cdot}5 = 25 \text{ lb. wt.}$$

GIRLS

EXAMPLES XXXIII

Components of a Force

1. A force of 10 lb. wt. acts along a line OA. Find its components along the lines OB and OC in the following cases :—

(*a*) $\angle BOA = 30°$; $\angle AOC = 30°$. (*b*) $\angle AOB = 30°$; $\angle AOC = 60°$.

(*c*) $\angle AOB = 110°$; $\angle AOC = 45°$.

2. The ends of a cord 5 ft. long are fastened to two points 4 ft. 6 in. apart and the same height above the ground. A weight of 20 lb. is suspended from the centre of the cord. Find the force the cord exerts on each support supposing it does not stretch. What difference would it make if the cord stretched by 20 per cent. of its original length ?

(*G.S.*)

3. A load is originally on a ledge at A, and is to be transferred to the opposite side of a vertical shaft. A rope is knotted to the load at D, and one end is fastened to a support at B, the other being pulled along DC.

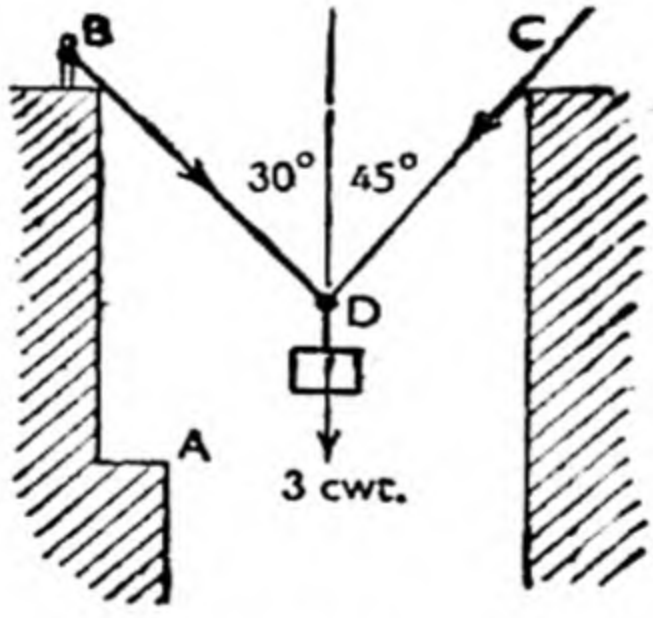

Fig. 190.

Find the forces exerted at B and C when DB and DC are inclined at 30° and 45° respectively to the vertical.

4. A load of 1,200 lb. hangs at a point B by a pair of ropes BA and BC, fastened to points A and C. Each rope is 16 ft. long. Rope BA is horizontal. The point C is 8 ft. higher than A and B. Find the pull in each rope.

5. The force exerted along the connecting rod on the crosshead C is 4,000 lb. wt. Calculate the force on the piston rod and the thrust on the guide bar.

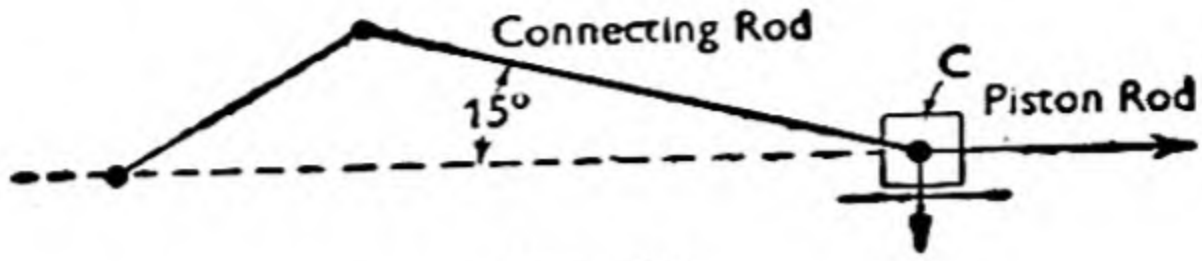

FIG. 191.

6. In a jib-and-tie arrangement the jib AC is 10 ft., the tie BC is 6 ft. and the post AB, which is vertical, is 6 ft. in length. If the jib will break under a force greater than 10 cwt., calculate the greatest load which can be suspended from C, so that the jib may not take more than half its breaking load.

7. A load of 1 ton is carried by a pair of sling chains AB and AC. What is the pull in each chain when the angle between them is 100°? What is the angle between the chains for which the pull in each is 1 ton?

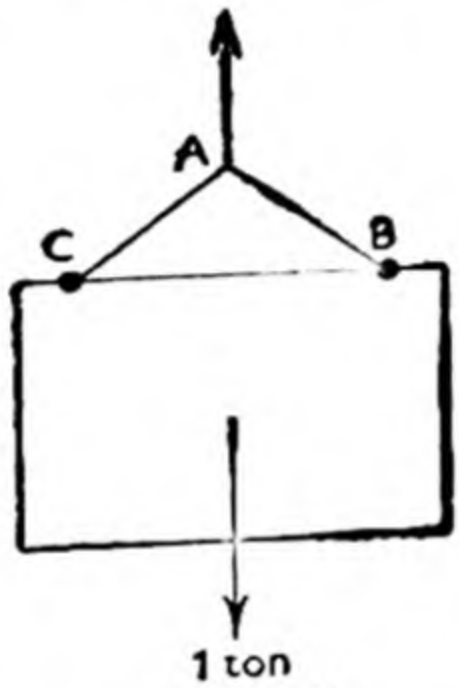

FIG. 192.

8. In the wall-crane arrangement shown in the figure, calculate the forces due to the load 800 lb. wt. produced at A along AB and at C along BC.

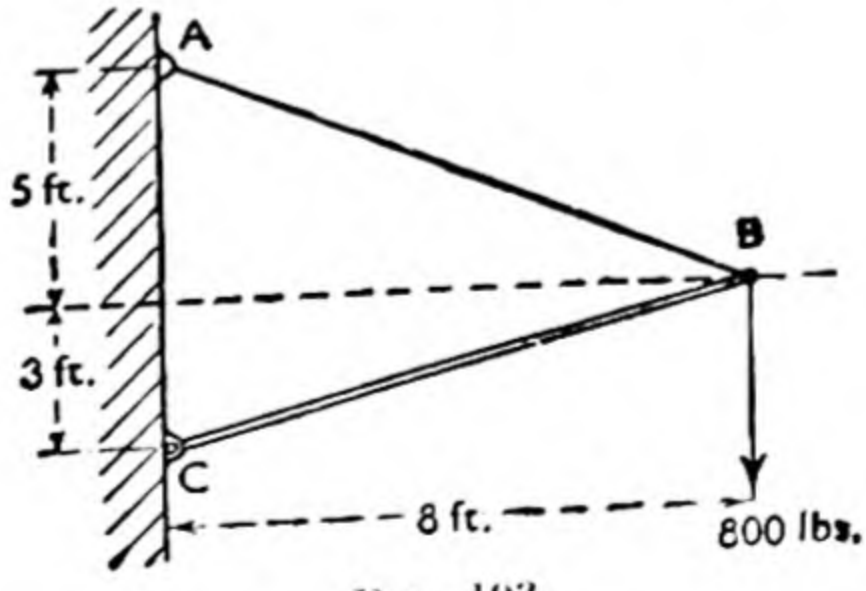

FIG. 193.

9. A rectangular mirror ABCD is supported by a string ANB, which passes over a nail N. The strings and mirror are in a vertical plane, and AB is horizontal. If the mirror weighs 20 lb., what is the resultant force on the nail? If AB is 2 ft. and the string ANB is 4 ft. long, find the tensions in the two parts of the string NA and NB. If ANB is shortened in length, show, by comparison of parallelograms of forces drawn to the same scale, that the tension in it is increased.

10. (*a*) The resultant of two forces each equal to P is also equal to P. At what angle do the forces act?

(*b*) If two equal forces P are inclined at an angle θ, show that their resultant is $2P \cos \frac{\theta}{2}$.

11. Forces P and 2P act at an angle of 60°, show that their resultant is $\sqrt{7}$. P and is inclined at an angle θ to the force P, such that

$$\tan \theta = \frac{\sqrt{3}}{2}.$$

CHAPTER XII

THE TRIANGLE OF FORCES

117. The Triangle of Forces.

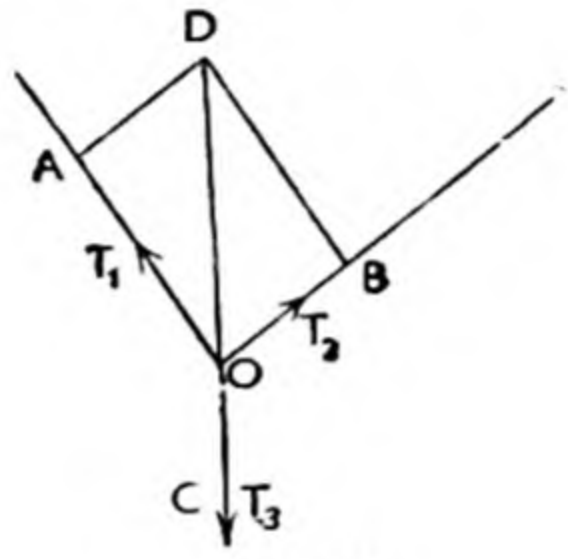

FIG. 194.

Referring to the figure used in demonstrating the parallelogram of forces, OA represents T_1 in magnitude and direction. AD = OB, since they are opposite sides of the parallelogram OADB, therefore AD represents T_2 in magnitude and direction.

It has been shown by experiment that D, O and C are in the same straight line, and that DO represents T_3 in magnitude and direction.

Consider the triangle OAD, OA, AD and DO represent the forces T_1, T_2 and T_3 respectively, acting at a point.

Now the above result was demonstrated in a number of different cases when the forces acting at O were given different values, hence we may state the result as follows :—

If three forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the three sides of a triangle taken in order.

Conversely, if three forces acting at a point can be represented in magnitude and direction by the three sides of a triangle taken in order, then the three forces are in equilibrium.

118. Since any two sides of a triangle are together greater than the third side, any two of three forces acting at a point which are in equilibrium are together greater than the third force in magnitude.

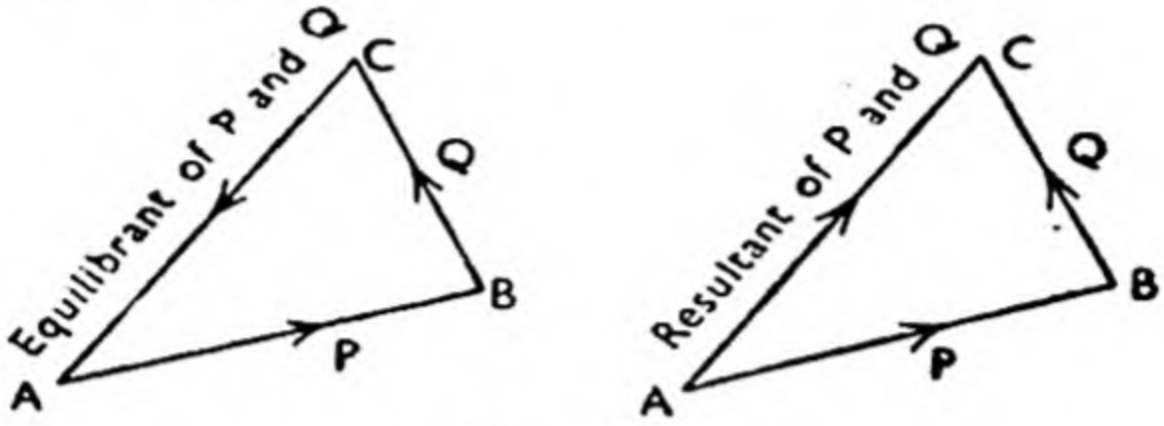

FIG. 195.

It should be noticed that the arrowheads, which indicate the directions of the forces in a triangle of forces, *follow each other round the triangle.*

Any side of a triangle of forces represents the equilibrant of the forces represented by the other two. In fact, if it is desired to find the equilibrant of two forces P and Q acting at a point we have only to draw two sides of a triangle *in order* to represent the two forces, *e.g.*, AB and BC in the above figure, in magnitude and direction, and *the third side of the triangle CA, following the same order as AB and BC, represents the equilibrant in magnitude and direction.*

Thus any side of a triangle of forces taken in its proper direction represents the equilibrant of the forces represented by the other two sides.

It follows from the last statement that the *resultant* of the forces represented by two sides of a triangle taken in order is represented by the third side of the triangle *reversed.* Thus AC represents in magnitude and direction the resultant of the two forces represented by AB and BC.

Example.—AB *and* AC *are two light strings of lengths* 4 *and* 5 *ft. respectively, knotted at* A *and fastened at* B *and* C *to two points on a*

horizontal beam 6 ft. apart. At A a weight of 10 lb. is suspended by a string. Find the tensions in the strings AB and AC in magnitude and direction.

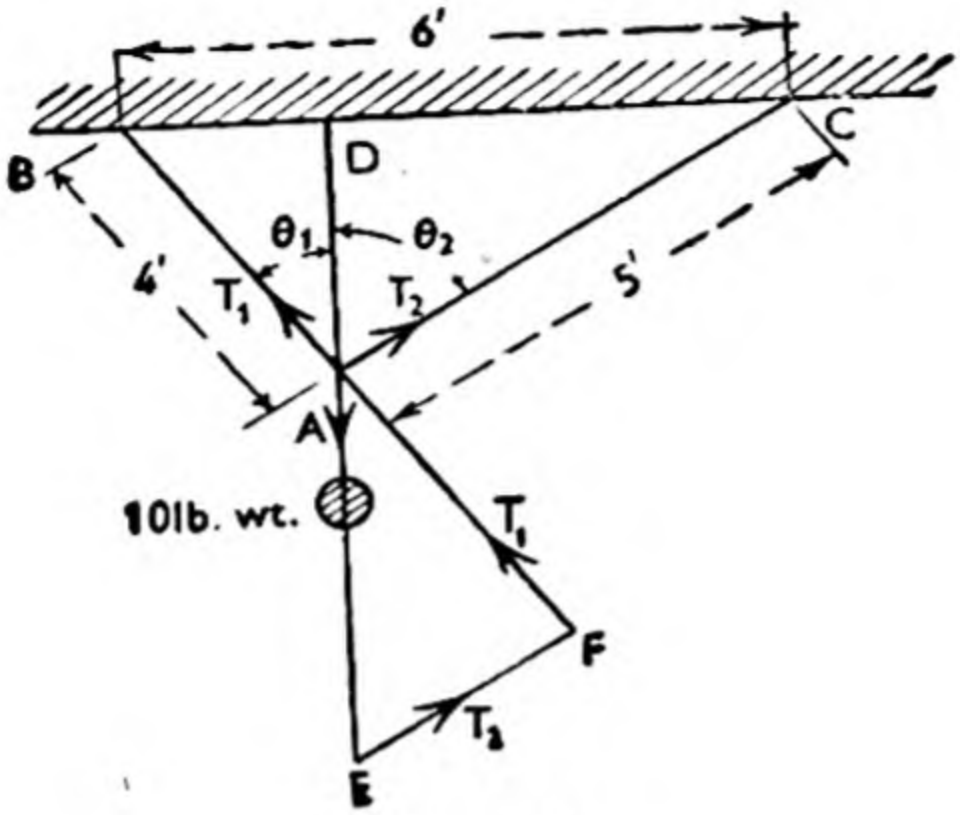

FIG. 196.

Draw BC 6 in. in length to represent 6 ft. Complete the triangle BAC, making BA and AC 4 and 5 in. in length respectively.

Draw DA perpendicular to BC.

Since BC represents the horizontal beam, DA represents the direction of the vertical string supporting the weight of 10 lb.

Along DA produced mark off AE to represent 10 lb. wt., *i.e.*, 5 in. long.

Draw EF // T_2 the tension in AC as it acts on the knot A,

FA // T_1 the tension in AB as it acts on the knot A.

Hence AEF is a triangle of forces.

EF = 2·83 in. Therefore T_2 = 5·66 lb. wt.

FA = 3·76 in. T_1 = 7·52 lb. wt.

θ_1 = 34·25°. θ_2 = 48·75°.

T_1 and T_2 are inclined at 34·25° and 48·75° to the vertical.

It should be observed that in the above problem the tensions T_1 *and* T_2 *are not affected by the lengths of the strings* AB *and* AC, *provided that the angles which they make with the vertical are unaltered.*

119. Experiment with a Jib-and-tie Arrangement.

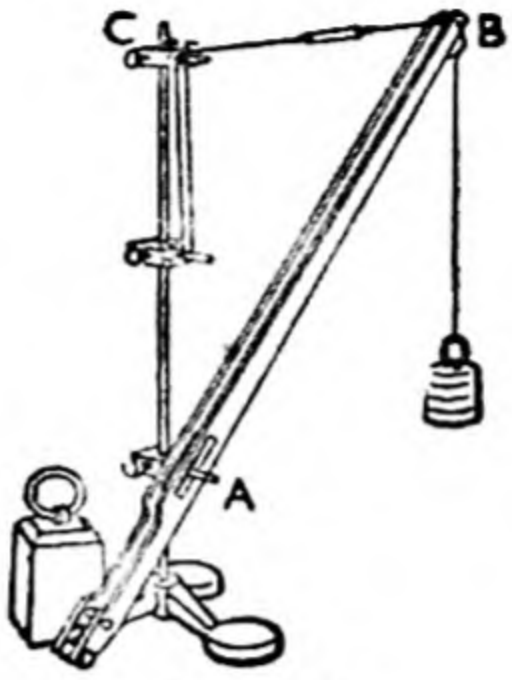

Fig. 197.

In the apparatus shown above, AB is the jib, BC is the tie.

The steel pin at A is supported horizontally and is secured in position by the capstan block. This pin takes the force acting along the jib, and the magnitude of the force is registered by the spring balance fastened to the pin at one end and to the end of the jib at its other end. The jib and spring balance are in one straight line.

Consider the forces acting at B. The load W acts vertically downwards; the tension T in the tie BC acts along BC; the force F in the jib acts along AB. The three forces act at B as they are shown in the figure below.

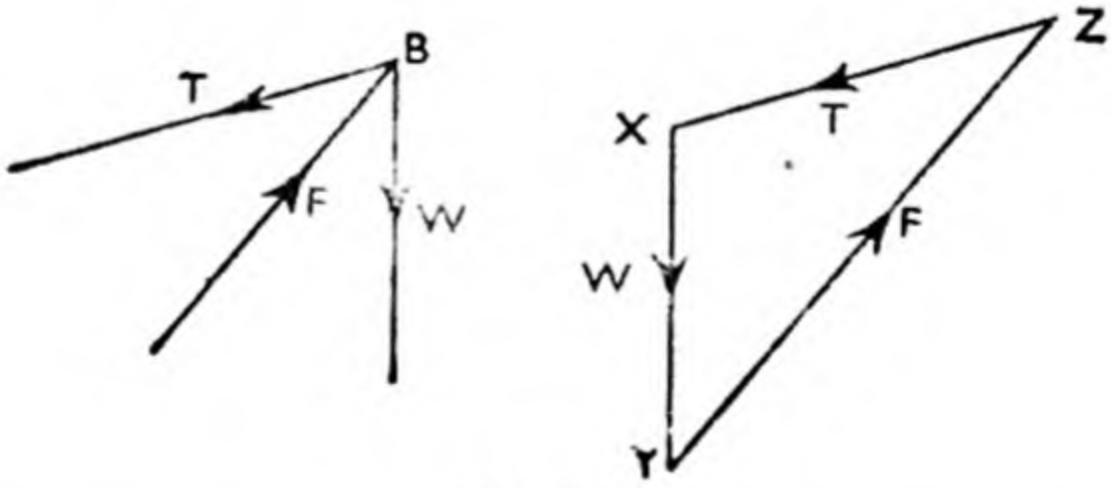

Fig. 198.

The force along the jib is difficult to obtain accurately from an experiment of this kind, for in order to read its value directly, as given by the reading of the spring balance at A, we have to neglect the weight of the jib. To keep the error as small as possible the values of the load should be made as large as the apparatus will reasonably allow.

Draw XY to represent W, YZ to represent T, and, finally, *join* ZX.

From the triangle of forces XYZ, ZX represents the equilibrant of W and T.

Measure ZX, and hence find the force which it represents.

Compare the force along ZX in magnitude and direction with the magnitude of the force registered by the spring balance at A and the direction of the jib.

120. The Inclined Plane.

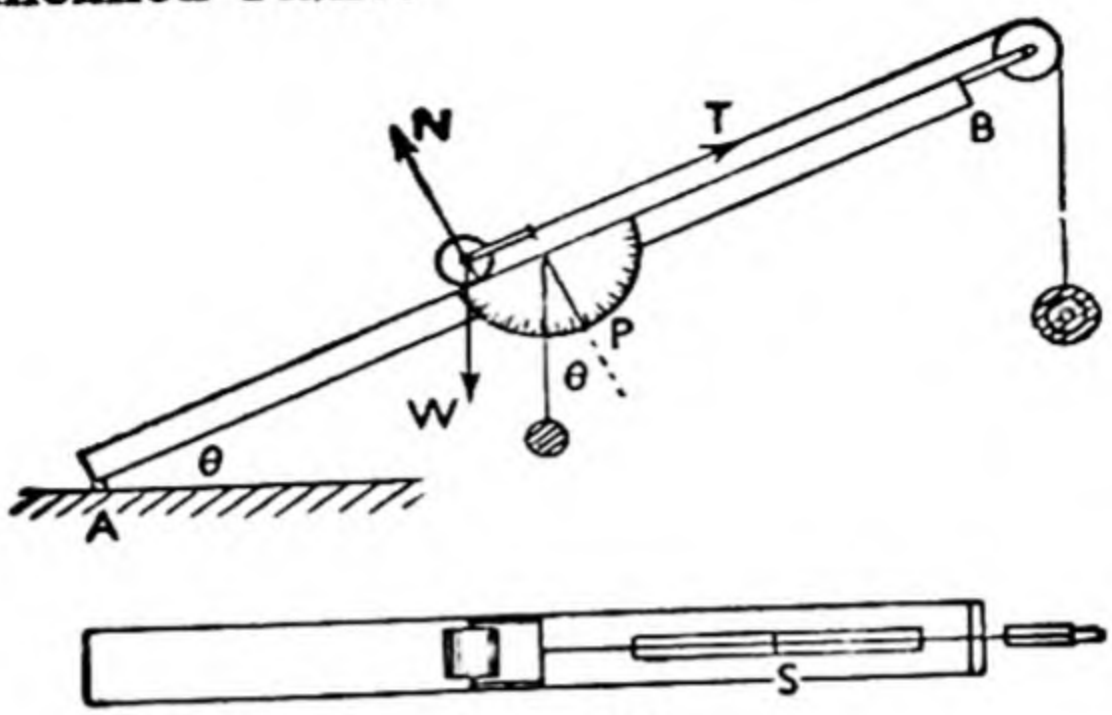

FIG. 199.

The above figure shows a form of inclined plane which is convenient for a number of experiments. It consists of a plane straight piece of wood AB, which carries a smooth pulley at B. P is a protractor fixed to the plane. From the centre of the protractor a plumb-line is suspended, and the angle of inclination of the plane with the horizontal may be read directly by noting the point at which the string of the plumb-line crosses the scale of the protractor. The inclined plane is capable of being fixed at any angle to the horizontal. For the experiments described below we make use of a solid brass cylinder to which is attached a light framework by means of pins fixed to the framework, which act as bearings for the cylinder. A string is fastened to the framework, by means of which the cylinder may be held at rest on the plane in any desired way. A slot S is cut in the plane so that the string may be passed through the plane.

121. EXPERIMENT.—Arrange the cylinder on the inclined plane so that it is held at rest by the string, which first passes over the pulley at B and is then attached to a weight W. The weight W is adjusted in value until the cylinder responds equally to a slight force, which is applied to it either up or down the plane.

The cylinder is now at rest under the action of three forces: the weight of the cylinder vertically downwards; the tension in the string T, which is arranged to act parallel to the plane; the force exerted by the plane on the cylinder, or the *reaction* of the plane on the cylinder.

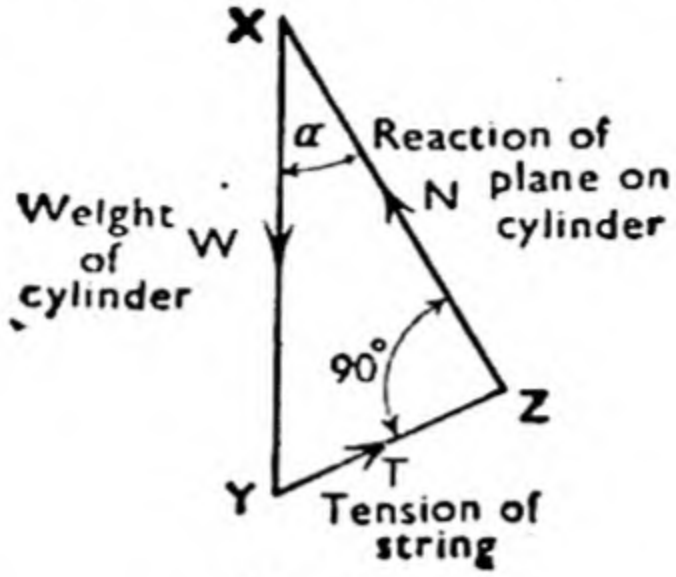

FIG. 200.

Draw XY // line of action of the weight of the cylinder, and to represent this weight.

Draw YZ // line of action of the tension T, and to represent it in magnitude.

Join XZ.

ZX represents the reaction N of the plane on the cylinder.

Notice that the angle YZX is very nearly equal to one right angle, thus indicating that the reaction of the plane on the cylinder is *perpendicular to the plane*.

If we now assume that there is no frictional force at the pin bearings, then the reaction of the plane on the cylinder may be assumed to be perpendicular to the plane. In this case, from the right-angled triangle XYZ,

$$\cos a = \frac{XZ}{XY} = \frac{N}{W},$$

$$N = W \cdot \cos a.$$

$$\text{Sin } a = \frac{YZ}{XY} = \frac{T}{W},$$

$$T = W \sin a.$$

Thus N balances the component of W perpendicular to the plane, and T balances the component of W parallel to the plane.

EXPERIMENT.—Arrange the cylinder so that it is held at rest with the string horizontal, as shown in the figure, and by drawing the triangle of

forces, using the known forces W and T, show that the reaction of the plane on the cylinder is again perpendicular to the plane, and find its magnitude.

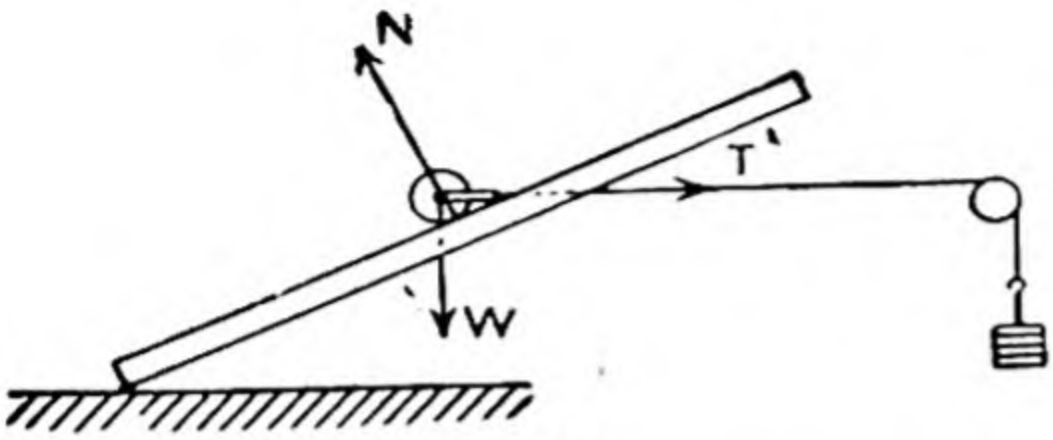

FIG. 201.

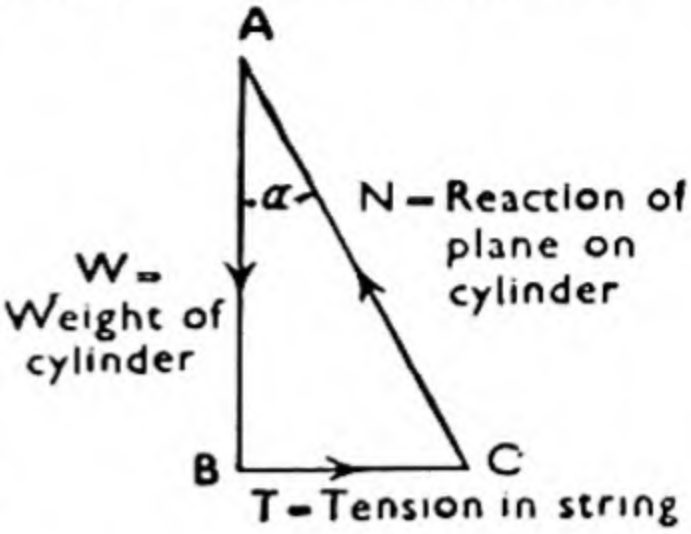

FIG. 202.

Let it be supposed that the reaction is perpendicular to the plane, then using the right-angled triangle ABC,

$$\frac{AB}{AC} = \cos a,$$

$$\frac{W}{N} = \cos a,$$

$$\therefore \quad W = N \cos a \text{ and } N = W \sec a.$$

Again,

$$\frac{BC}{AB} = \tan a, \text{ or } \frac{T}{W} = \tan a,$$

$$\therefore \quad T = W \tan a.$$

122.

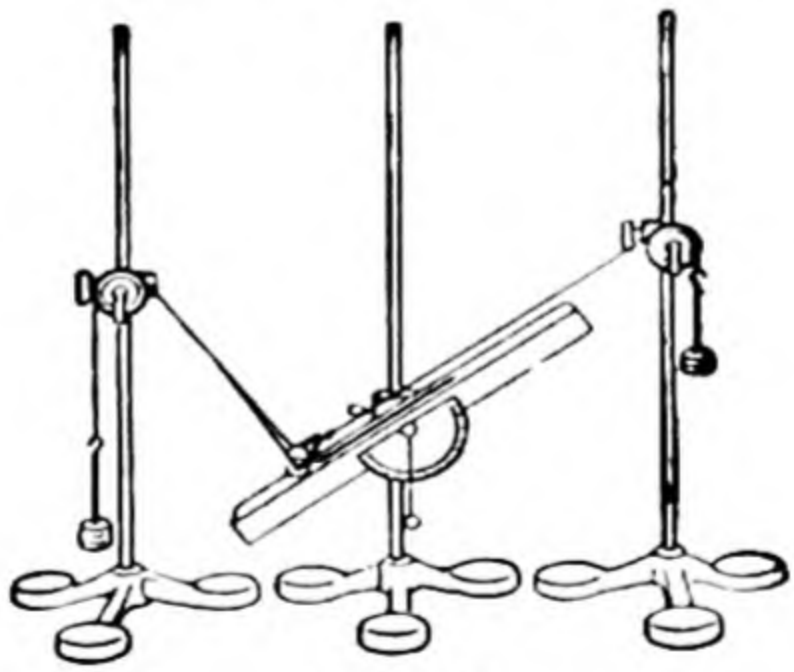

FIG. 203.

We may further demonstrate that the reaction of the plane on the cylinder is perpendicular to the plane in each of the last two experiments, for the cylinder may be just raised from the

plane by means of the tension in a string attached to the framework, as shown in the figure. Compare the tensions in the string which replace the reactions of the plane, with the reactions as found by the previous experiments.

123. Simple Frameworks.

The laws which have been described as the *Triangle of Forces* and the *Parallelogram Law* are the basis of the methods used for calculating the stresses in the " members " of frameworks used in structures such as cranes and bridges. According as a member of a framework is in tension or compression, it is said to act as a *tie* or a *strut*. If a member of a framework is acting as a tie only, then it may be replaced, so far as the stress which it has to take is concerned, by a rope or chain.

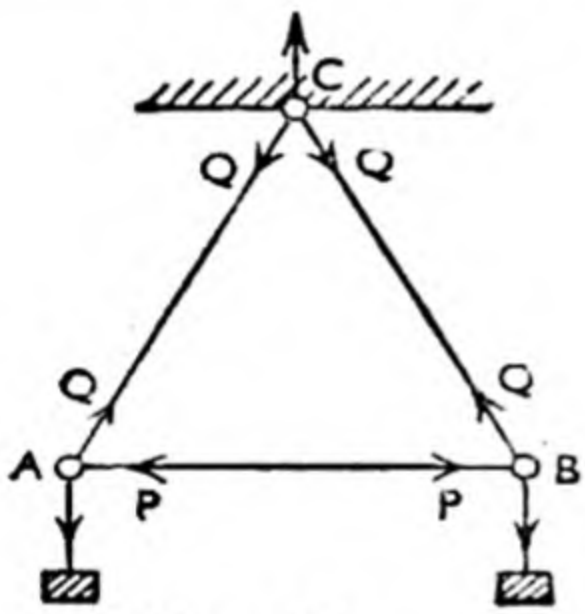

FIG. 204.

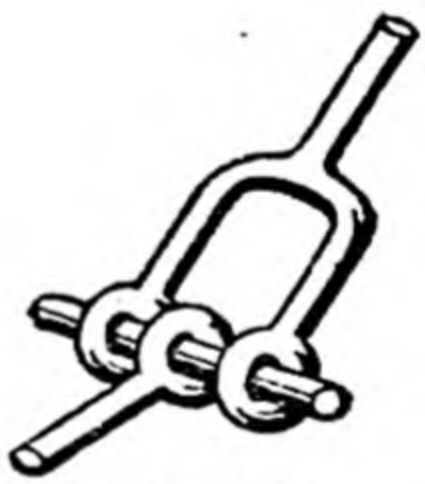

FIG. 205.

Consider the simple framework of three rods shown in the figure above.

The weights of the rods will be supposed too small to take into account in comparison with the stresses which the rods are called upon to bear.

The rods are of equal length and are freely jointed at A, B and C, *i.e.*, there is no friction at the hinges. A simple pin-joint of this type is shown.

The framework is supported at C and carries equal loads at A and B.

AB is acting as a strut, and is in compression. The arrow-

heads on the figure show the direction in which the rods act on the hinges. Thus the rod AB acts on the hinge B with a force P along the rod towards the hinge. At A the same rod acts with an equal force P along the rod towards the hinge. AC and BC are in tension and are therefore acting as tie-rods. The rod BC acts at B with a force Q along BC away from the hinge. The same rod acts at C with an equal force Q along the rod in the direction CB, away from the hinge C. The hinge B is therefore acted upon by three forces W, P and Q which are in equilibrium. If the force W is known, and the directions of P and Q are known, as in the above case, the triangle of forces may be drawn for these three forces, and the magnitudes of P and Q can therefore be found.

Example.—*A load of* 4 *cwt. is being lifted by a crane. The arrangement for this purpose is shown in the diagram.* ADE *and* BCE *are light chains attached to a small steel ring at* E, *and the ring is supported by the vertical chain* EH. AD *and* BC *are vertical and equal in length, and* DC *is a light but rigid wooden bar. Find the tensions in* BC, CE *and* EH, *and the stress in* CD.

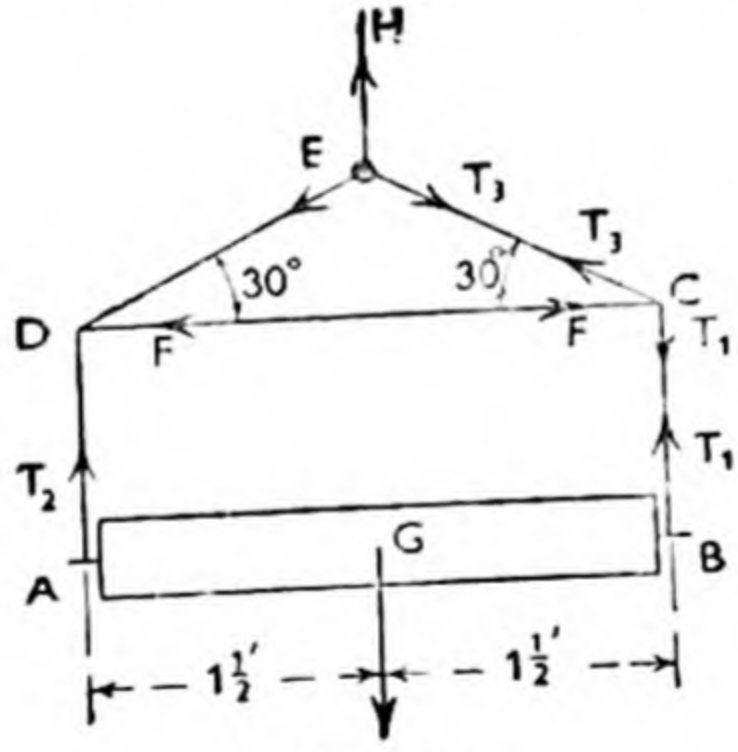

FIG. 206.

Consider the equilibrium of the load.

Referring to the figure, which shows the forces acting on the load of 4 cwt., if T_1 and T_2 cwt. are the tensions in BC and AD,

$$T_1 + T_2 = 4.$$

Take moments about G,

$$1\tfrac{1}{2}T_1 - 1\tfrac{1}{2}T_2 = 0.$$

$$T_1 = T_2 = 2 \text{ cwt.}$$

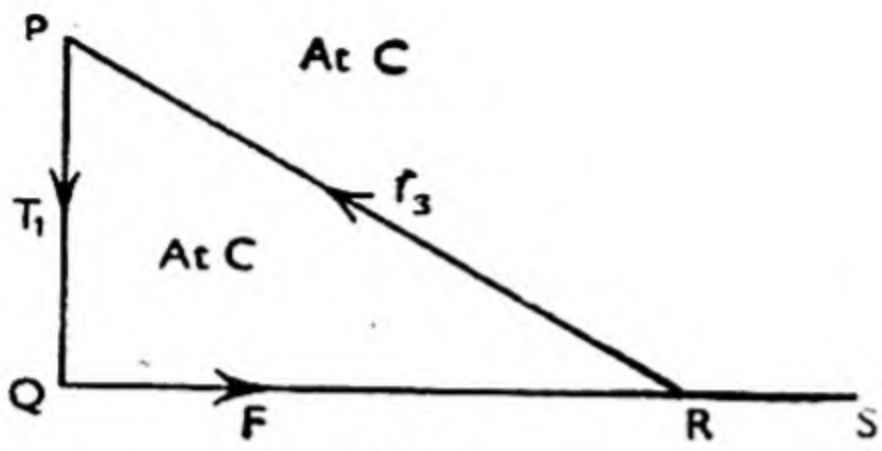

FIG. 207.

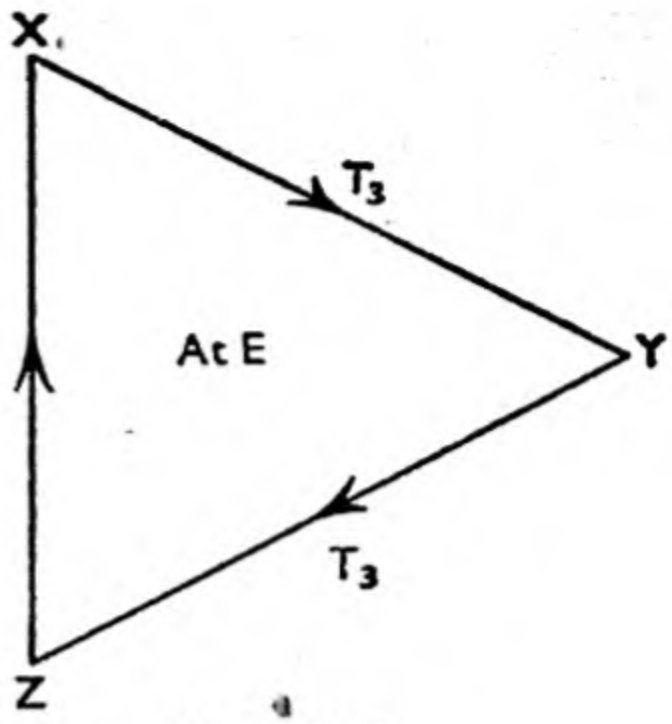

FIG. 208.

Consider the forces acting at C.

T_1 acts along CB, T_3 acts along CE, therefore F, the force in the rod DC, must act along DC at the point C, *i.e.*, the rod is acting as a strut.

Draw PQ to represent 2 cwt.,

QS // DC and PR // EC to meet QS at R.
PQ // line of action of T_1,
QR // " " F,
RP // " " T_3.

PQR is a triangle of forces.

By measurement,

$F = 3{\cdot}45$ cwt. $T_3 = 4{\cdot}0$ cwt.

Consider the forces acting at E.

By the symmetry of the arrangement the tension in DE is equal to the tension in CE.

Draw XY // EC to represent T_3 and YZ // ED to represent T_3.

ZX represents the tension in EH. Now the tension in EH supports the 4-cwt. load since the weights of chains and rod have been neglected, therefore this tension must be equal to 4 cwt. and provides a check upon the accuracy of the drawing.

124. Bridges.

Before a complicated structure is built the stresses likely to occur in its members are calculated, and the structure is designed so that each member can safely stand up to these stresses.

Even when extreme precautions are taken to ensure absolute safety, disasters sometimes occur. Thus in the case of the Quebec Bridge, completed in 1917, two accidents took place. In the first accident some of the members of the structure refused to

take the stresses to which they were subjected, and collapsed. On the second occasion the central span, which had been separately built, was towed on pontoons to the middle of the river, ready to be raised into position. While the span was being raised it broke away from the supports and fell into the river. A second span was built, and the bridge duly completed fifteen years after its commencement. In the case of the Sydney Harbour Bridge two years were spent in making the necessary preliminary calculations before the building was begun.

The Girder Bridge.—The simplest type of bridge is formed by supporting a plank at its ends. The plank sags a little under its own weight and still more when loaded. The sagging of the plank causes the upper part of the material of the plank to be compressed and the lower part to be stretched. Naturally, the uppermost parts of the plank are compressed most and the lowest parts are stretched most. The amount of sagging before

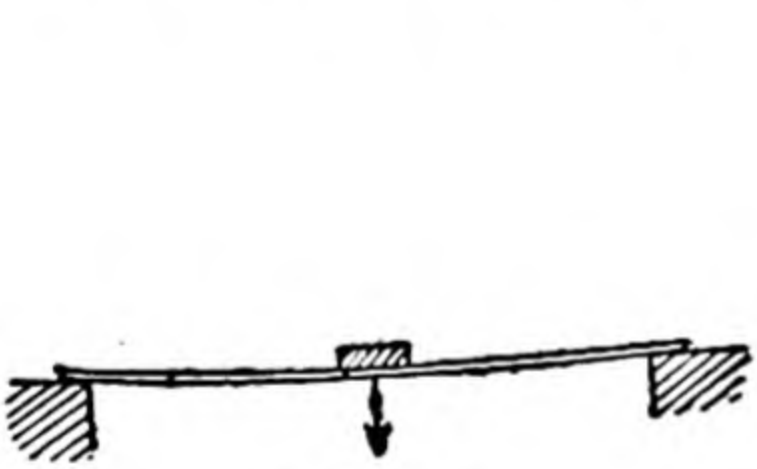

FIG. 209.

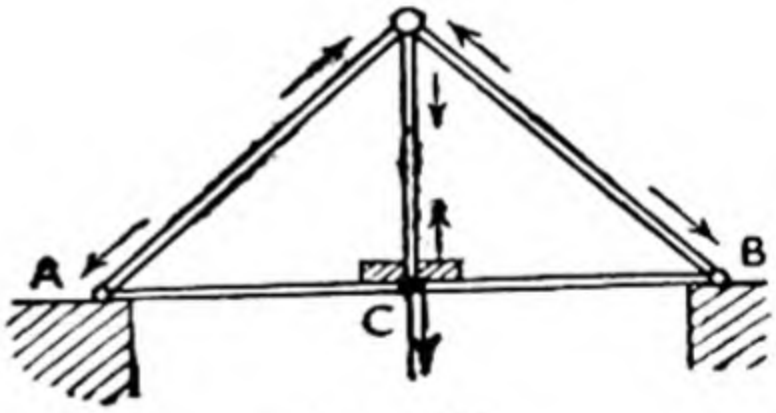

FIG. 210.

the beam snaps is dependent upon the maximum amount of stretching which the lowest parts of the plank can stand. To test this, bend an elastic piece of wood until it breaks, and notice how it breaks. See that the grain of the wood is parallel to its length, for if the grain is perpendicular to the length it will probably snap in a totally different way from that described.

To prevent a simple plank bridge from sagging too much we may stiffen it by the addition of a few more rods, such as AD, DB and DC. Consider the structure which is shown in the figure. When a load is carried by the plank AB it tends to sag.

The sagging is largely prevented, however, by the tie-rod DC, which transfers some of the load to the point D, where it is taken by the struts AD and BD. The structure is thus much stronger than the simple plank alone.

The simple structure which has been described may be regarded as a girder bridge in its most elementary form. It must be remembered that bridges have to take moving loads as well as stationary. The stiffening of the bridge by cross bracing may be observed in the picture. Girder bridges are particularly suitable where comparatively small spans may be used.

125. *The Suspension Bridge.*

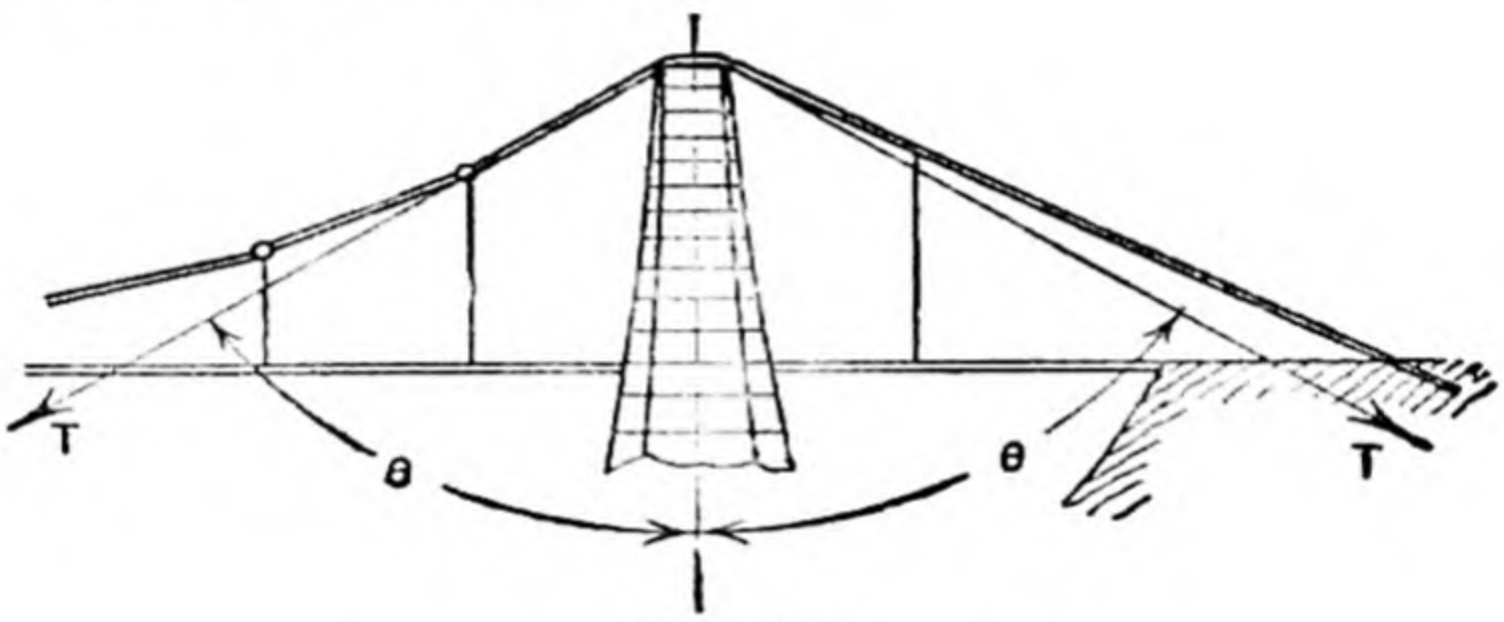

FIG. 211.

The suspension bridge consists of steel chains or cables supported from strong masonry piers or steel towers. The horizontal roadway is suspended from these chains or

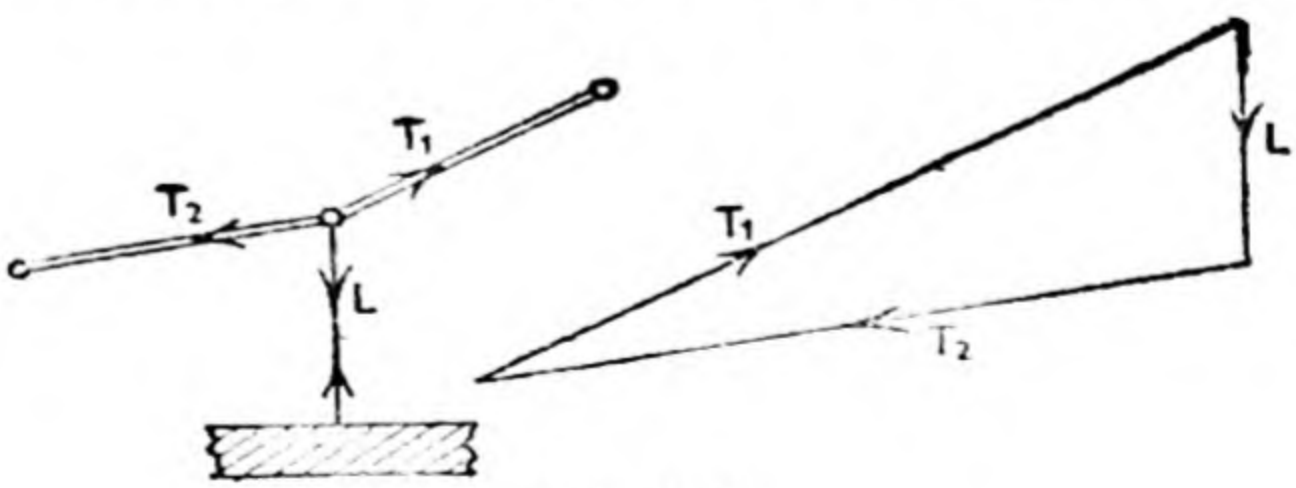

FIG. 212.

cables by vertical steel tie-rods. Two points are worth noticing: firstly, that since the chain links at a point where a tie-rod is supported make a small angle with each other, the tensions in the links are large in comparison with the vertical load taken by the tie-rod, as may readily be seen by drawing the triangle of forces at such a point; secondly, that the chains are so arranged at the supporting towers that the resultant force on the tower is vertically downwards, and in this way the tower is called upon to take a force of compression only, since the horizontal components of the tensions in the chains balance each other's effect on the tower.

126. *The Arch Bridge.*—It is sometimes difficult to find suitable foundations in a river-bed for the support of piers. In such a case it may be necessary to build a very large span. The arch type of structure is suitably strong and rigid for this purpose. The arch consists of a rigid structure to which vertical tie-rods are attached, and from these rods the roadway is suspended.

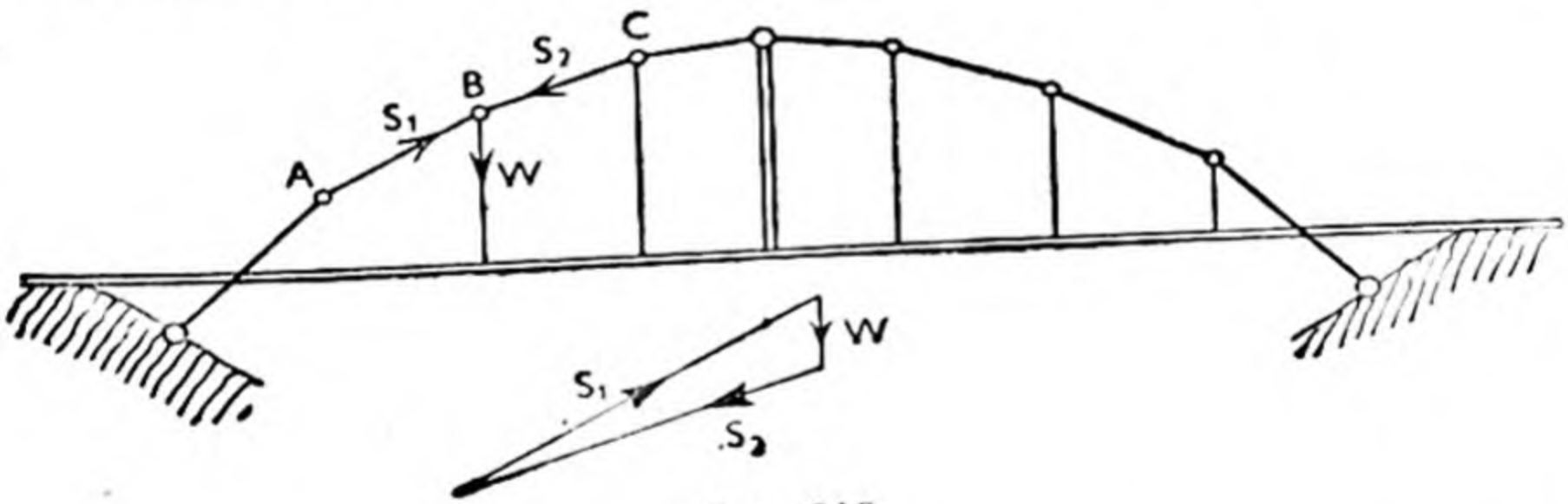

FIG. 213.

The skeleton diagram above ABC . . . shows the way in which the structure takes the load. For the sake of simplicity the bars which make the structure rigid are omitted.

Consider a point such as B. The rods AB and BC are in compression and therefore act as struts. The triangle of forces is shown for the forces W, S_1 and S_2 acting at B. It will be noticed that the struts take stresses which are large in comparison

with the load which they carry. At the towers which support the arch the end thrusts are exceedingly large and are therefore carried by huge masonry piers or steel towers of enormous strength. The central span of the Sydney Harbour Bridge is 550 yds. wide, and is the largest single-span arch in the world. Consider the structure and see if you can decide which members are ties and which are struts.

127. *The Cantilever Bridge.*

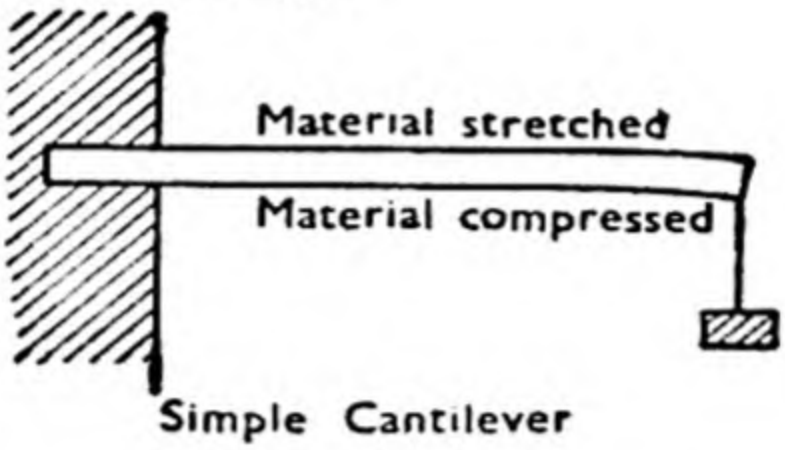

FIG. 214

A simple cantilever consists of a beam fixed at one end, as shown in the above figure. Bridges which make use of the cantilever principle, however, scarcely look like this simple structure. The cantilever parts of the Firth of Forth Bridge, which is taken as an example, are built out from a central tower of great strength, as indicated in the skeleton diagram below. The upper part of the cantilever structure, which corresponds to

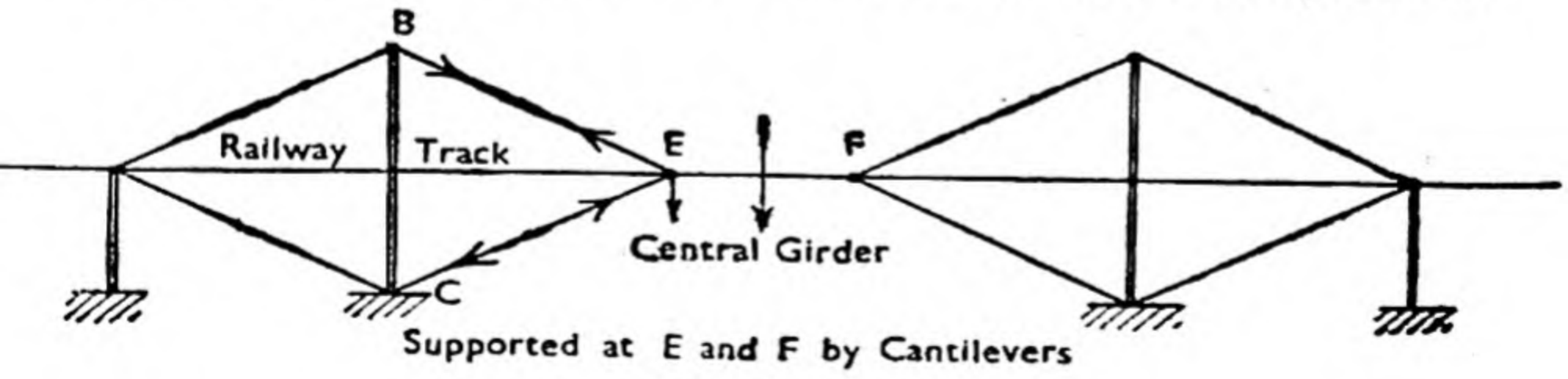

FIG. 215.

BE, is in tension, and the lower part, which corresponds to CE, is in compression. The directions of forces acting on one part of the structure are shown in the figure (see also Frontispiece).

128. Lami's Theorem.*

If three forces acting at a point are in equilibrium, each is proportional to the sine ratio of the angle between the other two.

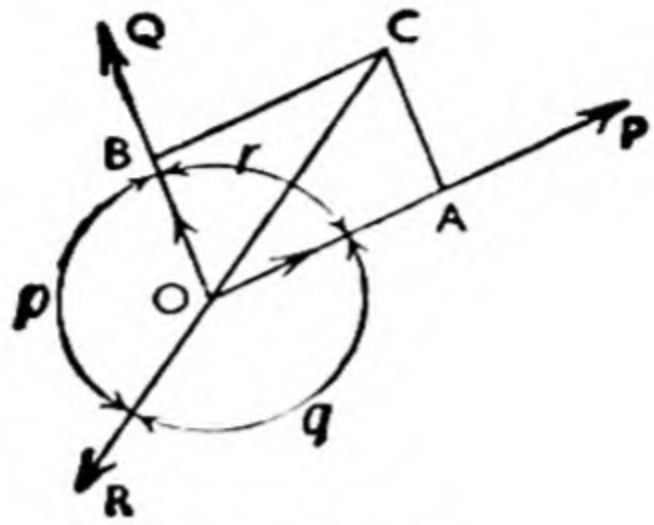

FIG. 216.

Let P, Q, R be the three forces acting at O.

Let OA represent P, and OB represent Q. CO represents R in magnitude and direction.

Let p, q, r be the angles between Q and R, R and P, P and Q respectively.

In the triangle OAC,

$\angle ACO = \angle BOC = 180° - p$, therefore $\sin ACO = \sin p$.

$\angle COA = 180° - q$, ,, $\sin COA = \sin q$.

$\angle CAO = 180° - r$, ,, $\sin OAC = \sin r$.

Also, by the sine rule,

$$\frac{OA}{\sin ACO} = \frac{AC}{\sin COA} = \frac{CO}{\sin OAC}.$$

Hence
$$\frac{P}{\sin p} = \frac{Q}{\sin q} = \frac{R}{\sin r}.$$

* So called since it was first stated by Father Bernard Lami in 1687.

Example.—*A weight of* 10 *lb. is supported by two strings* AC *and* BC *inclined respectively at* 45° *and* 55° *to the vertical, the weight being attached at* C. *Find the tensions in the strings,* (1) *by means of a scale diagram,* (2) *by calculation.*

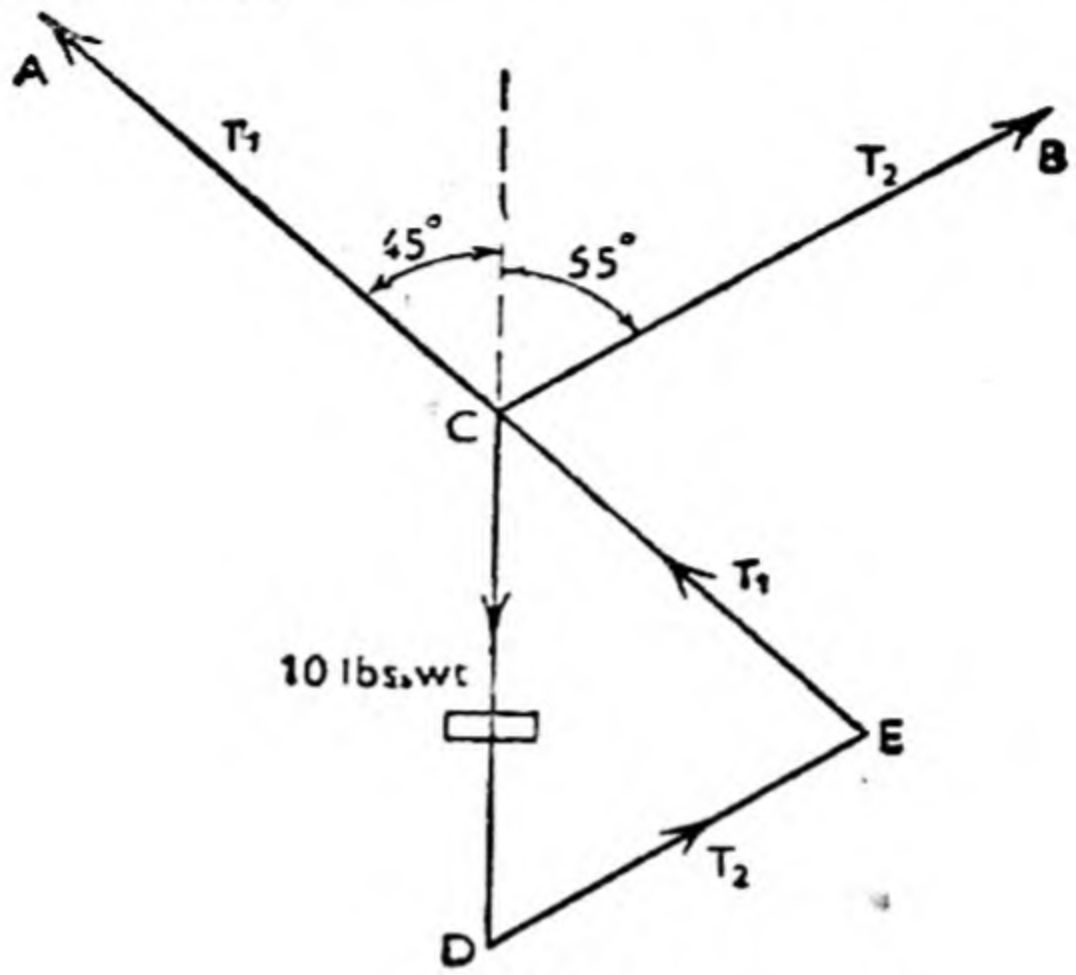

FIG. 217.

(1) Let the tensions in AC and BC be T_1 and T_2 respectively.

Draw CD, 10 cm. in length, to represent the weight, 10 lb.

Draw DE and EC parallel to the lines of action of T_1 and T_2 respectively.

CDE is a triangle of forces.

DE represents T_1 in magnitude and direction.

EC ,, T_2 ,, ,, ,,

By measurement,

$$DE = 7{\cdot}2, \text{ therefore } T_1 = 7{\cdot}2 \text{ lb. wt.}$$

$$EC = 8{\cdot}3, \quad ,, \quad T_2 = 8{\cdot}3 \text{ lb. wt.}$$

(2) *By calculation.*

The angle between the lines of action of T_1 and W is $(90° + 45°) = 135°$.

,, ,, ,, W and T_2 is $(90° + 35°) = 125°$.

,, ,, ,, T_2 and T_1 is

$$(360° - 260°) = 100°.$$

Applying Lami's theorem,

$$\frac{W}{\sin 100°} = \frac{T_1}{\sin 125°} = \frac{T_2}{\sin 135°}.$$

$\sin 100° = \sin 80°$, $\sin 125° = \sin 55°$, $\sin 135° = \sin 45°$.

To find T_1,

$$\frac{T_1}{\sin 55} = \frac{10}{\sin 80}.$$

$$T_1 = \frac{10 \cdot \sin 55}{\sin 80} = \frac{10 \times 0{\cdot}8192}{0{\cdot}9848}.$$

$$\begin{aligned} \text{Log } T_1 &= \log 8{\cdot}192 - \log 0{\cdot}9848 \\ &= 0{\cdot}9134 - \bar{1}{\cdot}9934 \\ &= 0{\cdot}9200 \\ &= \log 8{\cdot}316 \\ T_1 &= 8{\cdot}32. \end{aligned}$$

The tension in AC is 8·32 lb. wt.

Similarly, it may be shown that the tension in CB is 7·18 lb. wt.

EXAMPLES XXXIV

Triangle of Forces

1. Three strings, OA, OB and OC are knotted together at O. OA and OB are on opposite sides of the vertical through O, and are inclined to it at 40° and 60° respectively. Find the tensions in OA and OB when OC hangs vertically and supports a weight of 10 lb.

2. Three strings OA, OB, OC are knotted at O, and the tensions in them are 6, 7 and 8 lb. wt. respectively. If OC is vertical, find the directions at which the strings OA and OB are inclined to the vertical.

3. AB and BC, the jib and tie of a jib crane, are inclined at 45° and 60° to the vertical, and $\angle ABC$ is 75°. A load of 5 cwt. is carried at B. Find the forces exerted by the jib and tie-rod at B.

4. A rope supports a load of 60 kgm. A second rope is knotted at a point in the first, and the first rope is drawn aside by the tension in the second. If the second rope is horizontal and the tension in it is 30 kgm., find the angle at which the first rope is now inclined to the vertical and the tension in this inclined part of it.

5. The rods AB, BC are light, they are smoothly hinged at B. A

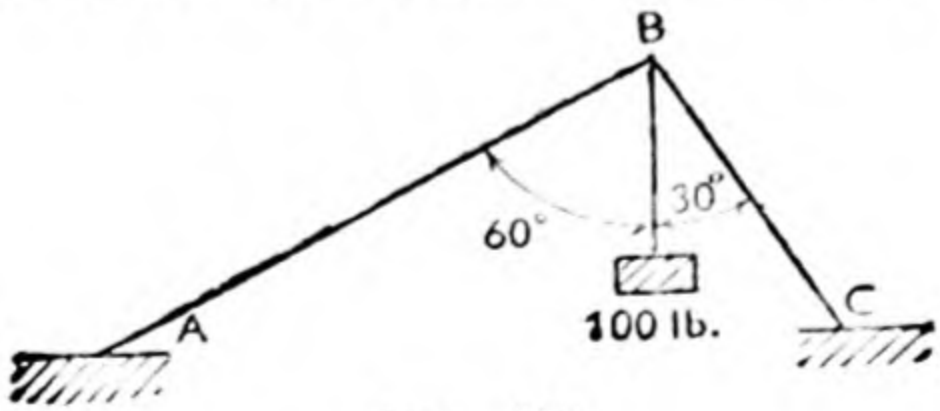

FIG. 218.

100-lb. load is carried at B. Find the forces exerted at B along the rods AB and CB.

6. Find in the following cases the single force which will balance two forces of 7 and 9 lb. wt. respectively, whose lines of action are inclined at (*a*) 60°, (*b*) 120°.

7. A body of weight 5 lb. rests on a smooth plane inclined at 30° to the horizontal, and is kept at rest by a string which is parallel to the plane. Find the reaction of the plane on the body and the tension in the string.

8. A body is held at rest on a smooth plane by a horizontal string. The tension in the string is 1 kgm., and the weight of the body 2·5 kgm. Determine the reaction of the plane on the body, and hence the angle at which the plane is inclined to the horizontal.

9. A rope ABCD is fastened to fixed points at A and D, and supports

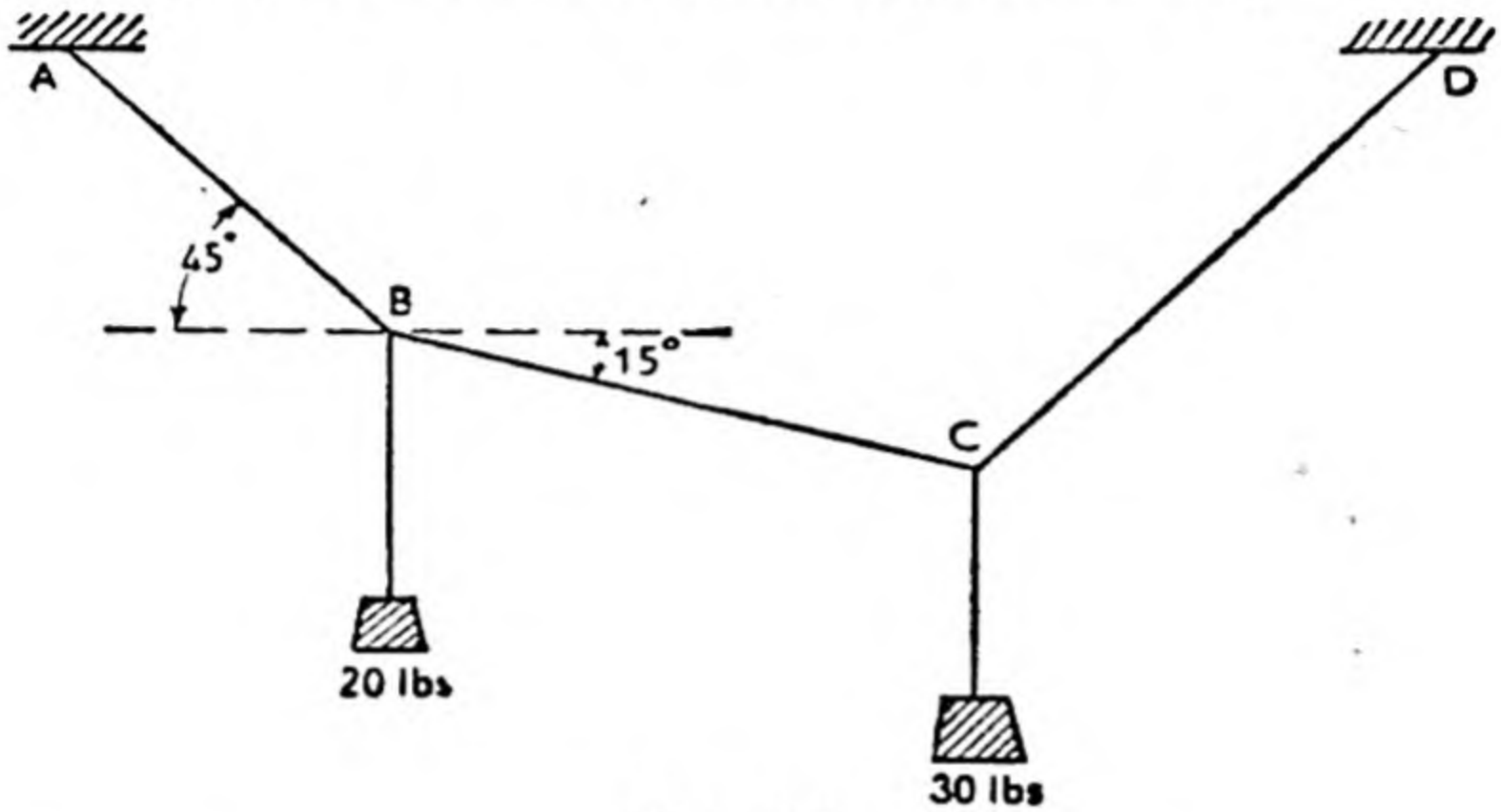

FIG. 219.

weights of 20 and 30 lb. at B and C respectively. AB and BC are inclined at 45° and 15° to the horizontal. Find the tension in AB, BC and finally in CD, and the angle at which CD is inclined to the horizontal. (The figure shown is not intended to be accurate.)

10. AB, BC are two links in a suspension bridge inclined at 10° and

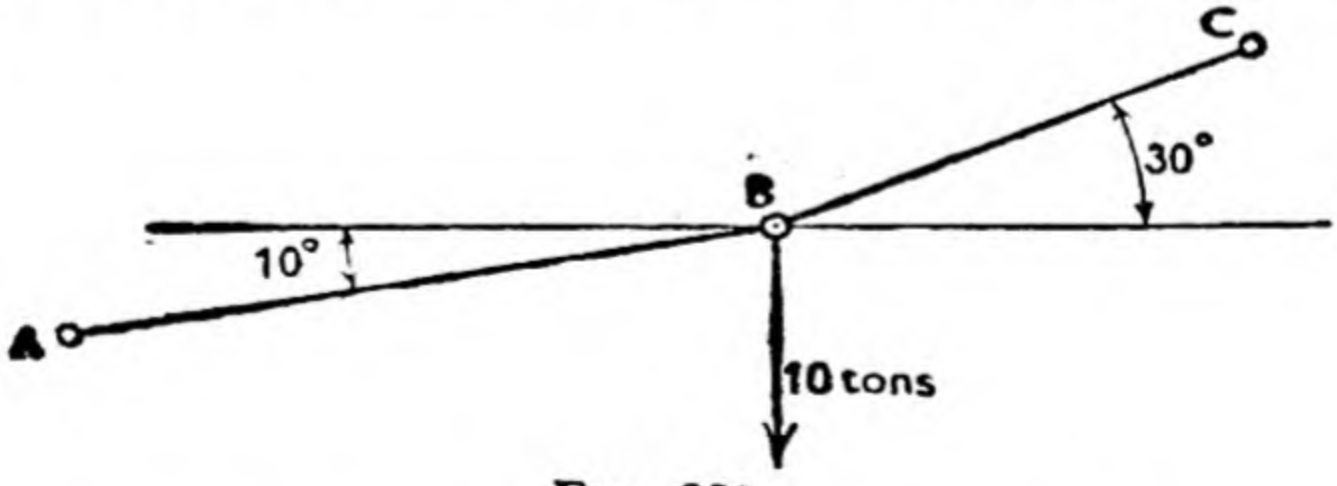

FIG. 220.

30° to the horizontal respectively. If a vertical load of 10 tons is carried at B, determine the tensions in AB and BC.

11. The two girders AB, AC are anchored at A, and they are each inclined at 45° to the horizontal. If the resultant force on the foundation

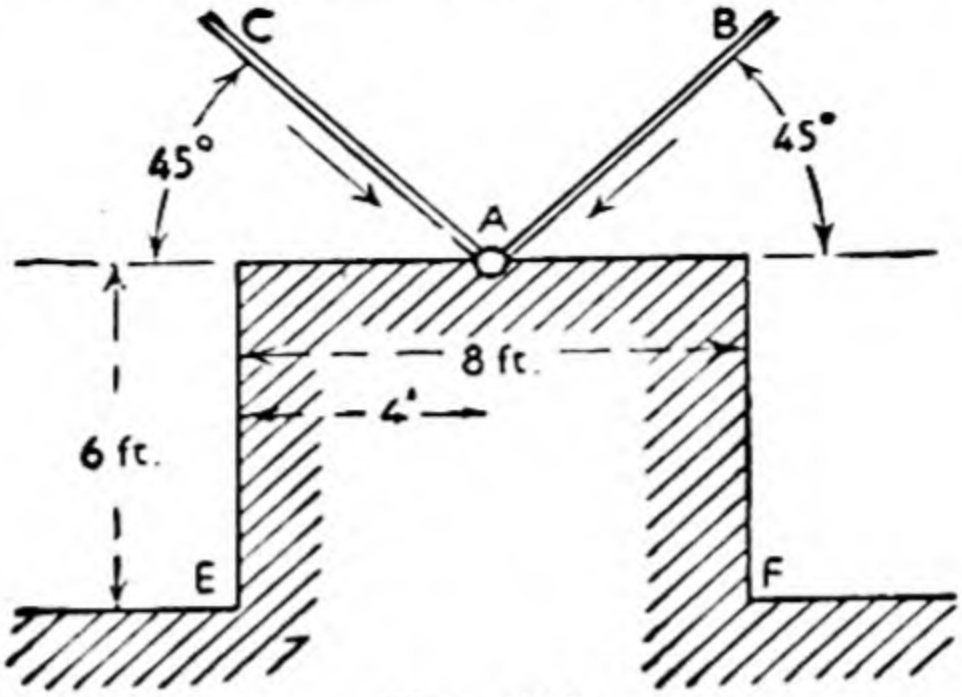

FIG. 221.

must not exceed 400 tons and must not pass outside the base EF, find the greatest amount by which the thrust in one bar may exceed that in the other.

12. Show that the resultant of two forces which act at a point diminishes as the angle between their lines of action increases.

A 28-lb. wt. is supported by two cords, one of which makes an angle of 60° with the vertical. What must be the direction of the second cord so that the tension in it may be the least possible? Find the tensions in the two cords in this case. (*G.S.*)

13. The ends of a string 2 ft. long are fastened to two nails A and B 16 in. apart horizontally. A weight of 10 lb. is attached to a point C in the string 10 in. from A, and hangs supported by the tensions in CA and CB. Determine graphically these tensions. (*G.S.*)

14. A 1 kgm. weight is suspended by a string 70 cm. long from a 2 kgm. weight, itself suspended from a fixed point by a string 50 cm. long. A horizontal force of 500 gm. wt. is applied to the lower weight. How far will the upper weight be displaced horizontally? Find also the tensions in each of the strings. (*G.S.*)

15. State the rule known as the triangle of forces.

Make use of it to determine the mechanical advantage of a plane inclined at an angle of 30° to the horizontal when the effort is (*a*) parallel to the length of the plane, (*b*) horizontal. Considering the plane as smooth, calculate the reaction of the plane in each case. (*L.M.*)

16. A uniform rod is 8 ft. long and weighs 10 lb. Draw a large-scale diagram showing the rod supported by two strings of lengths 4 and 6 ft., which are attached to the ends of the rod and to a common support. Find, graphically or otherwise, the tension in each string. (*L.M.*)

17. Explain how the triangle of forces is used to test the equilibrium of a body which is acted upon by three forces which pass through a point in it. A light string ABCD is supported at A and D, which points are in the same horizontal plane. At B is a mass of 20 lb., and at C a mass M. The string AB is at 45° to the vertical, DC is at 30° to the

vertical and BC slopes upwards at 45° to the horizontal. Find the tensions in the different parts of the string and the value of M. (*L.M.*)

18. State the theorem of the triangle of forces.

A uniform rod AB of mass 500 gm. is supported by two strings AC, BC fastened to a nail at C, and the strings make angles with the vertical of 45° and 30° respectively. By using the triangle of forces or otherwise, find the tensions in the strings. Find by diagram the angle the rod makes with the vertical. (*J.*)

19. Three strings are knotted together at one end, the other ends being attached to weights of 5, 7 and 8 lb. The 8-lb. weight hangs vertically and is supported by the other two, the strings of which pass over smooth pulleys. Draw a force triangle, from it obtain a structure diagram showing the strings in their true position. (*L.M.*)

20. State the proposition known as the triangle of forces. From the bob of a simple pendulum another simple pendulum is suspended. The lower bob is drawn aside by a horizontal string, and when the lower pendulum makes an angle of 45° with the vertical the upper is inclined at 30° to the vertical. If the lower bob weighs 10 lb., find the weight of the upper. (*L.M.*)

21. Find in tons the tension of the tie BC and the thrust in the strut

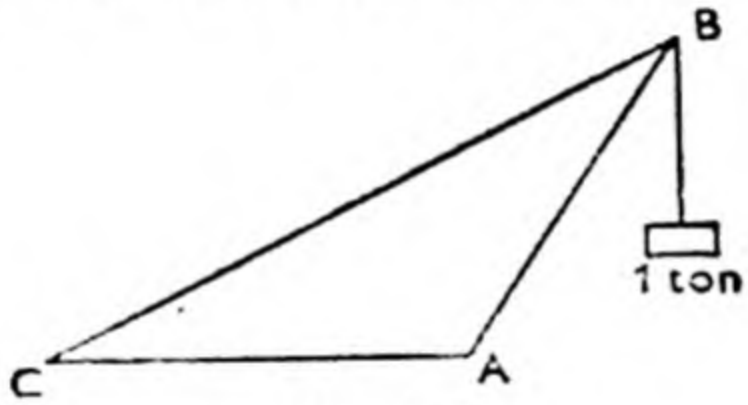

FIG. 222.

AB shown in the figure, which represents (not to scale) the framework of a derrick crane. AB = 20 ft., BC = 50 ft., AC (horizontal) = 40 ft. (*L.M.*)

22. The figure represents a jointed framework of four light rods

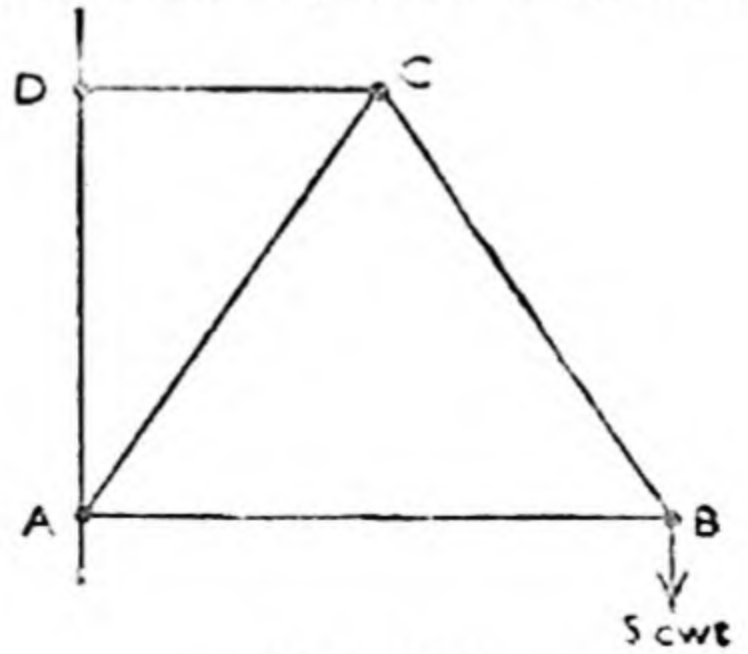

FIG. 223.

AB, BC, CA, CD, of which AB, BC, CA are of equal lengths and CD and AB are horizontal, attached to a vertical wall at A and D and carrying a load of 5 cwt. at B. Determine graphically or otherwise the stresses in the rods. (*O.C.*)

23. State the proposition known as the triangle of forces. Find the least force which, applied to a pendulum bob weighing 25 gm., will maintain it at an angle of 30° to the vertical. Find also the tension in the string. (*G.S.*)

24. Two weights of 20 and W gm. are attached at D and E to a string DAE, which passes over a smooth pulley at A, the common apex

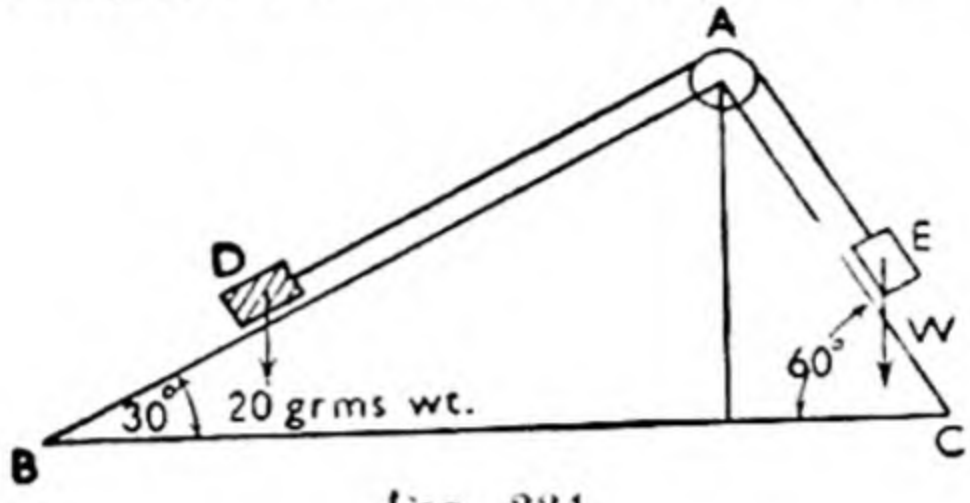

FIG. 224.

of two inclined planes, the two weights resting on the smooth planes BA and CA, inclined at 30° and 60° respectively to the horizontal. Find the tension in the string and the value of W.

129. The Polygon of Forces.

If any number of forces, acting at a point, can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.

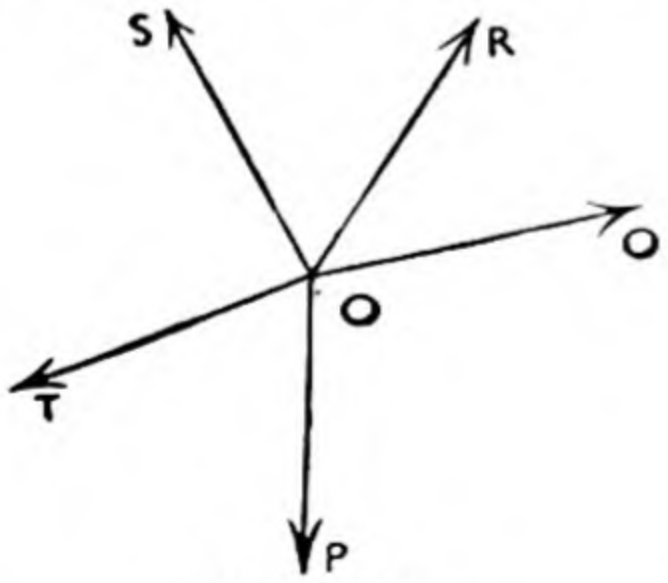

FIG. 225.

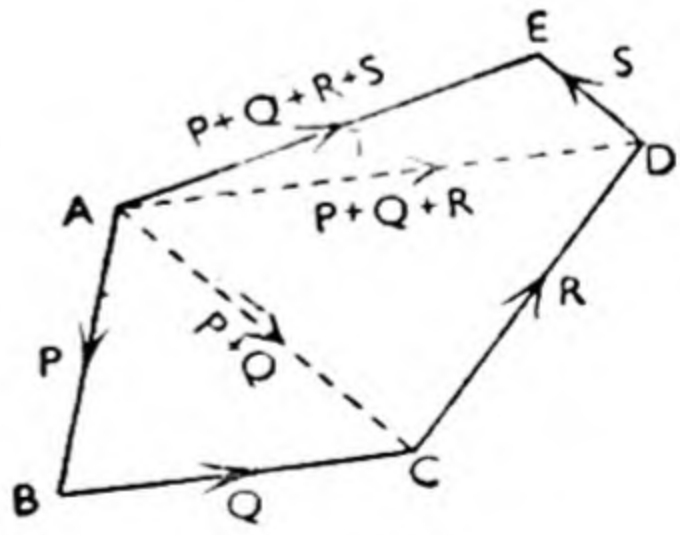

FIG. 226.

Let the sides AB, BC, CD, DE, EA of the polygon ABCDE represent the forces P, Q, R, S, T acting at a point O on a body.

The resultant of P and Q is represented by AC in magnitude and direction.

The resultant of the forces represented by AC and CD is

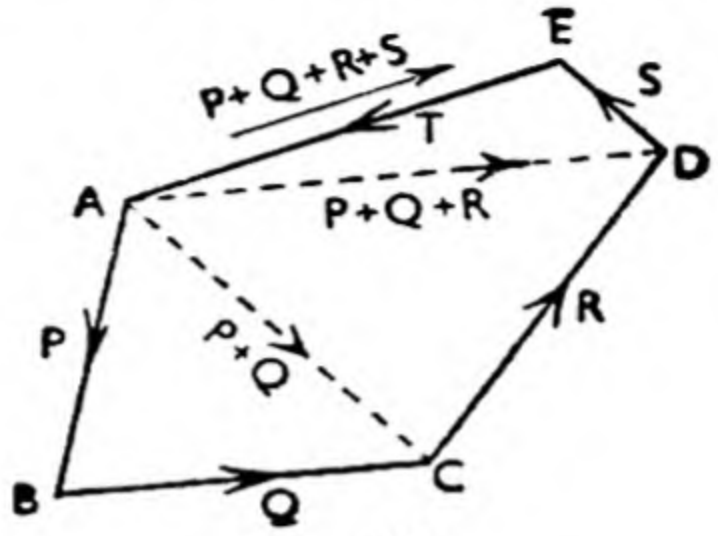

FIG. 227.

represented by AD. Therefore the resultant of P, Q and R is represented in magnitude and direction by AD.

Similarly, the resultant of P, Q, R, S is represented in magnitude and direction by AE, hence the resultant of P, Q, R, S is equal in magnitude but opposite in direction to the force T, and therefore the forces P, Q, R, S, T are in equilibrium, for they are known to act at a point.

The method of proof evidently applies whatever the number of forces.

If a number of forces act at a point, and we draw a polygon with its sides proportional and parallel to these forces (always drawing the next side of the polygon from the point at which the previous side ended), then if the polygon is closed we know that the forces are in equilibrium, but if the polygon does not close, as, for example, in the case of the figure ABCDEF shown below,

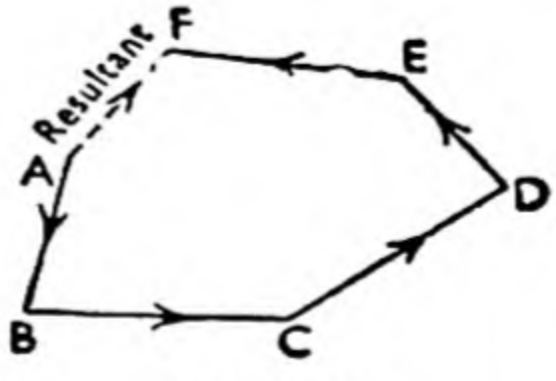

FIG. 228.

then it has also been shown above that the line which closes the polygon taken in the direction *away from the initial point of the polygon*, represents the resultant of the forces.

The order in which the forces are taken when the polygon is being drawn does not affect the resultant.

This method of finding the resultant of a number of forces acting at a point is much easier than repeating the application of the parallelogram of forces.

Example.—*The figure below shows an arrangement of a jib crane.*

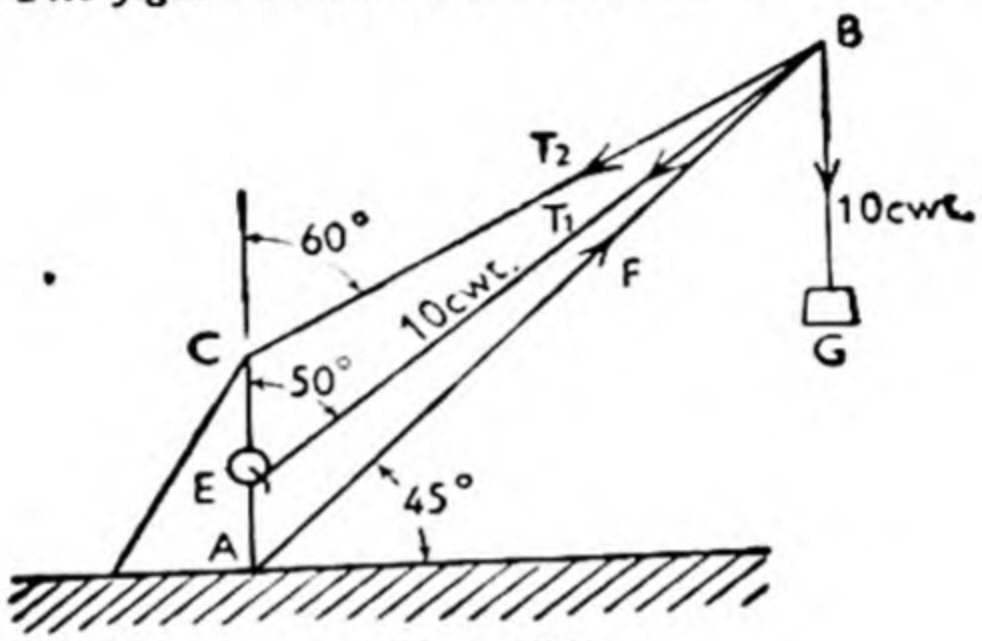

FIG. 229.

The load is supported at the end of the jib AB *by the rope* BG, *which passes over a pulley at* B *and so to the drum of a winch at* E.

BC *is the tie-rod.*

The directions of all the forces acting at B *are known, and the tensions in* GB *and* BE *are each equal to the weight of* 10 *cwt.*

It is required to find the magnitudes of the tension in the tie-rod BC *and the thrust in the jib.*

The forces acting at B are :—

(1) The tension along BG equal to 10 cwt.
(2) The tension along BE equal to 10 cwt. (T_1).
(3) The thrust F in the jib along AB.
(4) The tension in the tie-rod BC along BC. (T_2.)

Draw PQ to represent the vertical force of 10 cwt. (W.)
Draw QR to represent the force of 10 cwt. parallel to BE. (T_1.)
Draw RS parallel to AB to meet PS drawn parallel to CB at S.

Then PQRS is a polygon of force.
RS represents F in magnitude and direction.
SP " T_2 " " "

Note (1).—The known forces are taken first in drawing the force polygon.

Note (2).—If all the forces acting at a point are given in direction and all but two in magnitude, the magnitudes of the remaining two can be

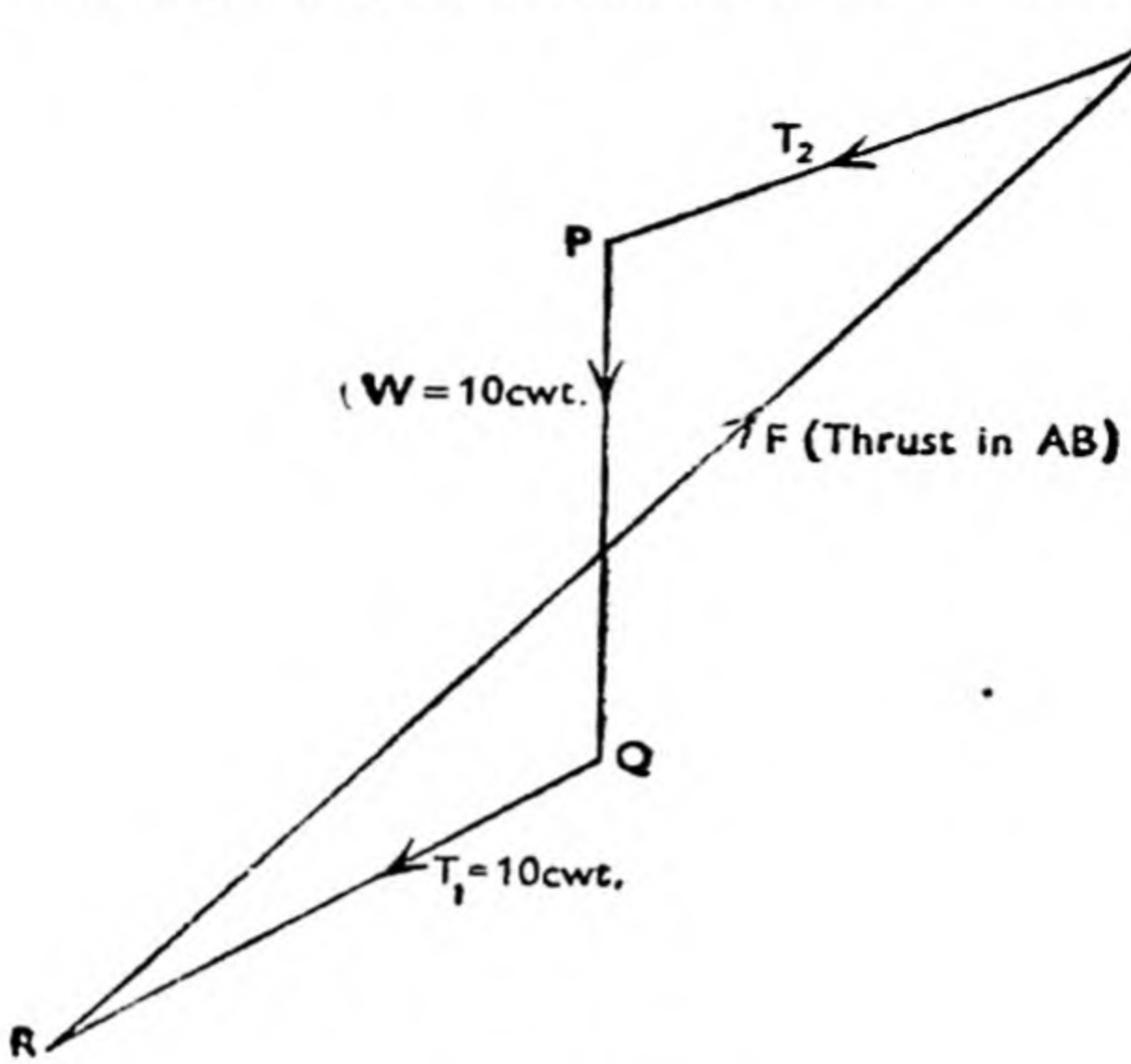

FIG. 230.

found by means of a force polygon, so that the system of forces may be in equilibrium.

EXAMPLES XXXV

POLYGON OF FORCES

1. Forces of 3, 4 and 5 lb. wt. act along OA, OB and OC. $\angle AOB = 30°$, $\angle BOC = 40°$. Find the magnitude of the resultant of these forces and state the angle which it makes with OA.

2. Forces of 100, 150 and 200 lb. wt. act along ropes OA, OB and

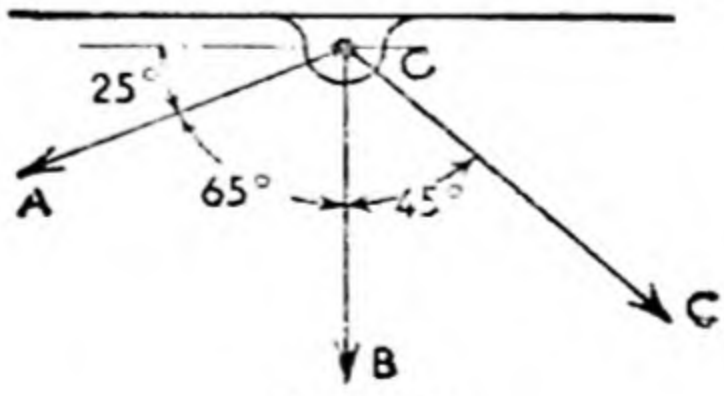

FIG. 231.

OC. OA is inclined to the horizontal at 25°, $\angle AOB = 65°$ and $\angle BOC = 45°$. Find the resultant force on the support at O.

3. Five forces 1, 2, 3, 4, 5 kgm. act at a point, and their lines of action make angles of 72° with each other. Find the magnitude of their resultant and the angles which it makes with the lines of action of the 3 and 5 kgm. forces.

4. There are four overhead wires attached at a point, and the directions of three of them are 35° west of north, 30° east of north, due east. The tensions in these three wires are 50, 75 and 30 lb. wt. respectively. Determine the direction of the fourth wire and the tension in it.

5. Two equal thin rods AB and BC, each of length 5 ft., are hinged at B and secured at A and C, which are at the same horizontal level, B being below the level AC. AC = 6 ft. A load of 400 lb. is carried at B, and a horizontal force of 100 lb. wt. is also applied at B. Find the magnitudes of the forces exerted by the rods at B and the directions along the rods in which they act.

6. If in Example 5 the horizontal force is increased to 400 lb. wt., find the forces exerted by the bars at B and the directions along the bars in which they act.

7. A load of 2 tons is carried at D by a jib crane. The rope supporting the load passes over a smooth pulley at D and passes then to a winch

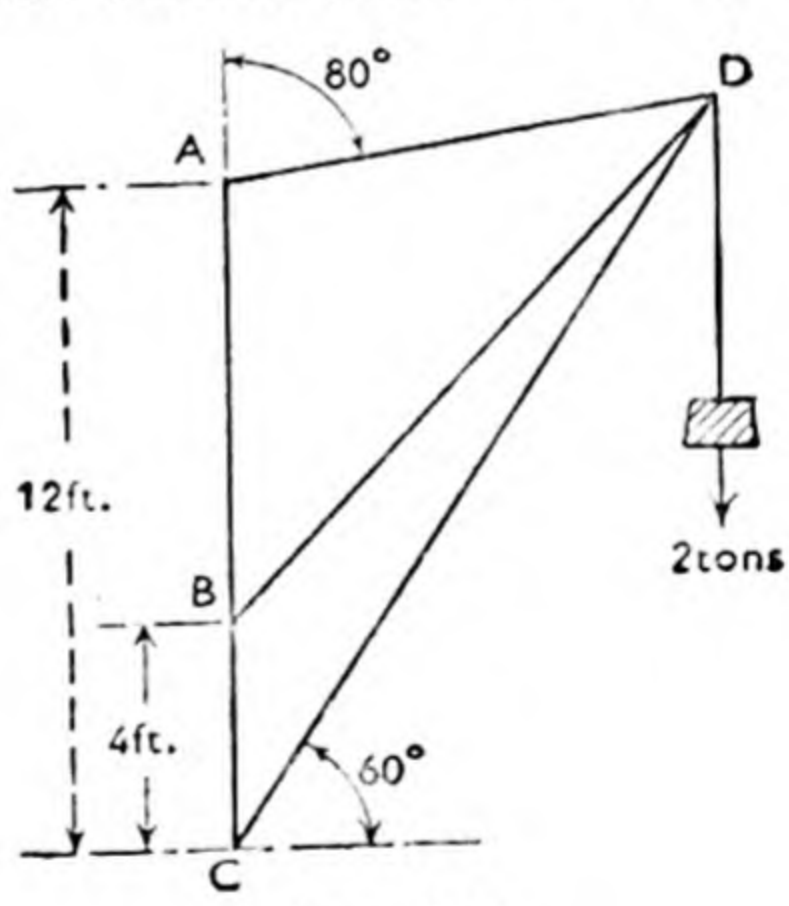

FIG. 232.

at B. The tie AD is inclined at 80° to the vertical and the jib CD at 60° to the horizontal. Find the forces exerted at D by the tie-rod AD and the jib CD.

8. Forces act at a point O in the directions OA, OB, OC and OD. $\angle COD = 120°$, $\angle DOA = 30°$, $\angle AOB = 75°$. The forces along OA, OB

are respectively 1 and 1·5 tons wt. Find the magnitude of the forces along OC and OD.

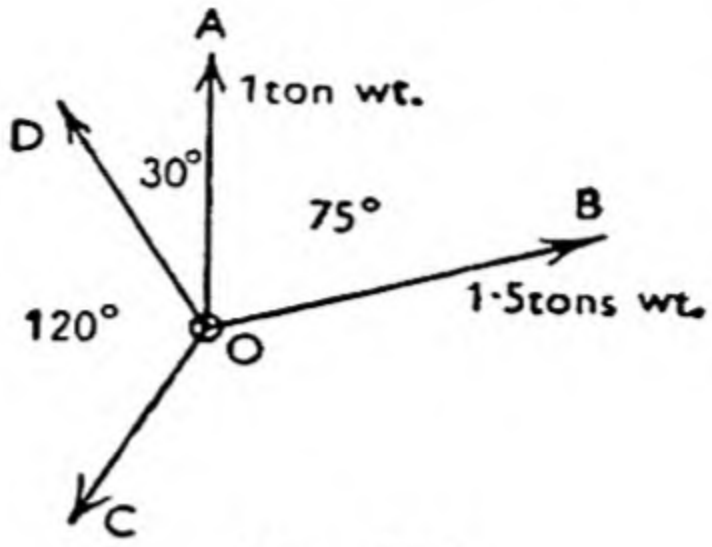

FIG. 233.

9. The figure shows a network of wires at rest. Determine the forces exerted at B by the three wires BA, BC and BD.

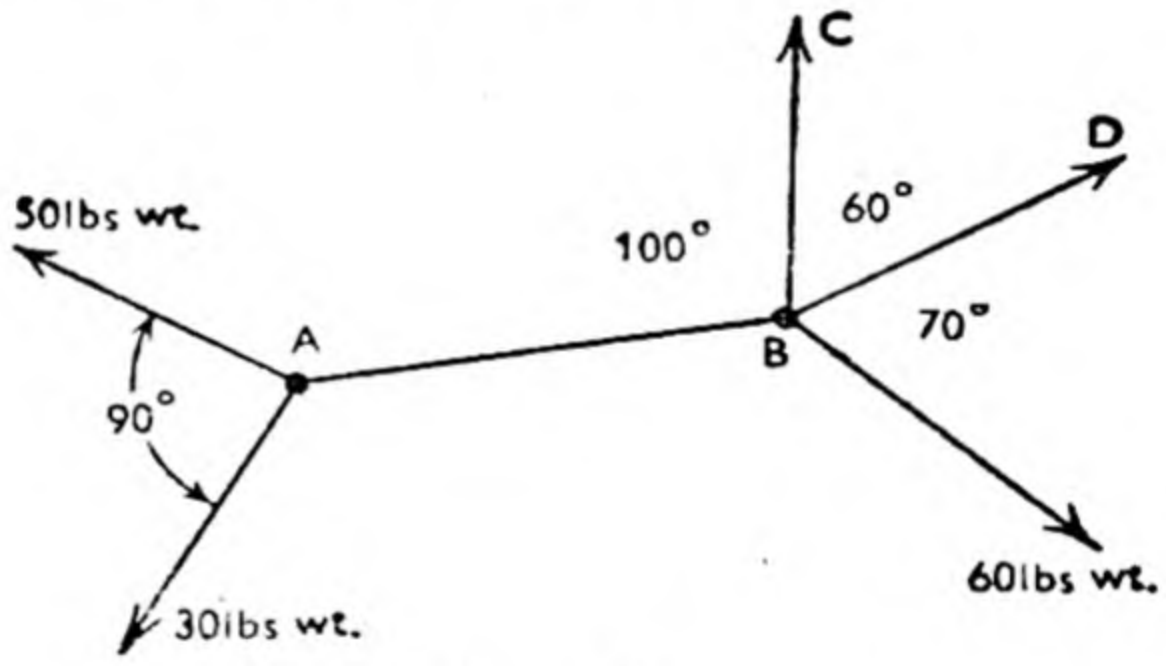

FIG. 234.

10. Four forces act at a point and are in equilibrium, 5 lb. along OA, 2 lb. along OB, $\angle AOB = 45°$, P lb. along OC, $\angle BOC = 105°$, Q lb along OD, $\angle COD = 90°$. Find the magnitudes of the forces P and Q.

11. Describe how you would ascertain whether a body would be in equilibrium under the action of three co-planar forces which pass through a point. Three forces pass through a point in a body and act outwards. The forces are : A, of 5 lb. wt. ; B, of 10 lb. wt. acting at 60° to A ; and C, of 15 lb. wt., at 210° to A, measured in the same direction. Is the body in equilibrium ? If not, what is the resultant of the forces and what single force will produce equilibrium ? (The problem may be solved graphically, but in any case a diagram is required showing the direction of the forces.)

(*L.M.*)

CHAPTER XIII

RESOLUTION OF FORCES

130. EXPERIMENT.—Three string are knotted together at A and two pass over smooth pulleys at B and C, and carry weights W_1 and W_2 at

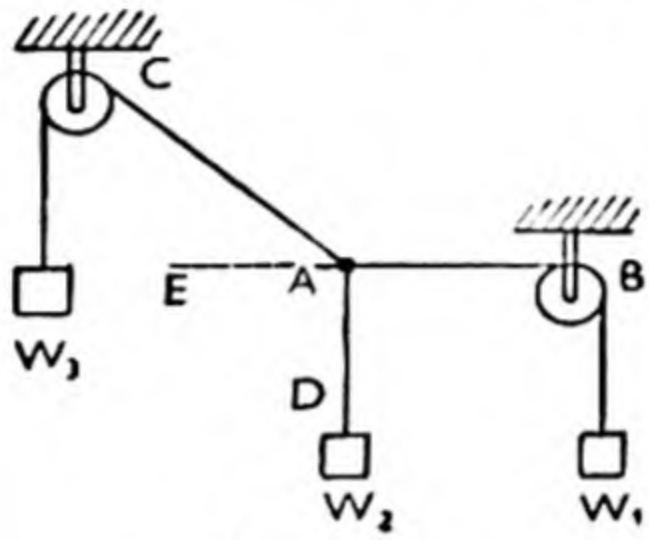

FIG. 235.

their other ends. The third string supports a weight W_2 at its end D. The tensions in the string are arranged so that the angle BAD is a right angle.

The three weights are noted and the angle made by AC with the horizontal, viz., $\angle CAE$.

Keeping W_1 constant, the other two weights are altered, together with the angle CAE, and the latter two are so arranged that $\angle BAD$ remains a right angle.

The experiment is repeated for other values of the angle CAE, and the values of W_3, W_2 and $\angle CAE$ are tabulated for all four experiments.

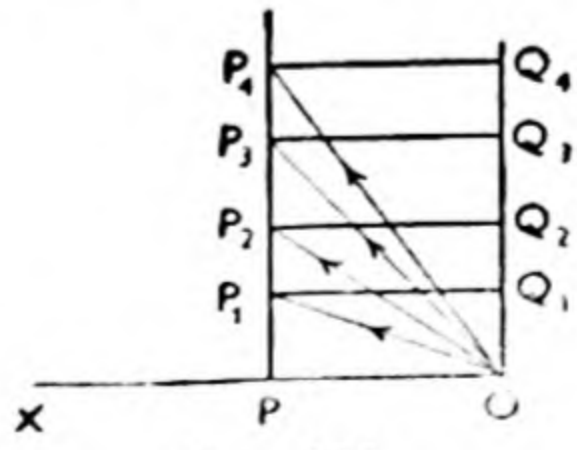

FIG. 236.

From a point O, set off OP_1, OP_2, OP_3 and OP_4 to represent the values of W_3 in their appropriate directions, taking a line OX to represent the horizontal direction. The rectangles having OP_1, etc., for their diagonals are completed, *i.e.*, OPR_1Q_1, etc.

From the figure the first thing to notice is that P_1, P_2, P_3 and P_4 are in one straight line perpendicular to OX.

Now OP represents the component of each of the forces represented by OP_1, OP_2, OP_3 and OP_4 horizontally. These components have now been shown to be equal, and by measuring OP we find that their common value is equal to W_1. This is what may be expected, for the horizontal component of W_2 acting at A is balanced by the same horizontal force W_1 in each case.

Moreover, the vertical components of the forces W_3 are represented by OQ_1, etc., and by measurement it is found that within the limits of experimental error these lines represent forces equal in magnitude to the forces W_2. The vertical components of the forces W_3 are therefore in equilibrium with the vertical forces W_2.

Where it is convenient a single force may be replaced by its two components in two mutually perpendicular directions acting at any point on the line of action of the force. The directions of the components may be chosen at will. These components of a force are sometimes referred to as the resolved parts of the force in the two perpendicular directions, and the force is said to be resolved in the two directions.

131. Resolved Parts.

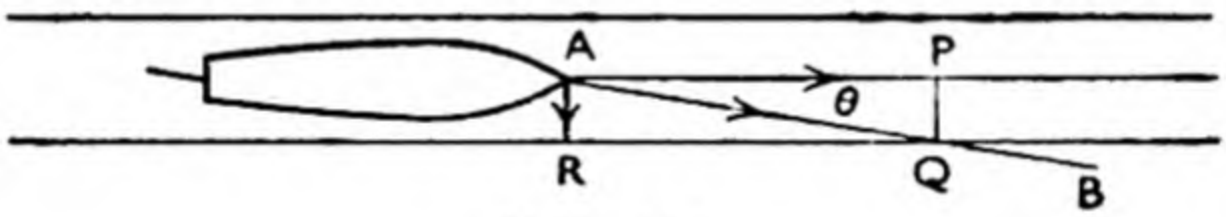

FIG. 237.

The above figure represents a barge drawn by a single horse. The barge does not move in the direction of the tension in the rope AB. The effect of the force acting on the barge *in the direction in which the barge is moving* may be found by drawing AQ to represent the tension in the rope, and completing the rectangle APQR, AP being in the direction in which the barge is moving and PQ perpendicular to AP.

The tension in the rope may be looked upon as the resultant of the forces represented by AP and AR.

The force represented by AR only tends to draw the barge in a direction perpendicular to that in which it is moving, and its effect is counteracted by the action of the rudder. This force neither helps nor hinders the motion of the barge along the canal.

The force represented by AP is therefore the whole effect of the tension in the direction AP.

The whole effect of a force in a given direction is called the resolved part of the force in that direction.

Thus AP represents the resolved part of the tension in the direction AP.

From the right-angled triangle APQ,

$$\cos\theta = \frac{AP}{AQ},$$

$$AP = AQ\cos\theta.$$

The resolved part of T along AP making $\theta°$ with AQ is

$$\mathbf{T}\cos\theta.$$

Again,
$$\sin\theta = \frac{PQ}{AP},$$

$$PQ = AP\sin\theta.$$

Now PQ = AR, therefore the resolved part of T along AR is

$$\mathbf{T}\sin\theta.$$

132. Sailing into the Wind.

A ship is said to sail into the wind if it makes headway against the wind, *i.e.*, if the direction in which the ship moves makes an acute angle with the direction from which the wind is blowing.

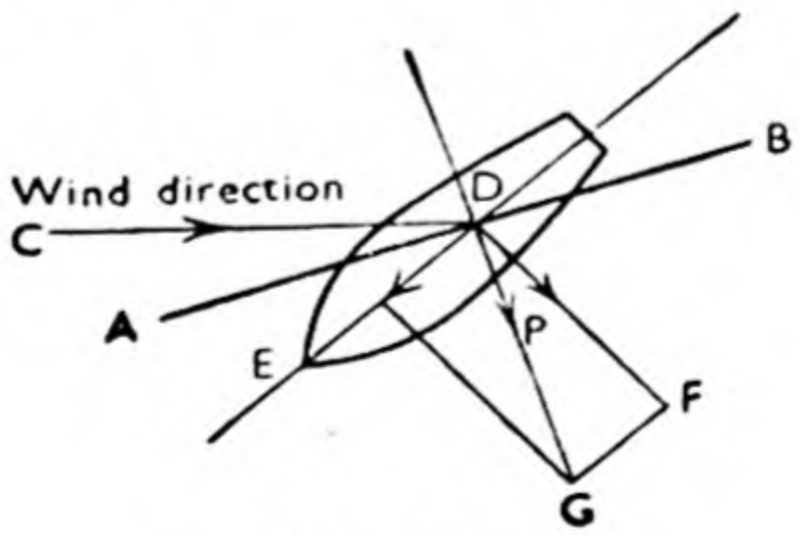

FIG. 238.

Let AB represent the plan of a sail of a ship and CD the direction of the wind.

The wind causes a difference of air pressure on the two sides of the sail, the greater pressure, as one might expect, being on the side of the sail on which the wind is incident. The effect of the difference of pressure is to produce a resultant force P acting at right angles to the sail along DG.

If we head the ship in the direction DE, so that $\angle CDE$ is acute, and the line of the sail AB is between CD and DE, we may imagine the force on the sail to be resolved into two components at right angles to each other, one *in the direction DE*, and the other perpendicular to this, along DF.

The resistance to the motion of the ship along DE is small compared with that which tends to prevent motion along DF. The result is that although the component of force along DE may be less than that along DF the resultant *motion* of the ship (itself compounded of components along DE and DF) makes an acute angle with the direction DC. The sideways motion of a yacht is largely prevented by a very deep keel.

133. The Action of Wind on an Aeroplane Wing.

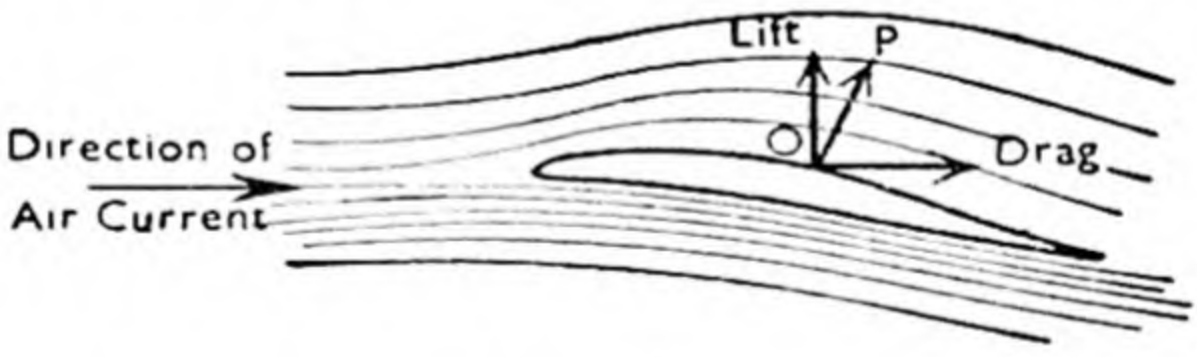

FIG. 239.

A cross-sectional view of an aeroplane wing is shown above. The " stream-line " shape helps to prevent the sucking effect, which tends to drag the wing backwards when it meets a rapid current of air, and at the same time provides a considerable difference of pressure on the top and bottom surfaces of the wing.

The air in contact with the upper wing surface is at a pressure less than atmospheric, and that in contact with the under surface is at a pressure greater than atmospheric. Thus the wing is

sucked up from above and pushed up from below. The suction effect is roughly twice the upward thrust, but both effects combine to produce a resultant force on the wing which may be taken to be in a direction sloping slightly backwards along OP. This resultant force may be resolved into a vertical component called the *lift* and a backward horizontal effect called the *drag*.

134. Application to the Inclined Plane.

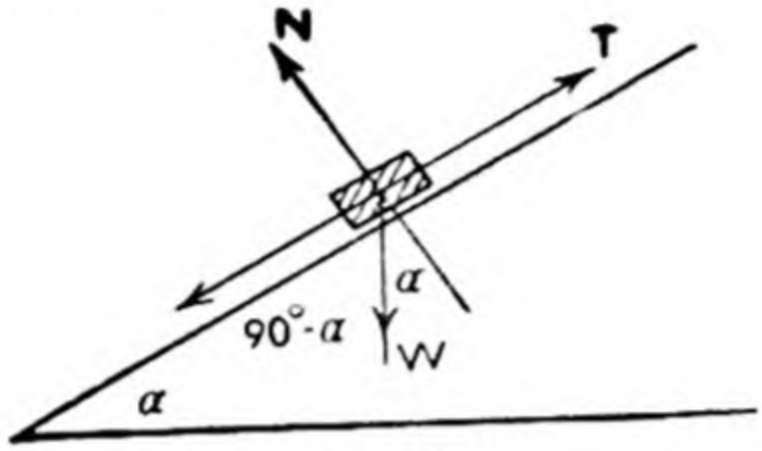

FIG. 240.

Suppose that a body is at rest on a smooth inclined plane acted upon by its weight W, the normal reaction N of the plane and the tension T in a string attached to it and parallel to the plane.

We have already shown that N and T may be found by drawing a triangle of forces. Since the lines of N and T are perpendicular to each other the problem may be very readily solved by resolving the force W into two components, one parallel to the plane, $W \cos(90 - \alpha)$, and one perpendicular to the plane, $W \cos \alpha$.

The forces acting on the body are in equilibrium, hence the tension in the string is balanced by the component $W \cos(90 - \alpha)$,

i.e., $$T = W \cos(90 - \alpha) = W \sin \alpha,$$

and the normal reaction is balanced by the component of W perpendicular to the plane, $W \cos \alpha$,

i.e., $$N = W \cos \alpha.$$

The method can be applied, however, when the forces to be determined are not perpendicular to each other. Thus when the

body on the inclined plane is acted upon by a horizontal force T in place of that parallel to the plane.

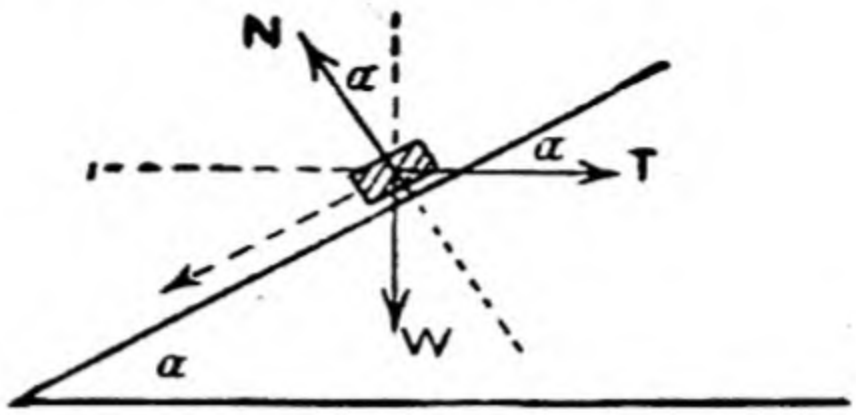

FIG. 241.

Choosing the horizontal and vertical directions as most convenient in considering the resolved parts of the forces acting, we have,

Vertically, $W = N \cos \alpha$ . . . . (1)

Horizontally, $T = N \cos (90 - \alpha) = N \sin \alpha$ . . (2)

From (1),

$$N = \frac{W}{\cos \alpha},$$

$$N = W \sec \alpha.$$

Substituting for N in equation (2),

$$T = \frac{W}{\cos \alpha} . \sin \alpha.$$

$$T = W \tan \alpha.$$

135. Application to Moments.

The moment of a force about a point is equal to the sum of the moments of any two component forces which have the given force as their resultant.* It is sometimes easier to resolve a force in two mutual perpendicular directions and to find the moments of these components about a point than to calculate the perpendicular from the point to the line of action of the force.

* See p. 296.

Example (1).—*Let* X *represent the force exerted by the roof of a building on the wall* AB. *What is the turning effect of the force* X *about the point* B *?*

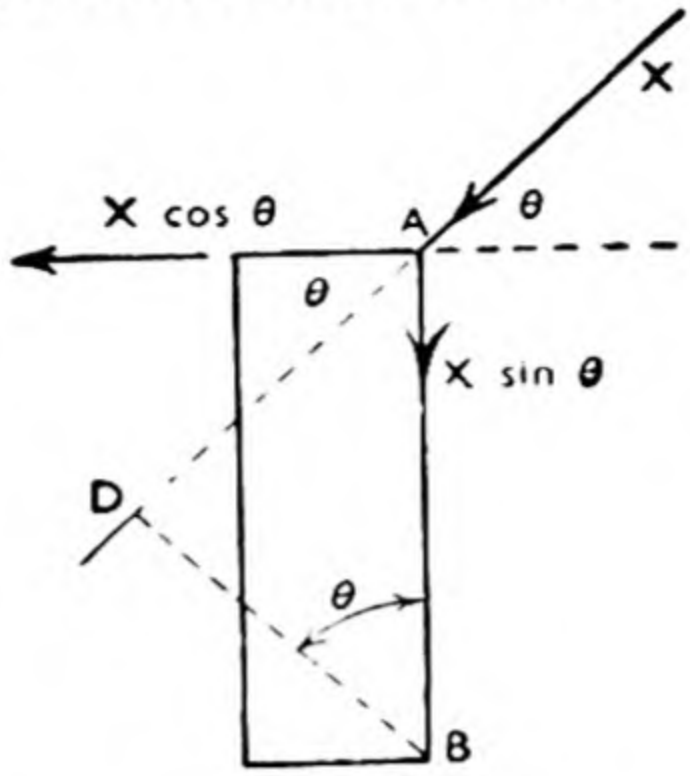

FIG. 242.

If the force X is resolved horizontally and vertically, it may be seen that only the horizontal component of force, X cos θ, has a turning effect about B, and the magnitude of the turning effect is therefore

$$X \,.\, \cos\theta \,.\, AB.$$

Now BD, the perpendicular from B to the line of action of X, is AB . cos θ, and the moment of X about B is therefore

$$X \,.\, AB \,.\, \cos\theta.$$

The two methods of finding the moment of X about B are thus seen to be equivalent.

Example (2).—ABCD *represents a block of stone. A force* P *is applied at the midpoint of the horizontal edge* A *at an angle of* 35° *to the horizontal.*

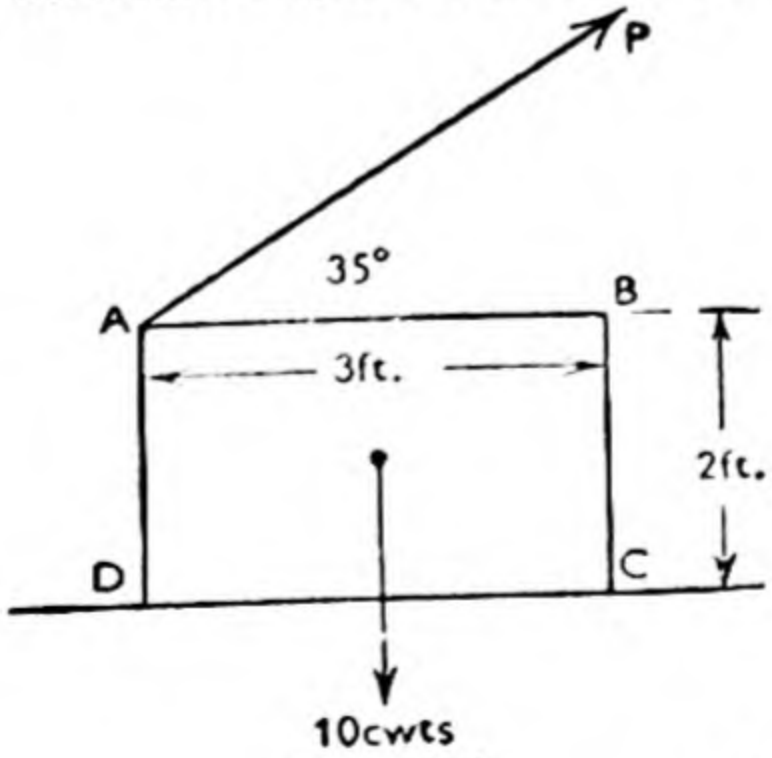

FIG. 243.

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If AB = 3 *ft.*, BC = 2 *ft. and the weight of the block is* 10 *cwt., find the value of* P *so that the block may begin to turn about the edge at* C.

The block will turn about C when the moment of P about C is equal to that of the weight of the block.

Resolve P vertically and horizontally.

The vertical component is $P \sin 35°$, and its moment about C is

$$3 \times P \sin 35°.$$

The horizontal component is $P \cos 35°$, and its moment about C is

$$2 \times P \cos 35°.$$

Hence, $$3P \sin 35 + 2P \cos 35° = 10 \times 1\tfrac{1}{2}.$$

Therefore $$P(3 \sin 35 + 2 \cos 35) = 15.$$

$$P = \frac{15}{3 \sin 35 + 2 \cos 35} = 4{\cdot}44 \text{ cwt.}$$

EXAMPLES XXXVI

Resolution, including Application to Moments

1. A force of P acts along OA at an angle θ to a line OX. OY is perpendicular to OX and is inclined at $90 - \theta$ to OA.

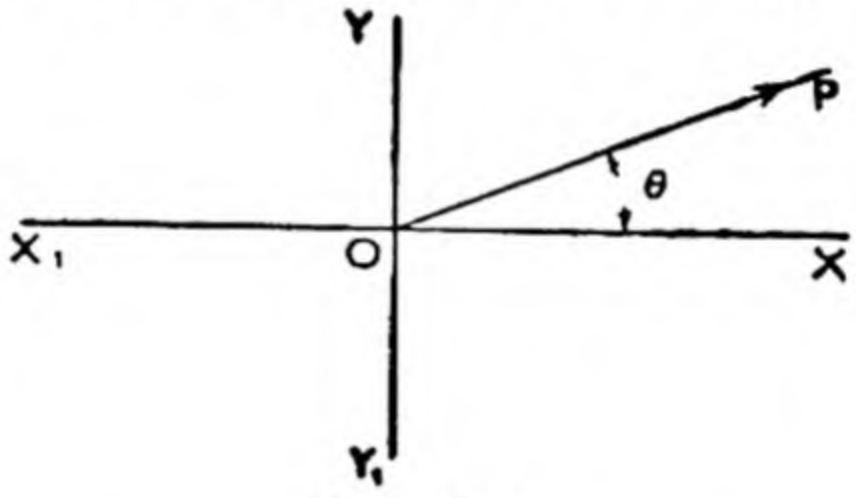

Fig. 244.

Find the resolved parts of the force P along OX and OY in the following cases :—

(*a*) P = 10 lb. wt., $\theta = 30°$.
(*b*) P = X lb. wt., $\theta = 30°$.
(*c*) P = 15 lb. wt., $\theta = 45°$.
(*d*) P = X lb. wt., $\theta = 45°$.
(*e*) P = 2 tons wt., $\theta = 37°$.

2. In question 1, XO is produced to X_1 and YO to Y_1. Find the resolved parts of the force P in the following cases :—

(*a*) Along OX, OY_1 ; P = 10 lb. wt., $\theta = 15°$.
(*b*) Along OX_1, OY ; P = 15 lb. wt., $\theta = 40°$.
(*c*) Along OX_1, OY_1 ; P = 2 tons wt., $\theta = 37°$.
(*d*) Along OX, OY ; P = 5 kgm. wt., $\theta = 120°$.
(*e*) Along OX, OY_1 ; P = 10 gm. wt., $\theta = 200°$.
(*f*) Along OX_1, OY ; P = 4 cwt., $\theta = 300°$.

3. A force of 4 lb. wt. is inclined at 40° to the horizontal. Find its horizontal and vertical components.

4. A force of 20 lb. wt. is inclined at $\theta°$ to the horizontal. If the vertical component of the force is 15 lb. wt., find the value of θ.

5. A wire stay is fastened to a point in vertical pole 12 ft. above the ground and to the ground at a point 6 ft. from the base of the poles. If the tension in the stay is 120 lb. wt., find the horizontal and vertical components of the force exerted on the pole.

6. A barge is drawn along a canal by a tow rope inclined at 15° to the direction of the canal. If the tension in the rope is 200 lb. wt., and the total weight of the barge is 30 tons, find the resistance to motion in lb. wt. per ton experienced by the barge in the direction of the canal.

7. A body of weight 50 lb. is held at rest on a smooth plane inclined at 20° to the horizontal by a rope parallel to the inclined plane. Find the reaction of the plane on the body and the tension in the rope.

8. A force of 4 lb. wt. acting parallel to a smooth inclined plane and directly up the plane upon a body maintains it at rest on the plane. The plane is inclined at an angle of 25° to the horizontal. Find the weight of the body and the normal reaction of the plane on it.

9. A horizontal force of 10 lb. wt. keeps a body of weight 20 lb. at rest on a smooth inclined plane. By resolving parallel to the plane, show that $\tan \alpha = \frac{1}{2}$, α being the inclination of the plane to the horizontal.

10. A body is being drawn up a plane inclined at 30° to the horizontal by a force P applied at an angle of 15° to the inclined plane. Neglecting

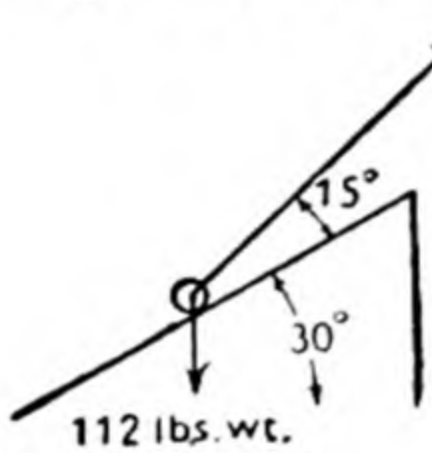

FIG. 245.

frictional resistance parallel to the plane, find the value of P. (Resolve parallel to the plane.)

11. A wooden sphere is supported by a fine wire and is pushed by

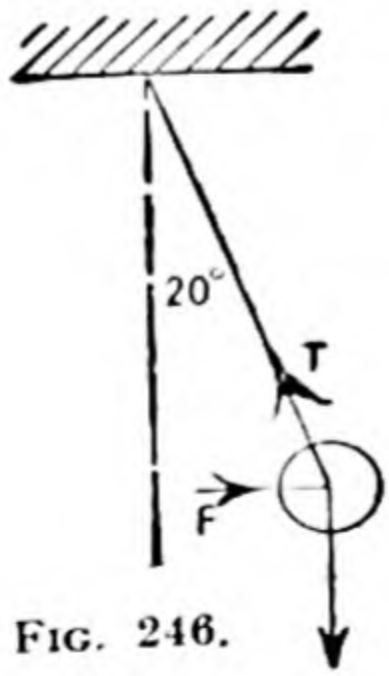

FIG. 246.

a horizontal force F until the wire is inclined at 20° to the vertical. Find the force F and the tension in the wire if the weight of the ball is 2 lb. wt.

12. A uniform heavy bar BC of weight 5 cwt. and length 8 ft. is kept in a horizontal position by a chain attached at a point D, 2 ft. from C, and to a point A, 3 ft. vertically above B. By resolving the tension in

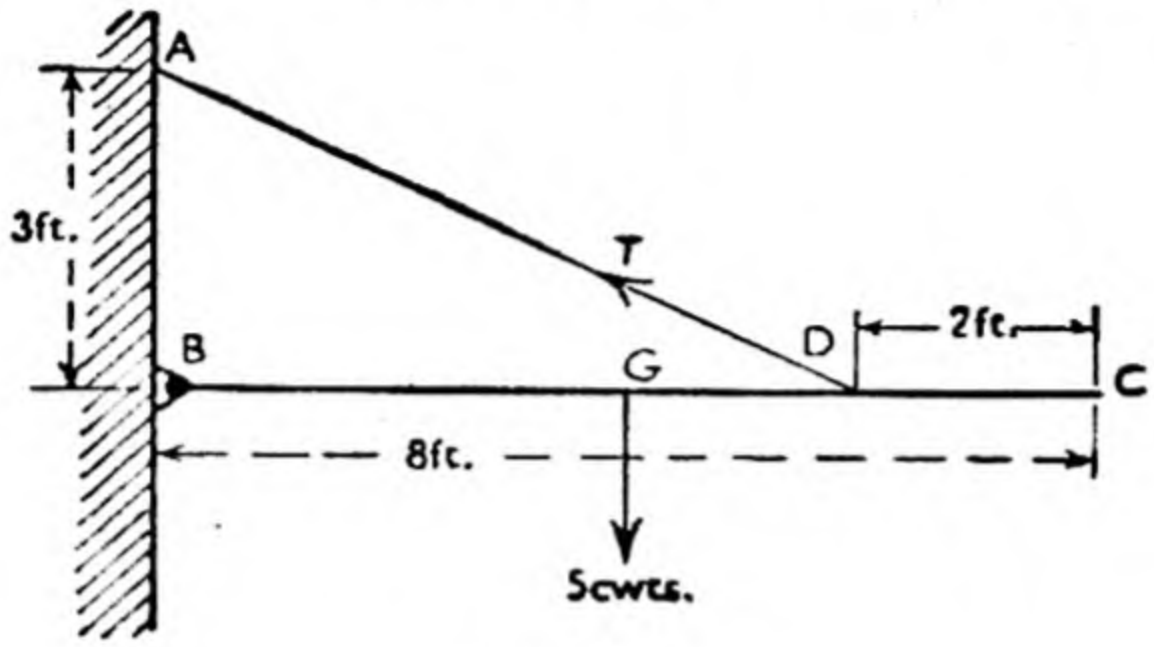

FIG. 247.

DA vertically and horizontally at D, find tension T by taking moments about B.

13. A horizontal force of F lb. wt. acts at A on a 200 lb. wt. vertical rod AB hinged at B. The rod is prevented from turning by the rigid bar CD inclined at 30° to the rod and fastened to it at a point 5 ft. from B.

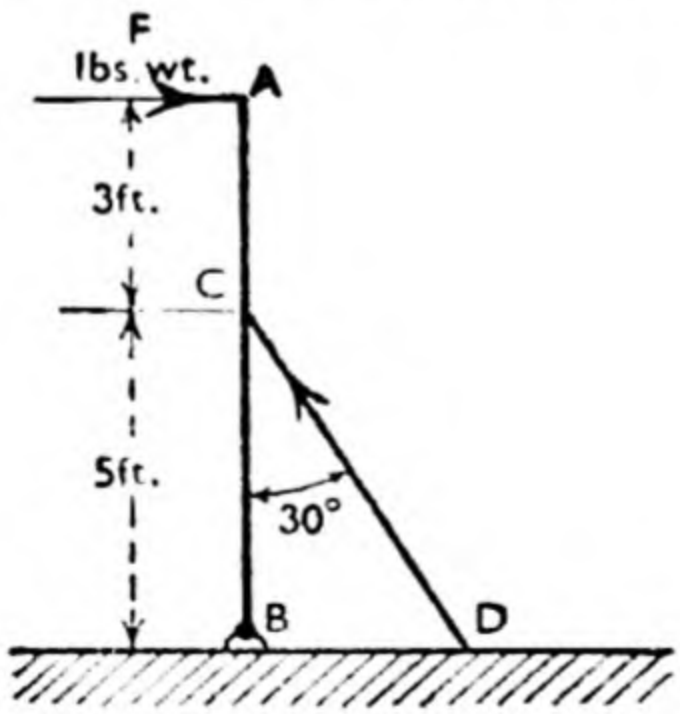

FIG. 248.

The force exerted by the bar DC at C is 200 lb. wt. Find the horizontal component of the 200 lb. wt. force, and by taking moments about B, find the value of F.

14. AB represents a uniform heavy bar of length 4 ft., smoothly hinged at B and prevented from turning about B by a force of 25 lb. wt. acting at an angle of 70° with AB. Find the component of the 25 lb. wt.

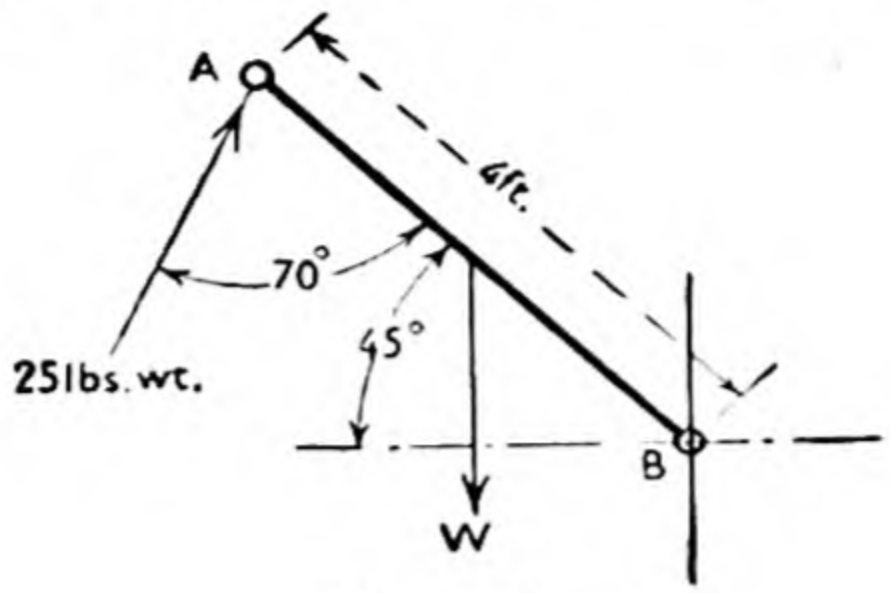

FIG. 249.

force perpendicular to AB. If the weight of the bar is W lbs. wt., what is the component of W perpendicular to AB? By taking moments about B, find W.

15. Show clearly how a force may be resolved into two components at right angles to each other.

A picture, 3 ft. long and 2 ft. wide, is suspended in a vertical plane with its longer side horizontal by a string 6 ft. long attached to its two top corners and passing over a nail. A second picture of the same size and weight is suspended in the same way by a string of the same length but with its shorter side horizontal. Compare the tensions in the two strings. (*G.S.*)

16. A light cord ABCD is fastened at A and D to two points in the same horizontal plane. At B a mass of 10 lb. is suspended, and AB is at 45° to the horizontal. At C is an unknown mass W, and the string BC is horizontal and CD at 30° to the horizontal. Find graphically the tensions in each part of the string and the value of W. Verify the results by calculation. (*L.M.*)

17. Explain what is meant by the resolution of forces.

A mass of 10 lb. is attached to a point B of a weightless string ABC. AB is inclined at 30° to the vertical and BC at 45° to the vertical, and AC is horizontal. Determine the pull in each string, expressing the results to two places of decimals. (*L.M.*)

136. The Sum of the Resolved Parts of Two Forces Acting at a Point in any given Direction is Equal to the Resolved Part of their Resultant in that Direction.

OA and OB represent the forces P and Q acting at a point. OC, the diagonal of the parallelogram OACB, represents their resultant.

Let OX be the given direction.

Draw AD, BE, CF perpendicular to OX, and AG perpendicular to CF.

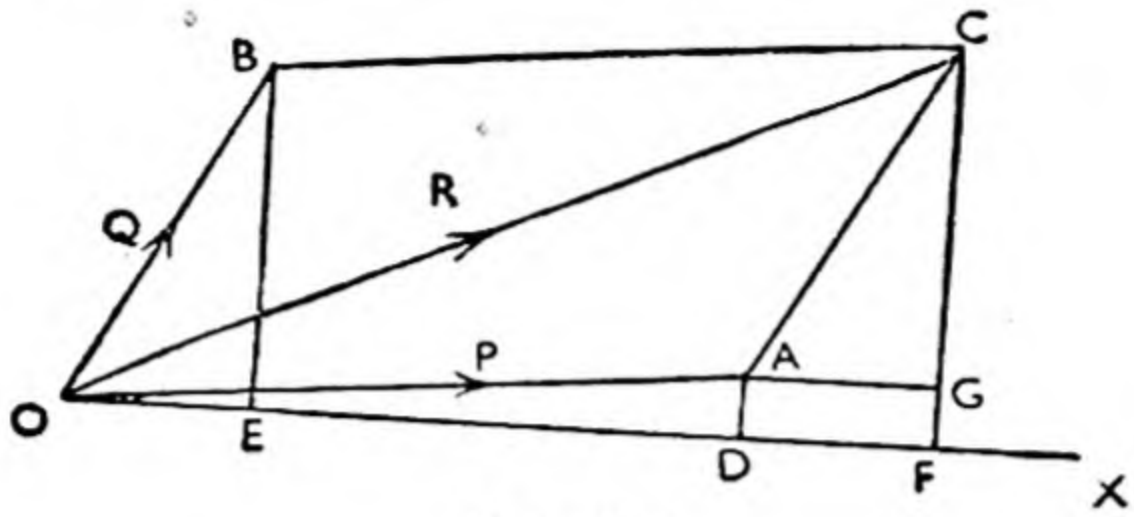

FIG. 250.

The triangles OBE, ACG are equal in all respects, therefore

$$OE = AG = DF,$$

and

$$OF = OD + DF = OD + OE.$$

Now OF represents the resolved part of the resultant of P and Q in the direction OX, and OD, OE represent the resolved parts of P and Q in that direction, therefore the above statement is proved.

If a third force R acts at the same point as P and Q, then the resultant of P and Q may be considered together with R in precisely the same manner as P and Q have been considered together, and it may be shown that the resolved part of the resultant of P and Q together with the resolved part of R in the direction OX is equal to the resolved part of the resultant of P, Q and R in that direction.

This result is true for any number of forces acting at a point, and we shall use it to find the resultant of a number of forces acting at a point.

137. The Resultant of a Number of Forces Acting at a Point.

By resolving the forces in two perpendicular directions.

Forces of 7, 5, 4 and 2 lb. wt. act at a point O in the directions shown in the figure. $\angle POQ = 60°$, $\angle QOR = 75°$, $\angle ROS = 85°$.

Axes OY and OX are chosen perpendicular to each other, the axis OY being along the 7 lb. wt.

The angles made by each force with the axes are found.

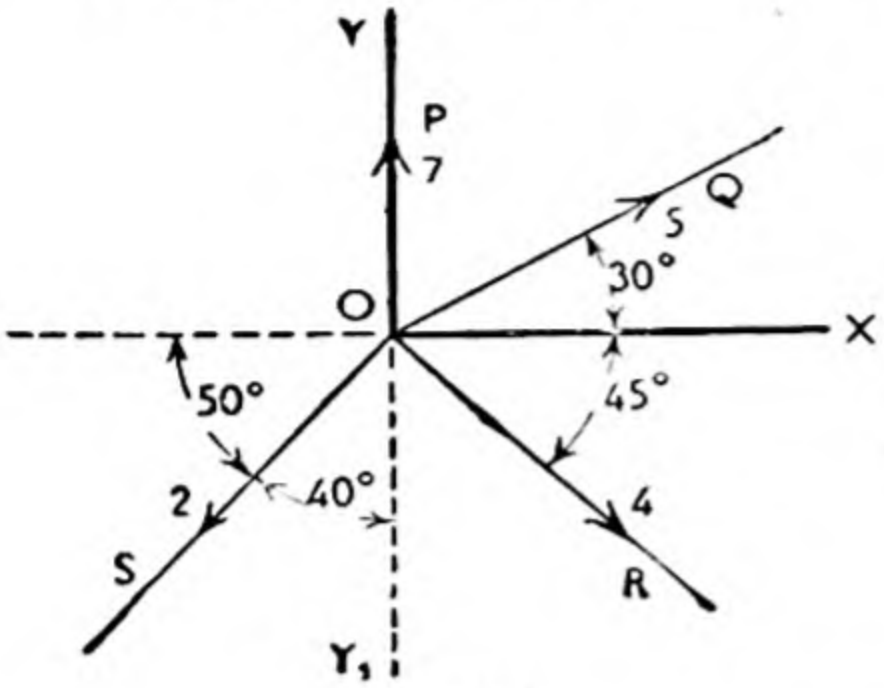

Fig. 251.

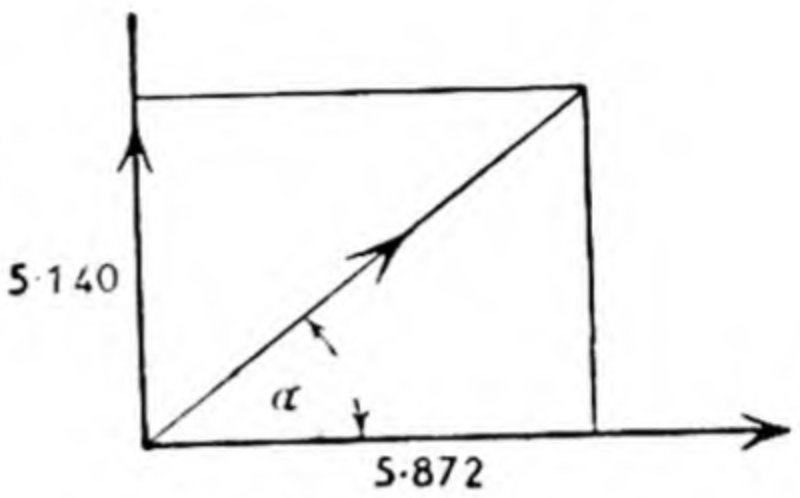

Fig. 252.

Where the component of a force is in the opposite direction to that chosen as the positive direction it is entered in the table below as a negative component.

Consider the 4 lb. wt. force,

$$\text{Component along OX} = 4 \cos \text{ROX} = 4 \cos 45°$$
$$= 4 \times 0{\cdot}7071 = 2{\cdot}8284 \text{ lb. wt.}$$

$$\text{Component along OY} = -4 \sin \text{ROX} = -4 \sin 45°$$
$$= -2{\cdot}8284 \text{ lb. wt.}$$

Force.	Component along OX.			Component along OY.		
		+	−			
7	0				7·000	...
5	$5 \cos 30° = 5 \times 0{\cdot}8660 =$	4·330	...	$5 \sin 30° = 5 \times 0{\cdot}5 =$	2·500	...
4	$4 \cos 45° = 4 \times 0{\cdot}7071 =$	2·828	...	$-4 \sin 45° = -4 \times 0{\cdot}7071 =$	...	2·828
2	$-2 \cos 50° = -2 \times 0{\cdot}6428 =$	...	1·286	$-2 \sin 50° = -2 \times 0{\cdot}7660 =$	...	1·532
	Sums	7·158	1·286		9·500	4·360

The sum of the components along OX = 5·872 lb. wt.

" " " " " OP = 5·140 lb. wt.

The resultant of the forces = $\sqrt{(5{\cdot}872^2 + 5{\cdot}140^2)}$

= 7·88 lb. wt.

If the direction made by the line of action of the resultant with OX is $\alpha°$,

$$\tan \alpha = \frac{5 \cdot 140}{5 \cdot 872}.$$

$$\alpha = 41° \ 12'.$$

138. The Equilibrium of a Number of Forces Acting at a Point.

Since the resultant of a number of forces acting at a point is zero if the forces are in equilibrium, it follows that the components of the resultant in such a case must be zero in any two mutually perpendicular directions.

Conversely, if the algebraic sums of the components of a number of forces acting at a point in two mutually perpendicular directions are zero the forces are in equilibrium, for in this case their resultant is zero.

In the example worked on the previous page the forces acting at a point have as their equilibrant a force equal but opposite to their resultant which was determined.

EXAMPLES XXXVII

Resultant and Equilibrant of Forces Acting at a Point by Resolution

1. Forces of 1, 2, 3, 4 lb. wt. act at a point, the angle between each pair of forces being 30°. Find the magnitude of the resultant of the forces and the angle which its line makes with the 1 lb. wt. force.

2. Three horizontal wires radiate from a point in directions north-east, 30° west of north and due south, the tensions in them being respectively 50, 60 and 80 lb. wt. Find the magnitude and direction of the resultant of these forces.

3. Show how to determine, in magnitude and direction, the resultant of a system of forces acting at a point.

Six strings, each string making an angle of 60° with the next string, are attached to a small body. If the tensions in the strings are 1, 2, 3, 4, 5 and 6 units respectively, find the magnitude and direction of the resultant force on the body. (*L.M.*)

4. Forces 2, 6, 10, 4 lb. wt. act at a point along the lines OA, OB, OC and OD. The angles AOB, BOC, COD are 90°, 45°, 90° respectively. Find the magnitude of the single force acting at O which will maintain equilibrium, and state the angle at which it is inclined to OA.

5. Five forces 2, 3, 4, 5 and 6 lb. wt. act outwardly from a point, and the angle between each pair of forces is 72°. Find in magnitude and direction the equilibrant of the system, giving the angle which it makes with the 4 and 6 lb. wt. forces.

6. The following forces, A, 560 lb. wt.; B, 196 lb. wt.; C, 364 lb. wt., all act outwards at the same point and make equal angles of 120° with one another. State completely what additional force is required to keep these forces in equilibrium.

7. Forces of 2, 4, 6, 4, 2 lb. wt. respectively act outwards from a point on a body with an angle of 60° between each pair. Find the magnitude and direction of the resultant.

8. Six forces, all in the same plane, act at a point O along lines OA, OB, OC, OD, OE and OF, as shown in the figure. Determine the

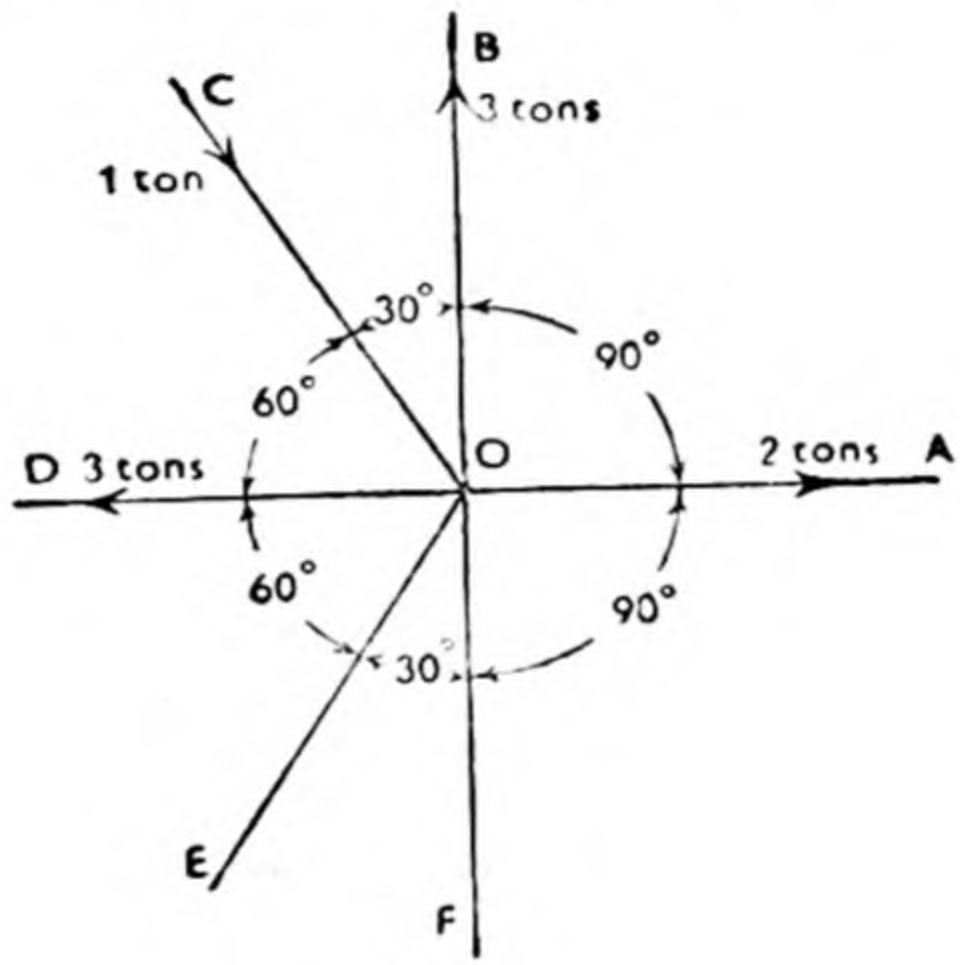

FIG. 253.

magnitude and sense of each of the forces acting along OE and OF in order to maintain the point O in equilibrium.

CHAPTER XIV

THREE FORCES ACTING ON A RIGID BODY

139. ***If three forces acting in one plane upon a rigid body are in equilibrium, their lines of action either meet in a point or are parallel.***

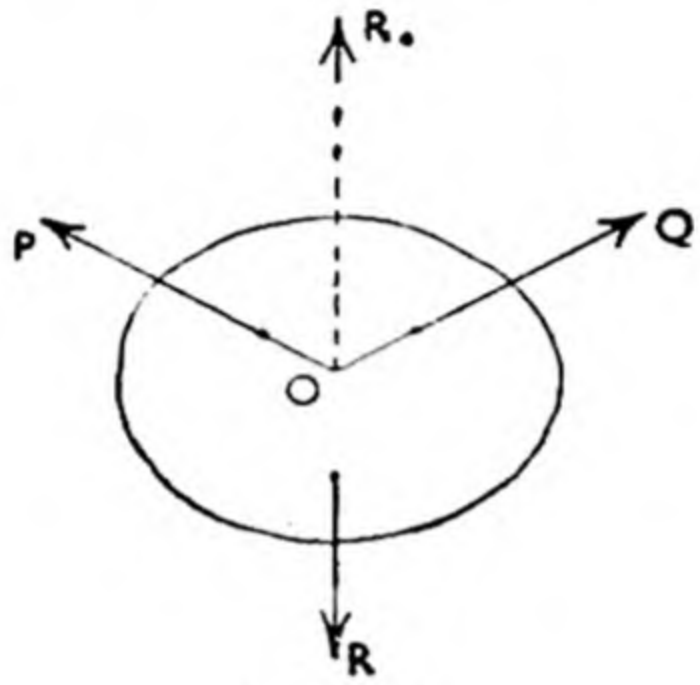

FIG. 254.

Let P, Q and R be three forces in equilibrium acting in one plane upon a body. Suppose that the lines of action of P and Q are not parallel, and when produced, therefore, they meet at a point O. A force may be considered to act at any point in its line of action. The forces P and Q may be considered to act at O. The resultant of P and Q acting at O is a force whose line of action passes through O. This resultant R_1 can only be balanced by an equal and opposite force, its equilibrant, *which has the same line of action.* Since P, Q and R are in equilibrium, R is the equilibrant of P and Q and therefore of their resultant R_1. Hence R and R_1 are equal and opposite forces which have the same line of action. The lines of action of P, Q and R therefore pass through a point.

If the lines of action of P and Q do not meet when produced they are parallel. The equilibrium of parallel forces is considered at a later stage from a theoretical standpoint, and from a practical standpoint we have already dealt with it.

If a body is at rest under the action of three non-parallel forces which act in one plane and are in equilibrium, (1) their lines of action pass through a point, (2) the forces may be represented in magnitude and direction by the three sides of a triangle taken in order.

140. The following experiments demonstrate the fact stated in italics above, but, in addition, show the way in which the forces act upon a body in typical three-force problems :—

EXPERIMENT (1).—Find the centre of gravity of a thin sheet of tin. Support the tin as shown in the figure below by two strings AP and BQ in the same plane as that of the lamina.

The lamina is at rest under the action of the two tensions T_1 and T_2 in AP and AQ respectively and the weight of the lamina itself.

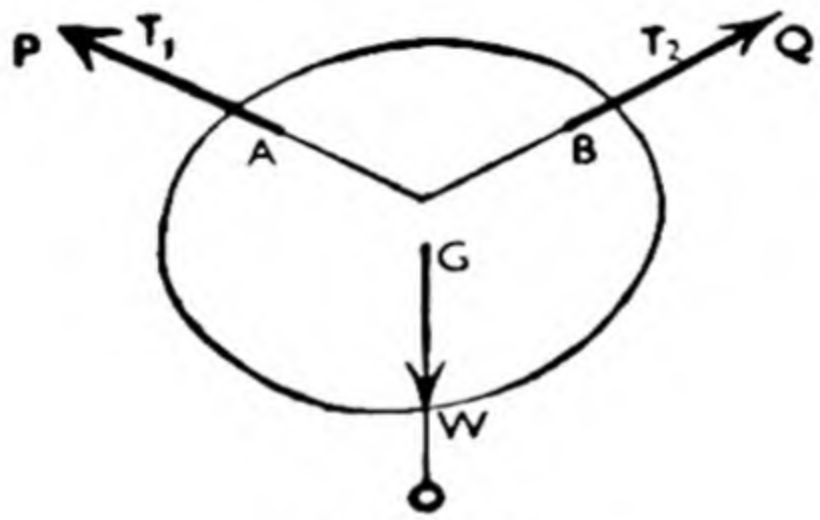

FIG. 255.

The lines of action of T_1 and T_2 are marked on the surface of the lamina, and the line of action of the weight of the lamina is found by supporting a plumb-line in front of the lamina to pass through a point immediately in front of its centre of gravity. The lines of action thus marked on the lamina should pass through a point.

EXPERIMENT (2).—Arrange the sheet used in the last experiment so that it rests on a horizontal plane supported by steel balls, on which it is free to roll. Let three strings be attached to the tin and three horizontal tensions be arranged to act upon the lamina in its own plane. Demonstrate as before that the lines of action of the three tensions pass through the same point. The vertical forces acting on the lamina, viz., its weight and the reactions of the steel balls on it, do not affect the equilibrium of the three horizontal forces.

EXPERIMENT (3).—Attach unequal lengths of string to the ends of a uniform rod AB. Fasten the other end of the string attached to A to a

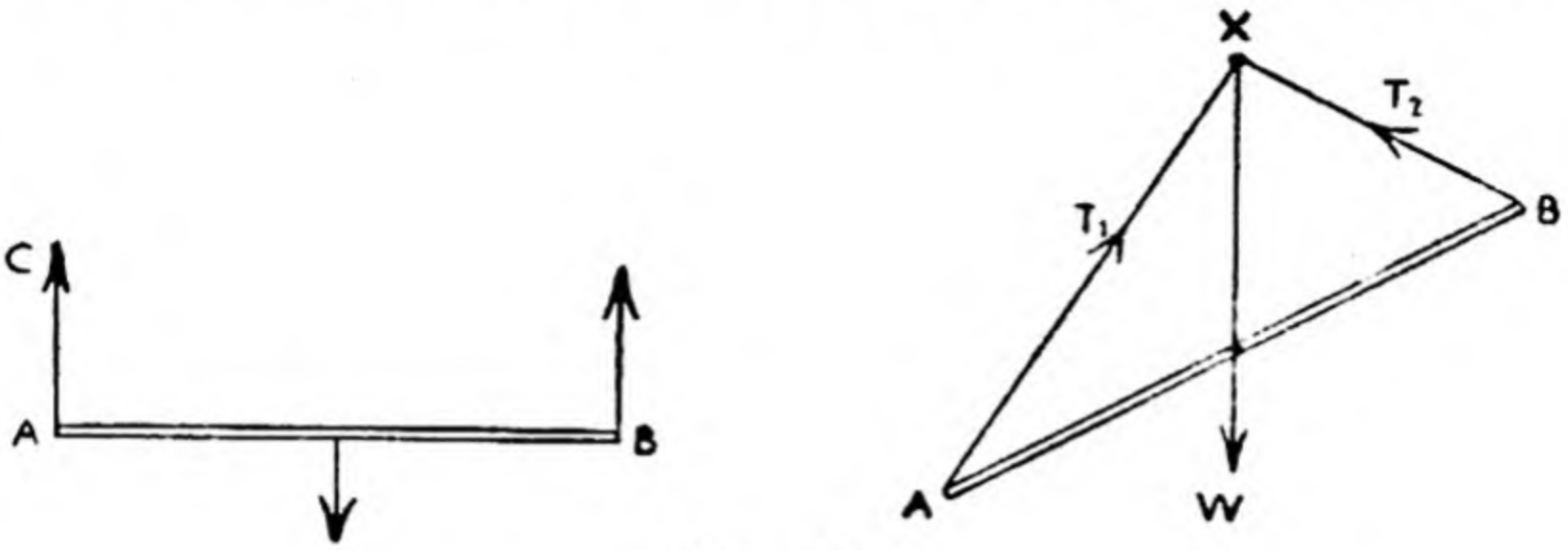

FIG. 256.

fixed point C. Pull the free end of the second string attached to B so that AC is vertical. What do you notice about the direction of the string at B ?

Fasten both strings to a point X and allow the rod to hang freely. From X suspend a plumb-line, and notice the point at which the vertical through X cuts the rod. The three forces acting on the rod are T_1, the tension in AX, T_2, the tension in BX, W, the weight of the rod acting through its centre of gravity.

Do the lines of action of these three forces pass through a point ?

EXPERIMENT (4).—Attach a string to the ends A and B of a non-uniform rod, and support the rod by passing the string over a smooth

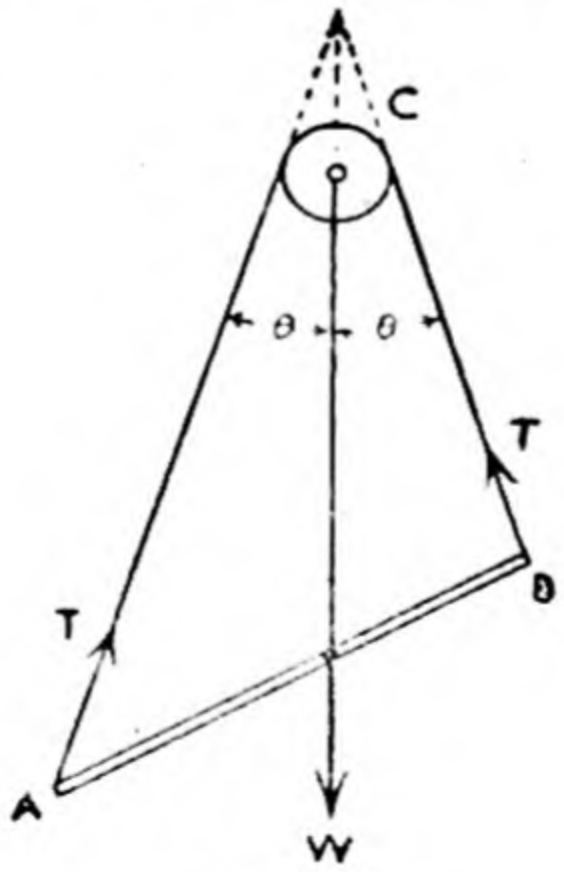

FIG. 257.

pulley C. Measure the angles which the strings AC and BC make with the vertical. Remembering the fact that the pulley is smooth, explain the result of your measurement. Arrange a plumb-line to pass through the axle of the smooth pulley and notice that it passes through the centre of gravity of the rod (which may be found by balancing it on a knife-edge).

Conversely, we may find the position of the centre of gravity of a rod by the above method, assuming that the lines of action of the tensions in AC and BC and of the weight pass through a point.

EXPERIMENT (5).—Hinge a fairly heavy uniform rod at its end A by means of a pin, secured to the capstan block C, which passes through a hole in the end of the rod. The pin should fit loosely into this hole.

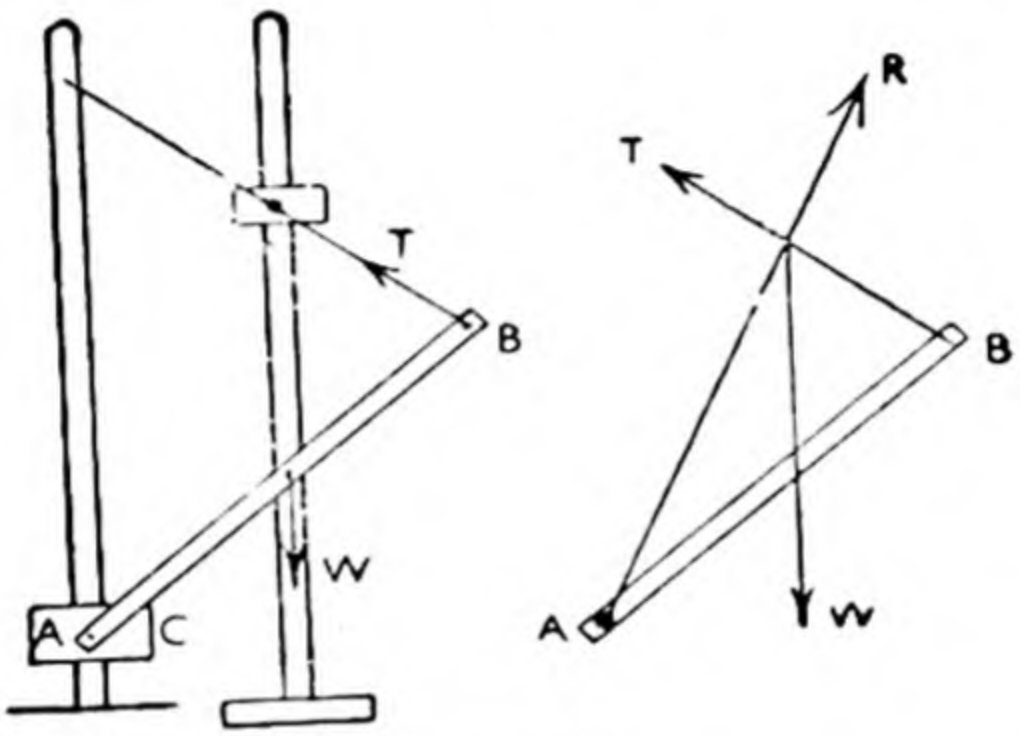

FIG. 258.

Support the end B of the rod by means of a string BD.
The rod is at rest under the action of three forces in equilibrium :—

(1) The weight of the rod which acts vertically downwards through the centre of gravity of the rod.
(2) The tension in the string acting at B.
(3) The reaction at the hinge A, *i.e.*, the reaction of the pin on the rod.

In order to find the direction of the reaction at A, attach a string to the pin, which is very light in comparison with the weight of the rod, loosen the screw which secures the pin, and arrange the tension and direction of the string until the pin may be completely removed from its support without its position being altered when the support is taken away.

It should be found that the two lines of the strings when produced intersect on the line of action of the weight of the rod which passes through its centre of gravity.

EXPERIMENT (6).—At the end B of the rod used in the above experiment attach a light smooth pulley.

Arrange the rod so that it is supported at B by the pulley which rests against a vertical wall, and at A by the pin, as in the last experiment.

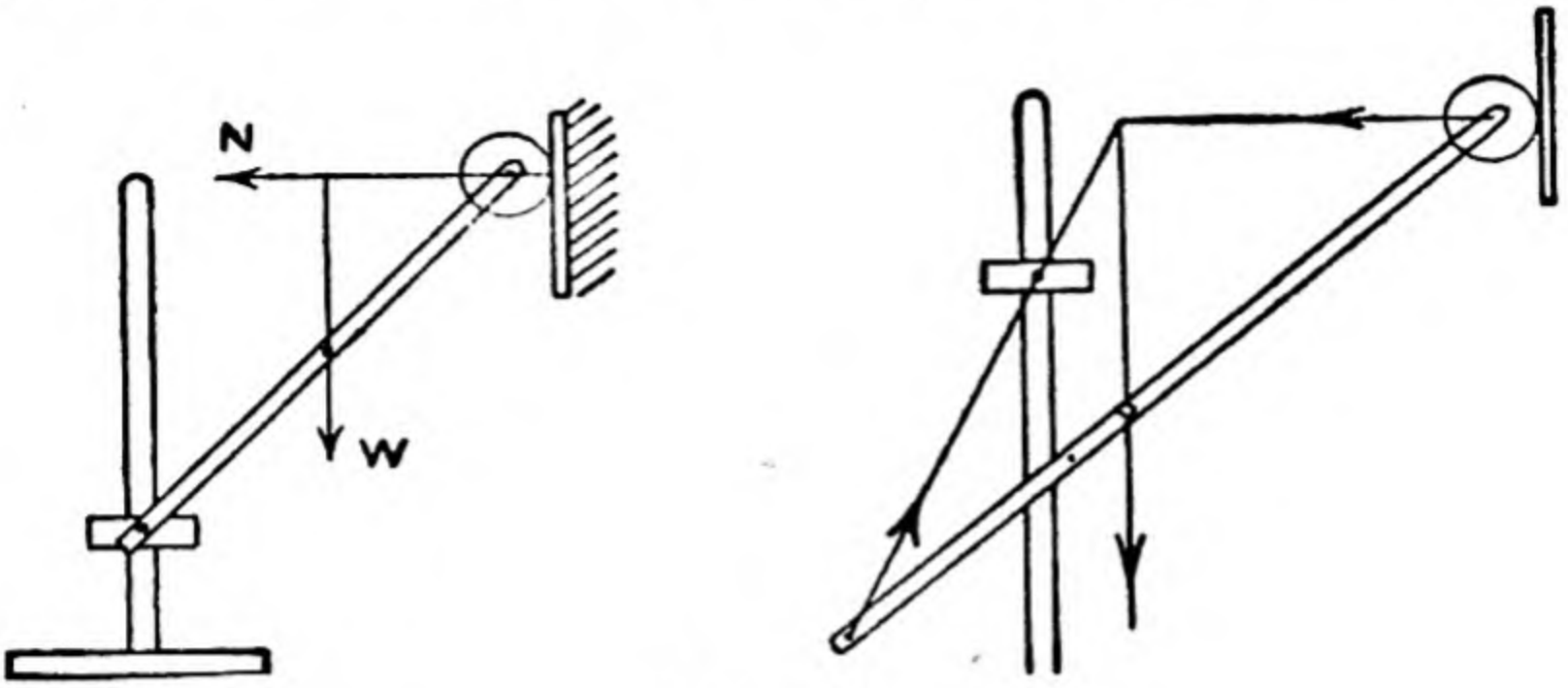

FIG. 259.

The reaction at B may be considered to be horizontal, as it would be if the end B of the rod were supported by a smooth vertical wall.

Verify that the reaction at the hinge A passes through the point of intersection of the lines of action of the weight of the rod and the horizontal reaction at B.

141. The following points, some of which have been mentioned before, must be carefully remembered :—

(1) The weight of a body acts vertically downwards through its centre of gravity.

(2) When a body is leaning against a *smooth* surface the reaction is normal to the surface, as N_1 in figure below.

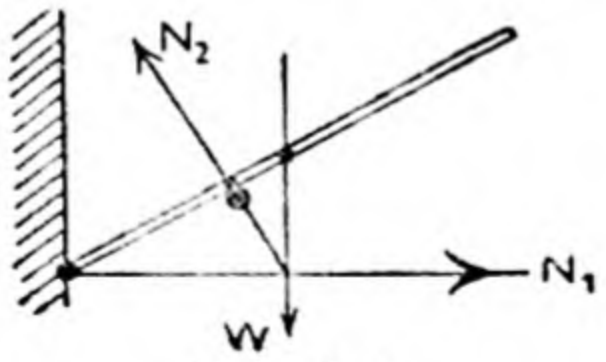

FIG. 260.

(3) When a rod is resting on a smooth *peg* the reaction of the peg on the rod is perpendicular to the rod, as N_2 in figure above.

(4) The tension in a light string is the same throughout the string and is unaffected by the string passing over *smooth* pegs or pulleys.

(5) The resultant of two equal forces T, T bisects the angle between them. Thus when a string passes over a smooth peg (or pulley) the resultant force on the peg bisects the angle between the parts of the string on each side of the peg (see the fifth of the experiments described above).

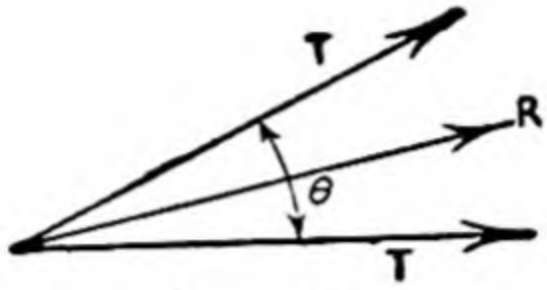

FIG. 261.

(6) When a rigid body is suspended from a fixed point, the centre of gravity of the body must lie in the vertical line through the fixed point. Referring to the third experiment described above, the resultant reaction at X must balance the weight, and for this to be possible the two forces must act in the same straight line. This

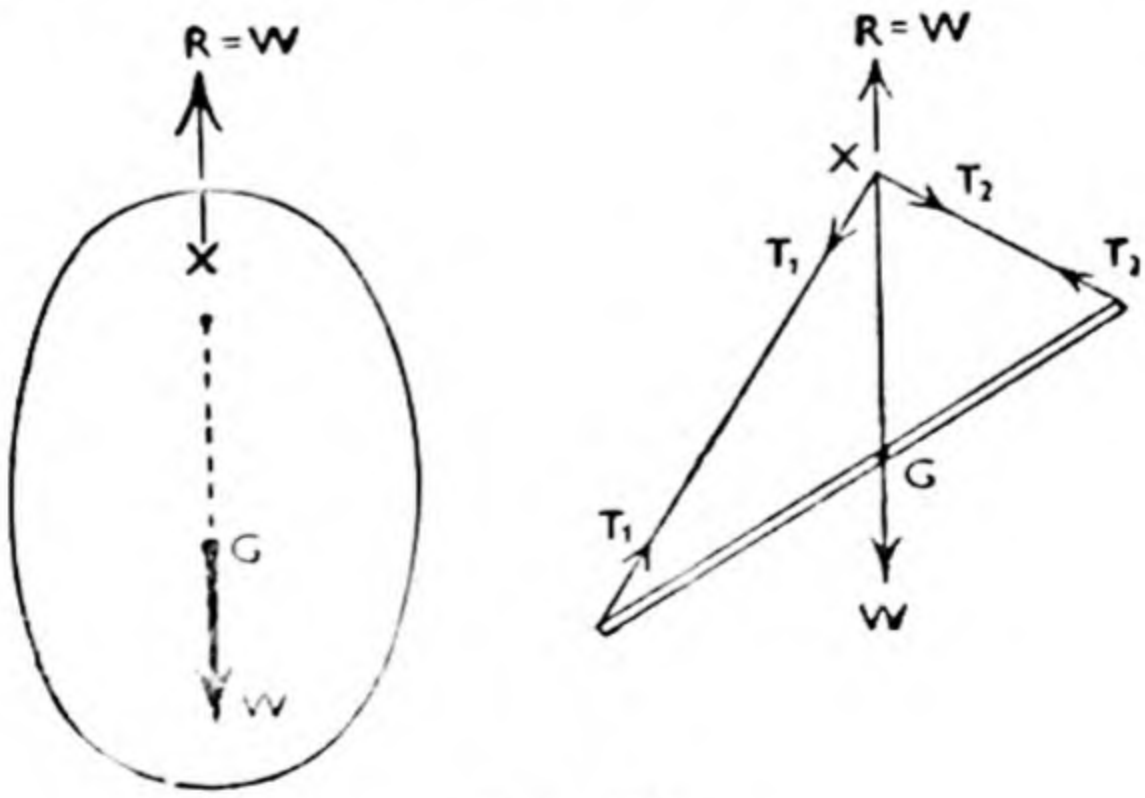

FIG. 262.

result is true whether X is a point in the body itself, as shown below, or whether the body is attached to X by two strings.

(7) The directions of forces as they act upon the body which is under consideration must be carefully noticed. This applies especially to reactions at a hinge where it is essential to distinguish between the action of the hinge on the hinged body as against that of the hinged body on the hinge, these forces being equal in magnitude and opposite in sense.

Example (1).—*A uniform horizontal beam* AB, *of length* 8 *ft. and weight* 100 *lb., is hinged at* A, *and is held in position by a chain* CB *attached at* C *to a support* 4 *ft. vertically above* A. *Find the tension in the chain and the reaction of the hinge* A *upon the rod, stating carefully the direction in which this reaction acts.*

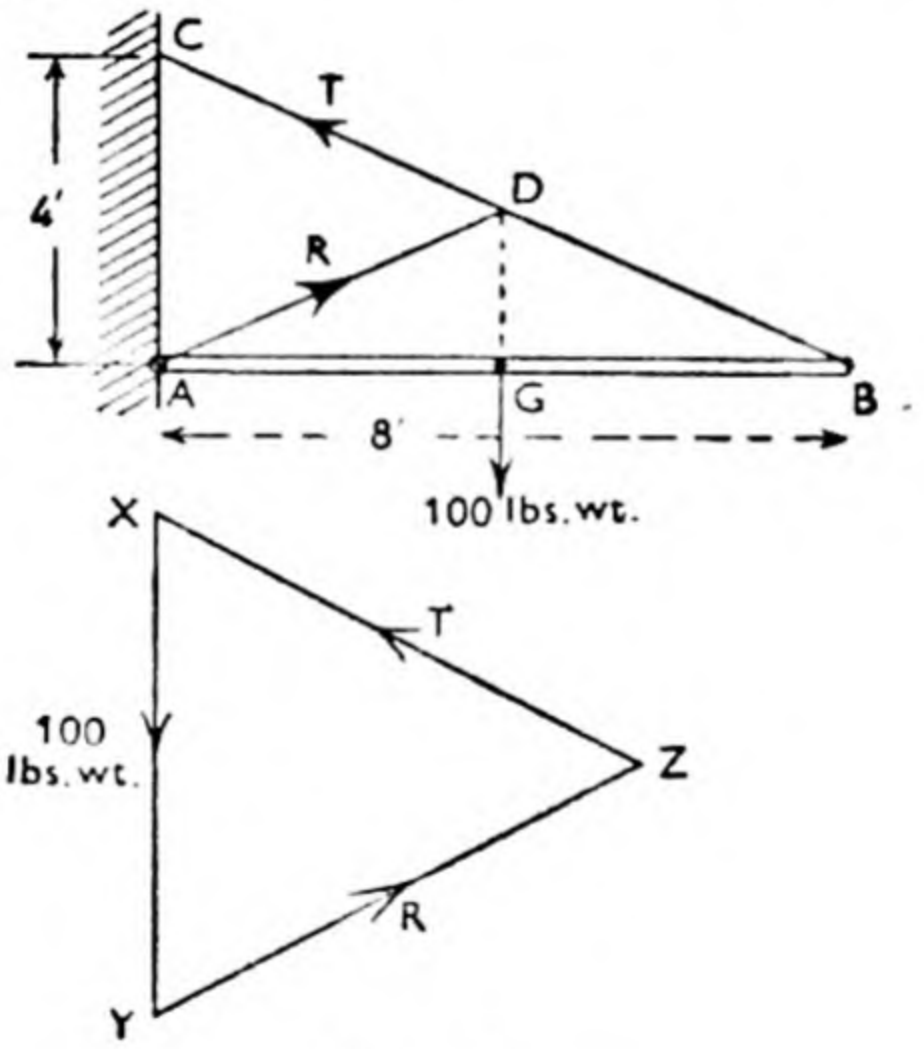

FIG. 263.

The weight of the beam may be taken to act at its midpoint since the beam is uniform.

The three forces acting on the beam are :—

(1) Its weight, which acts vertically downwards through its mid-point G.

(2) The tension T in the chain BC in the direction BC.
(3) The reaction R of the hinge A on the rod.

Since the lines of action of the weight of the beam and the tension T meet when produced at D, the line of action of the third force R must also pass through D.

If we join AD,

CA is parallel to the line of action of the 100 lb. wt.,
AD ,, ,, ,, ,, R,
DC ,, ,, ,, ,, T,

therefore CAD is a triangle of forces.

For an accurate graphical solution it is necessary to draw, to as large a scale as possible, a separate triangle of forces such as XYZ, then,

$$\frac{100}{XY}=\frac{R}{YZ}=\frac{T}{ZX}.$$

For a solution by calculation the triangle of forces CAD, already referred to in the structure diagram, may be used.

Since DG // AC and G is the midpoint of AB, DC = DB.

Again, CAB is a right angle, therefore the circle diameter CB passes through A, and AD = DC.

$$CB^2 = CA^2 + AB^2 = 4^2 + 8^2 = 80.$$

$$CB = \sqrt{80} = 4\sqrt{5}.$$

$$AD = DC = \tfrac{1}{2}.$$

$$\frac{100}{CA}=\frac{R}{AD}=\frac{T}{DC}.$$

$$\frac{100}{4}=\frac{R}{2\sqrt{5}}=\frac{T}{2\sqrt{5}}.$$

Hence, $R = T = 50\sqrt{5} = 112$ lb. wt. (Correct to three significant figures.)

$$\angle DAB = \angle DBA.$$

Since $$\tan \angle DBA = \tfrac{1}{2} = \tfrac{1}{2},$$

$$\angle DBA = 26^\circ\ 34'.$$

The reaction makes 26° 34′ with the horizontal.

Example (2).—*The swinging arm devised to carry a boiling pot over a fire is perhaps one of the earliest examples of the* Crane *used by man. The likeness to the long-necked British bird of the same name does not at once strike one when examining a modern jib crane, but it is a little more apparent in the figure shown below.*

The rod AB is hinged at A to a vertical rod, and the jib CD is hinged at A to the vertical rod and at D to AB.

At the end B is suspended a load of weight 20 lb.

What is the force in the jib CD and the reaction at the hinge A?

Consider the forces acting on the rod AB.

(1) The weight of 20 lb. at B.
(2) The force P in the jib CD acting along CD on the rod AB.
(3) The reaction R at the hinge A.

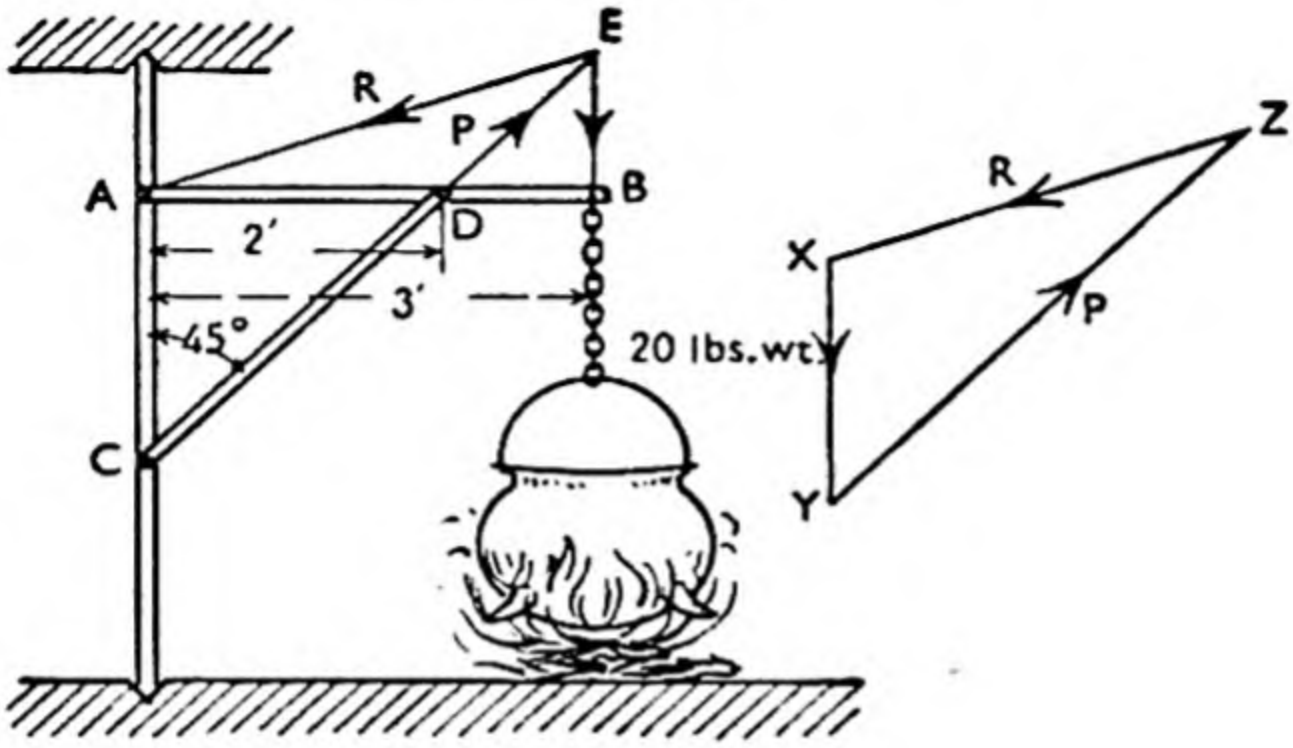

FIG. 264.

Since the lines of action of the weight of 20 lb. and the force P meet at the point E when they are produced, the line of action of the reaction at A must also pass through this point.

A triangle of forces XYZ may now be drawn, and from it

$$\frac{20}{XY}=\frac{P}{YZ}=\frac{R}{ZX}.$$

Thus if XY is drawn to represent 20 lb. wt. to a suitable scale, YZ represents P and ZX represents R. The direction of the reaction R on the rod AB makes $\angle EAB$ with the horizontal.

Solution by Calculation.—In the structure diagram,

EA // line of action of R.
AC // ,, ,, 20 lb. wt.
CE // ,, ,, P.

Hence EAC is a triangle of forces.

Using the right-angled triangle DBE, since $\angle EDB = 45°$, BE = 1 ft., and $ED = \sqrt{2}$ ft.

Using the right-angled triangle EAB,

$$EA^2 = AB^2 + BE^2 = 3^2 + 1^2 = 10$$

$$EA = \sqrt{10} \text{ ft.}$$

Using the right-angled triangle ACD, since $\angle ACD = 45°$, AC = 2 ft., $CD = 2\sqrt{2}$ ft.

Thus AC = 2 ft., $CE = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$ ft., $EA = \sqrt{10}$ ft.

Now $$\frac{20}{AC}=\frac{P}{CE}=\frac{R}{EA}.$$

Therefore $$\frac{20}{2}=\frac{P}{3\sqrt{2}}=\frac{R}{\sqrt{10}}$$

$$P=30\sqrt{2}=42{\cdot}4 \text{ lb. wt.}$$
$$R=10\sqrt{10}=31{\cdot}6 \text{ lb. wt.}$$

Since $\tan \angle EAB=\frac{1}{3}$,

$$\angle EAB=18^\circ\ 26'.$$

Hence R is inclined at 18° 26′ to the direction BA.

Example (3).—*A rectangular packing case is being raised by a chain* OAB, *which is attached to a hook at* O *and passes round smooth pulleys at* A *and* B, *which may be taken as the midpoints of two parallel, horizontal edges of the case. If the weight of the case is* 5 *cwt., find the tension in the chain and the force exerted on each pulley.* AB = 6 *ft., the total length of the chain* OAB *is* 16 *ft.*

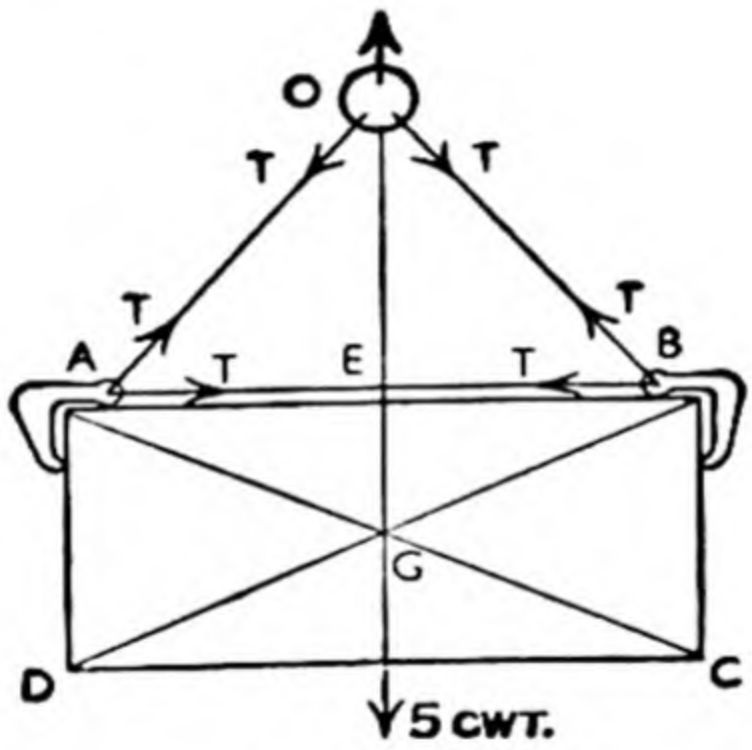

FIG. 265.

Let ABCD represent the packing case.

Since the pulleys at A and B are smooth, the tension throughout the chain (whose weight is neglected) is constant.

The resultant of the two tensions at O bisects the angle AOB, and it must balance the force on the hook at O, which is equal to the weight of the packing case. G, the centre of gravity of the case, is vertically below O. Let OG meet AB at E.

AE = 3 ft., and AO = 5 ft., since AO + AE = OB + BE = 8 ft.

The triangle of forces for the forces acting at O may now be drawn, and T, the tension in the chain, determined.

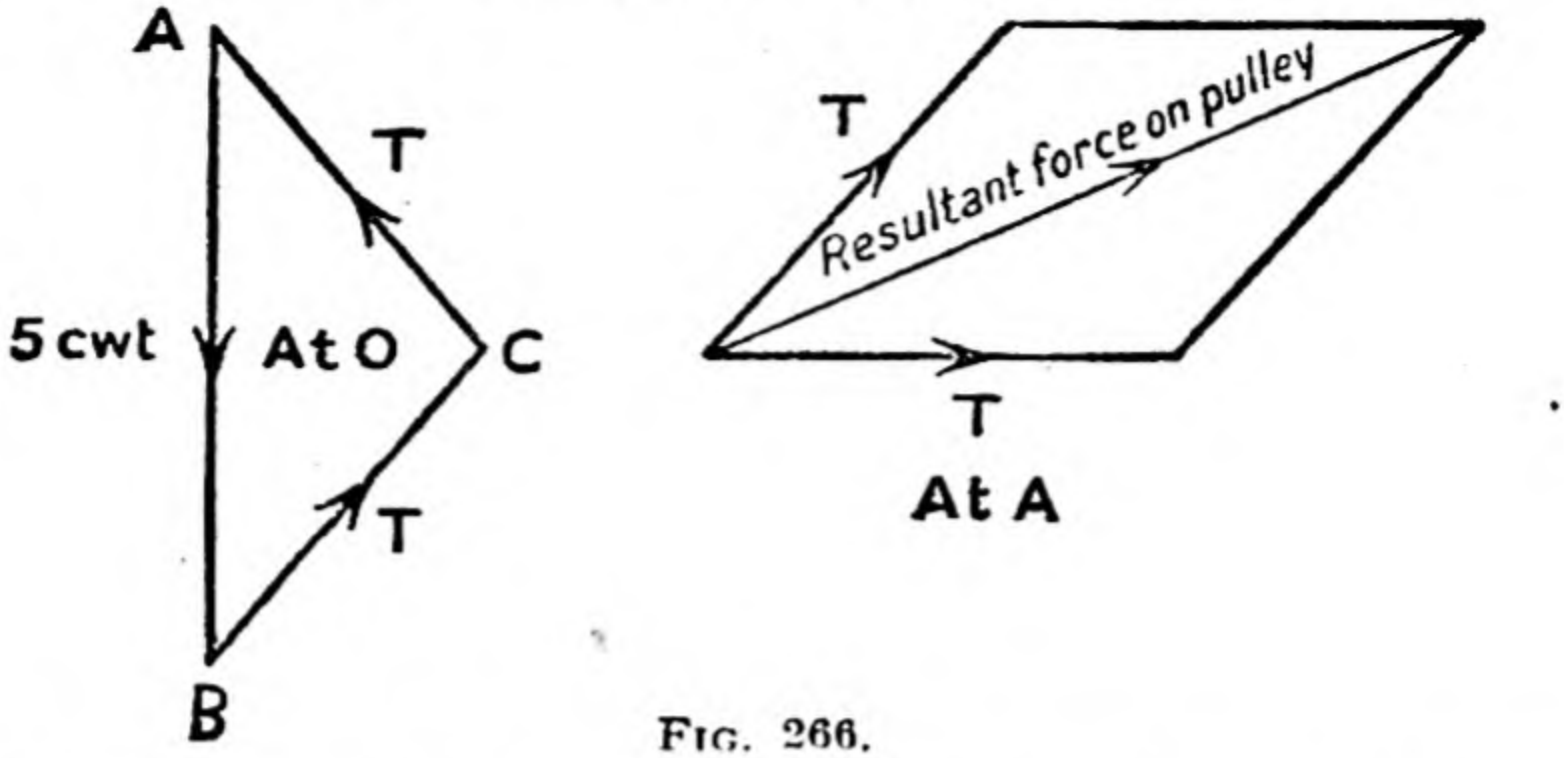

FIG. 266.

Having found T, the force on each pulley may be found by the parallelogram law.

Solution by Calculation.—Resolve vertically at O.

$$2T \cos AOE = 5.$$

$$2T \cdot \frac{4}{5} = 5.$$

$$T = 3{\cdot}125 \text{ cwt.}$$

Let R be the resultant of the two equal tensions acting at A upon the pulley.

Using $R^2 = P^2 + Q^2 + 2PQ \cos \theta$.

$$R^2 = \left(\frac{25}{8}\right)^2 + \left(\frac{25}{8}\right)^2 + 2 \cdot \frac{25}{8} \cdot \frac{25}{8} \cdot \frac{3}{5} = \left(\frac{25}{8}\right)^2 \cdot \frac{16}{5}.$$

$$R = \frac{5\sqrt{5}}{2} = 5{\cdot}59 \text{ cwt.}$$

Example (4).*—*A uniform ladder* AB, *of weight* W, *rests with the end* A *in contact with a vertical wall at which friction may be neglected, and the end* B *in contact with the ground, which is horizontal. If the angle of inclination of the ladder to the vertical is* θ, *and the ladder is on the point of sliding, prove that* $\tan \lambda = \frac{1}{2} \tan \theta$ *where* λ *is the angle of friction between the ladder and the ground. Find also the reactions at the ends* A *and* B *of the ladder in the case when* $\theta = 30°$.

Since the wall is smooth, the reaction of the wall on the ladder is perpendicular to the wall.

The weight of the ladder acts vertically downwards through its mid-point G.

The *resultant* reaction R at B must pass through the point of inter-

* This example should only be read after Chapter XVI. has been studied.

section of the lines of action of the weight of the ladder and the normal reaction at A, viz., O.

Since the ladder is in limiting equilibrium, OB is inclined at $\lambda°$ to the vertical, where λ is the angle of friction at B.

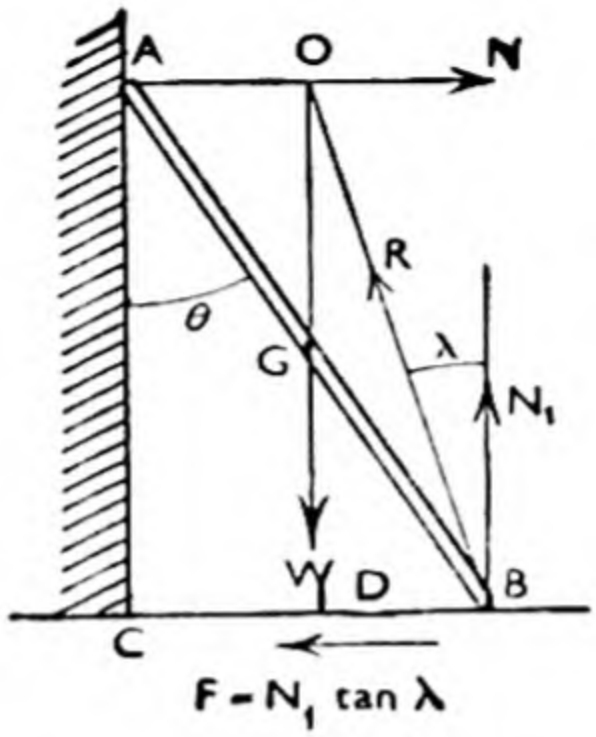

FIG. 267.

From the figure, GA = GB, GD // AC, therefore CD = DB.

$$\text{Tan } \theta = \frac{BC}{AC}.$$

$$\text{Tan } \lambda = \frac{BD}{OD} = \frac{BD}{AC} = \frac{1}{2} \cdot \frac{BC}{AC} = \tfrac{1}{2} \tan \theta.$$

Given the angle θ, the above figure may be accurately drawn and the angle λ measured.

When $\theta = 30°$, $\tan \lambda = 0{\cdot}5774/2 = 0{\cdot}2887$.

Therefore $\lambda = 16° 6'$.

OD is parallel to the line of action of W,
DB " " " " N_2,
BO is the line of action of R,
ODB is a triangle of forces.

$$\frac{R}{W} = \frac{OB}{OD} = \sec \lambda.$$

$$R = W \sec \lambda = 1{\cdot}04 \text{ W}.$$

$$\frac{N_2}{W} = \tan \lambda.$$

$$N_2 = W \tan \lambda = 0{\cdot}289 \text{ W}.$$

EXAMPLES XXXVIII

THREE FORCES ON RIGID BODIES (SOLVE GRAPHICALLY OR OTHERWISE)

1. A uniform bar AB, of length 5 ft., weighs 200 lb., and is smoothly hinged at A. The end B is held by a horizontal rope CB, of length 4 ft.,

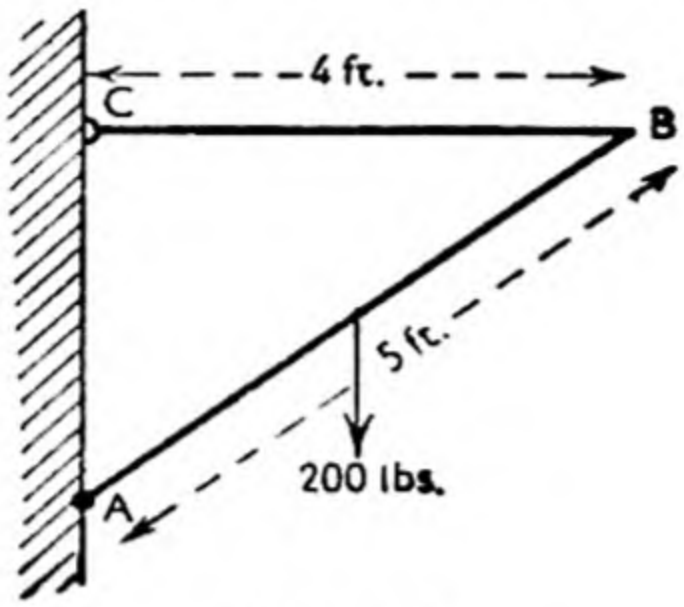

FIG. 268.

fastened at a point C vertically above A. Draw a careful diagram showing the way in which the tension in CB, the weight of the bar and the reaction of the hinge at A act upon the bar. Find the tension in CB and the reaction at A in magnitude and direction.

2. ABC represents a wall crane. AB is a horizontal rigid girder, of length 8 ft., hinged at A, and BC is a tie-bar secured at a point C, 4 ft.

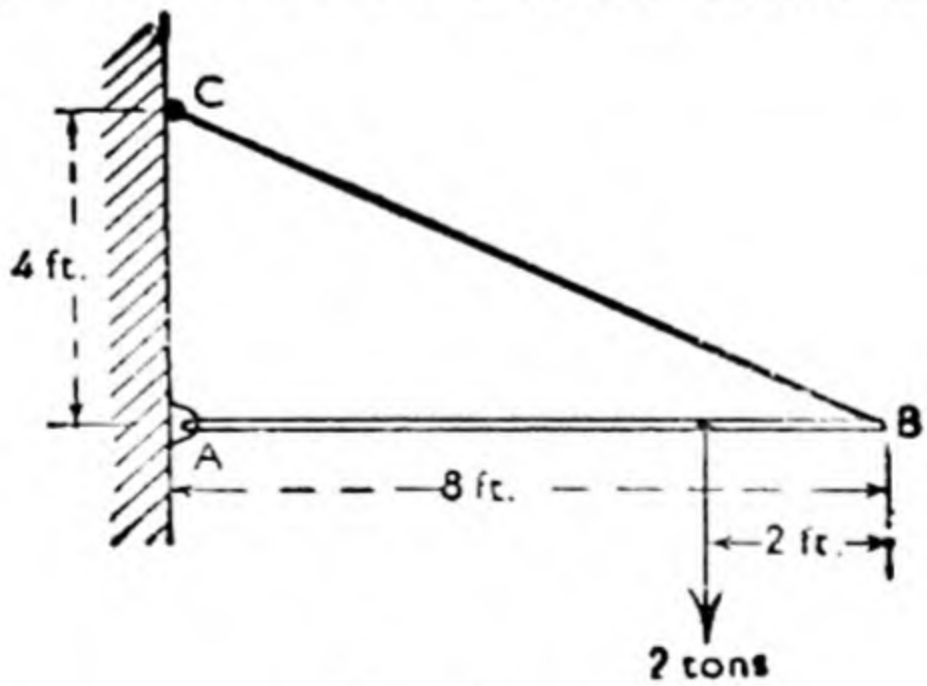

FIG. 269.

vertically above A. If the load on the girder (including its weight) is 2 tons and acts at a point 2 ft. from B, draw a diagram showing the forces acting on the girder, and determine the tension in BC and the reaction at A in magnitude and direction.

3. Find graphically or otherwise what force P applied to the handle of a garden roller weighing 3 cwt. (see figure, not to scale) will support

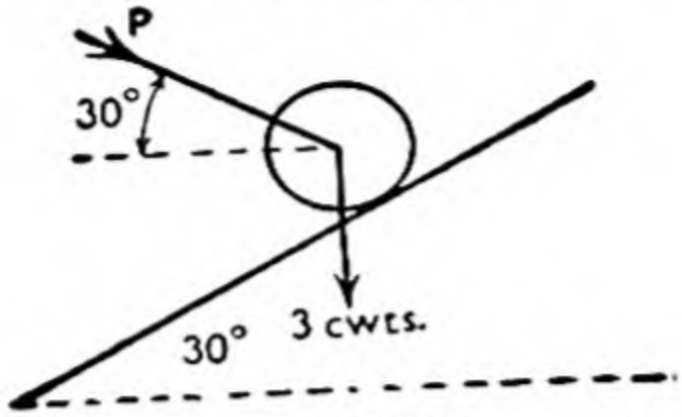

FIG. 270.

it on a slope of 30°. Find also the normal reaction of the roller on the slope.

4. A uniform bar of length 6 ft. 6 in. and weighing 50 lb. is suspended from a point by two cords of lengths 2 ft. 6 in. and 6 ft. respectively attached to its ends. Find the tension in each string and the angles which they make with the vertical.

5. AB (figure, not drawn to scale) gives the side view of the tailboard of a cart. It is supported by two chains (CD and a corresponding chain at the other side), C is vertically above A (the hinge), AB = 20 in. The

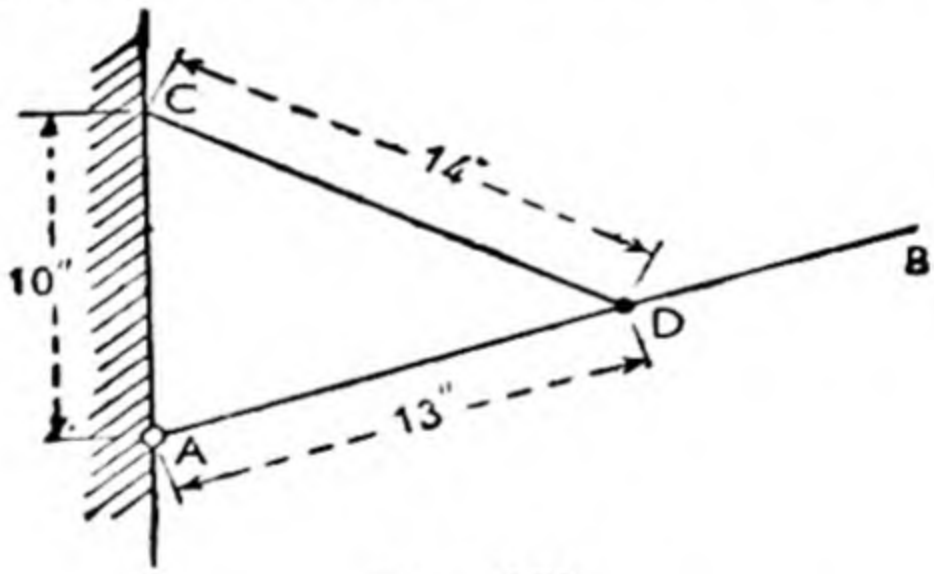

FIG. 271.

other dimensions are indicated on the figure. Draw the figure to scale, and thus find the additional tension of the chains and the reaction at A when a load of 12 st. is balanced on the edge B of the tailboard. What further information would be necessary for determining the distribution of this additional tension between the chains ? (*O.C.*)

6. A uniform rod AB, 2 ft. long, is hinged at A and hangs vertically. A horizontal force of 10 lb. wt. applied at the end B causes the rod to rest inclined at 45° to the vertical. Find the reaction at the hinge and the weight of the rod.

7. A uniform rod of length 1 ft. and weight 8 lb. is supported at both ends in a horizontal position by a cord 20 in. long which passes over a smooth peg. Find the tension in the cord.

8. A uniform bar 4 ft. long weighing 10 lb. is hinged to a wall and

held horizontal by a cord running from the end of the bar to a hook in the wall 3 ft. above the hinge. Find the tension in the cord. What will be the effect on it of hanging a weight of 10 lb. from the end of the bar? Explain carefully the steps in your calculation. (*L.M.*)

9. A five-barred gate weighing 1 cwt. hangs by hinges 2 ft. 6 in. apart. The centre of gravity of the gate is 4 ft. horizontally from the vertical line through the hinges. Assuming that the whole weight of the

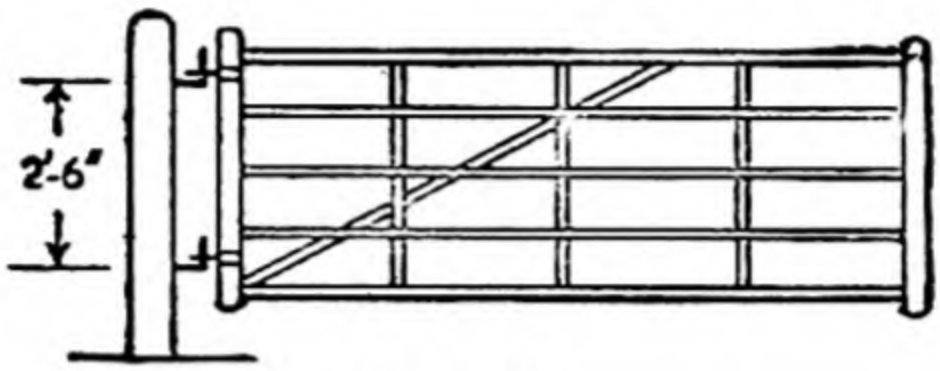

FIG. 272.

gate is taken (1) by the upper hinge, (2) by the lower hinge, find the reactions at the hinges in each case. Which do you consider the better arrangement? (*O.C.*)

10. A rigid body is in equilibrium under the action of three forces. State the necessary conditions of equilibrium.

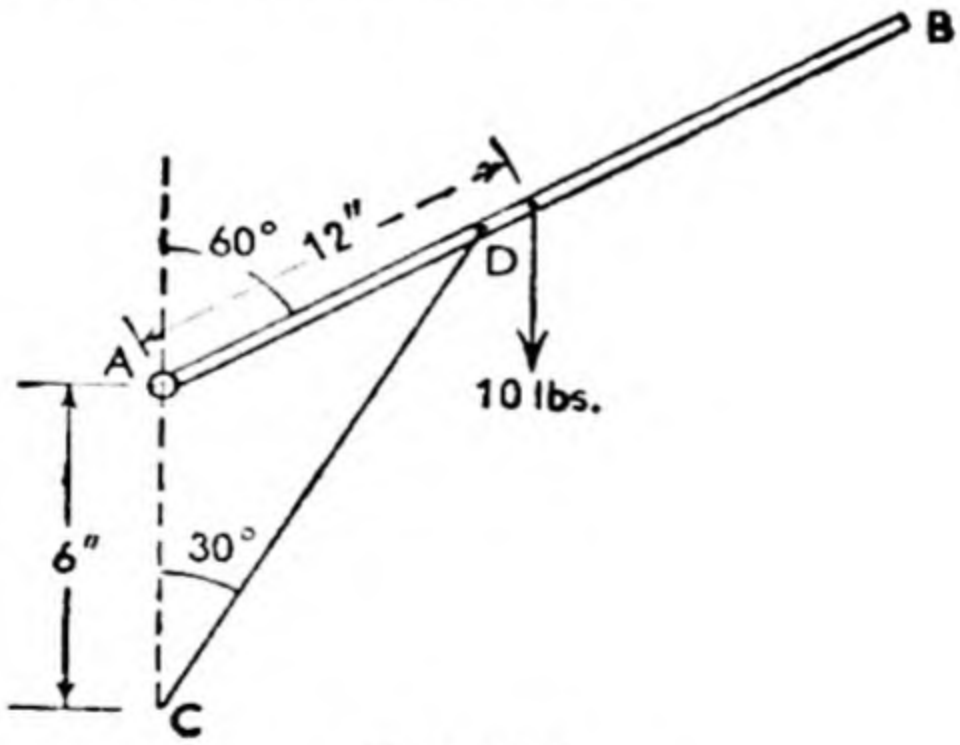

FIG. 273.

The heavy bar AB, hinged at A, is held at 60° with the vertical by the light stay CD. C is 6 in. below A, and the weight of the bar (10 lb.) acts at a point 12 in. from A. Find graphically or otherwise the thrust in the stay, and the magnitude and direction of the reaction at the hinge. (*O.C.*)

11. A uniform rod 100 cm. long is supported at one end by a hinge and at the other by a light cord which is attached at a point 50 cm. above the centre of gravity of the rod, which is thereby kept horizontal. If the mass of the rod is 100 gm., and a load of 800 gm. is placed at 75 cm.

from the hinge, find by any means the tension in the cord and the reaction at the hinge. Draw a diagram showing the directions of the forces involved. [Hint: Combine the weight of the rod and the load to form one force acting vertically through their centre of gravity.] (*L.M.*)

12. Explain how you would determine whether three coplanar forces acting at a point are in equilibrium.

Two smooth balls, each having a radius of 10 cm. and weighing 100 gm., are suspended from the same hook by threads 10 cm. long attached at the surfaces, so that the balls hang in contact. What is the reaction between the balls and the tension in each thread? Indicate clearly on a diagram the forces on one of the balls. (*L.M.*)

13. The figure represents a line diagram of a lean-to roof in which

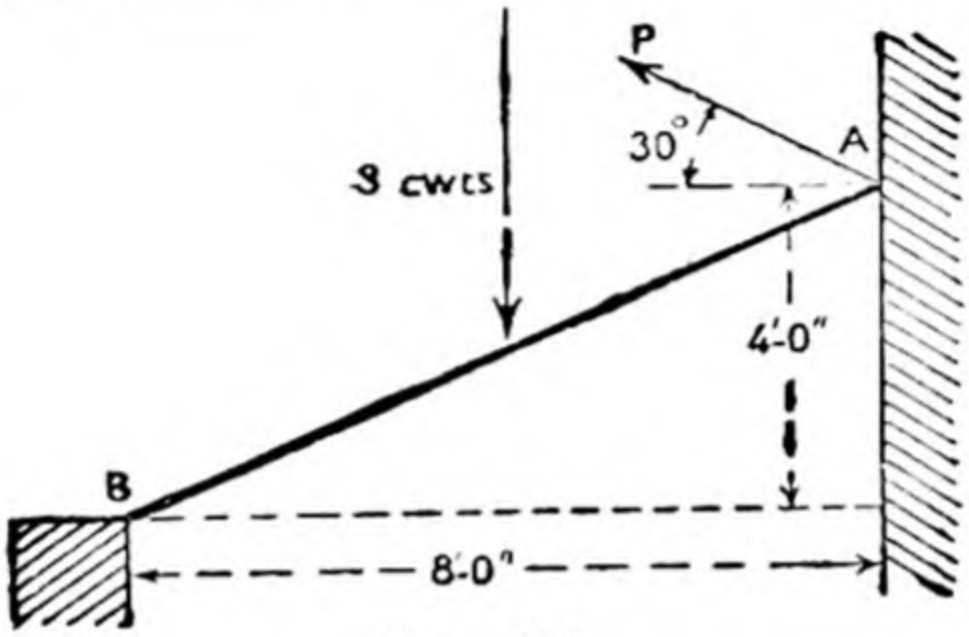

FIG. 274.

the reaction from the main wall at A is assumed to be in the direction shown. Determine the reaction at B and the value of P.

14. State the conditions which must be satisfied by three non-parallel forces which keep a rigid body in equilibrium.

A ladder rests with one end on rough ground and the other against a smooth vertical wall. The foot of the ladder is 5 ft. from the wall and the top 12 ft. above the ground. Determine graphically or otherwise the forces exerted by the wall and the ground on the ladder when the total load at its centre is 13 st. (*L.M.*)

15. Two smooth spheres, each of radius $2\frac{1}{2}$ in. and weight W, rest

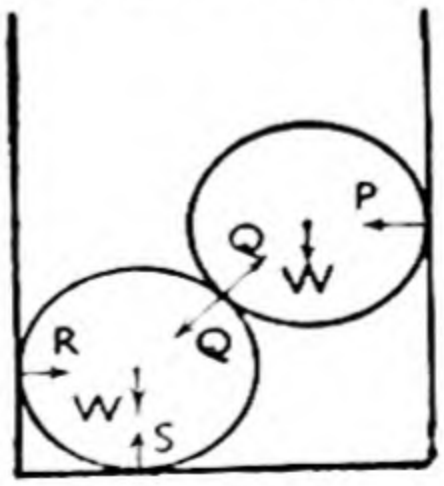

FIG. 275.

one on the other inside a smooth cylindrical tin of internal radius 4 in. Find the reactions P, Q, R and S between the spheres and between the spheres and the vertical wall of the tin. [Hint: Find P and Q first, then proceed to lower sphere, resolving vertically and horizontally to find S and R.]

16. A rigid bent lever ABC is hinged at B, and forces of 100 and F lb. wt. act perpendicular to the parts CB and AB at C and A respectively,

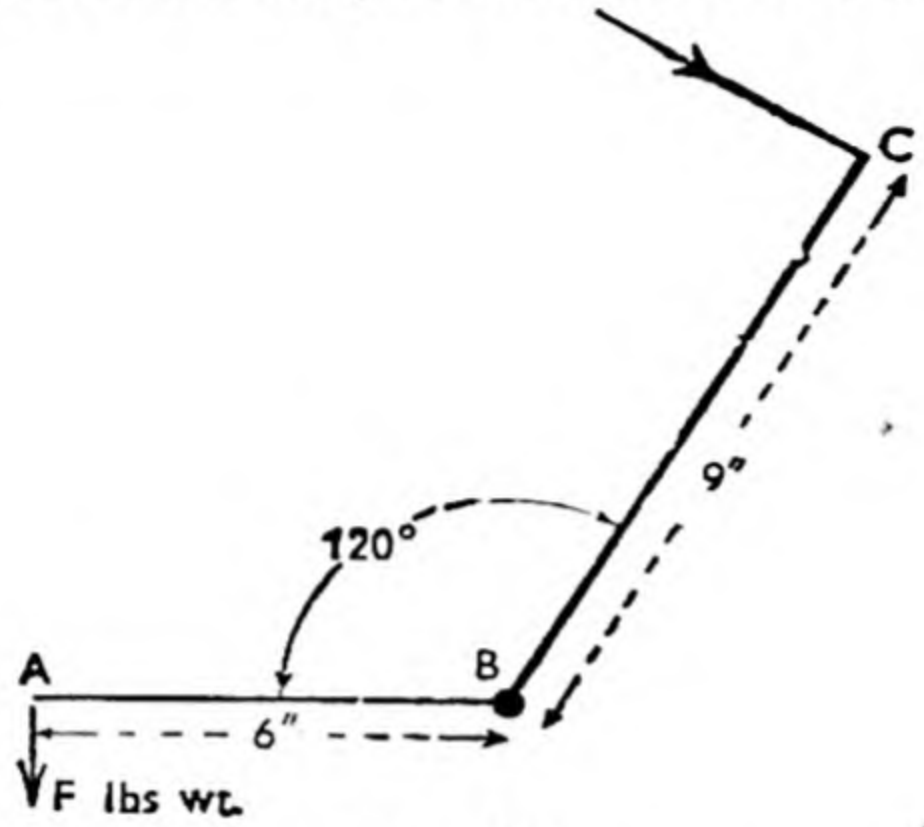

FIG. 276.

and maintain the lever at rest. Find the force F and the reaction at B of the hinge on the lever.

17. The luggage carrier AB of a bicycle is hinged to the frame at A, and supported in a horizontal position by the stay CD. AC = 10 in.,

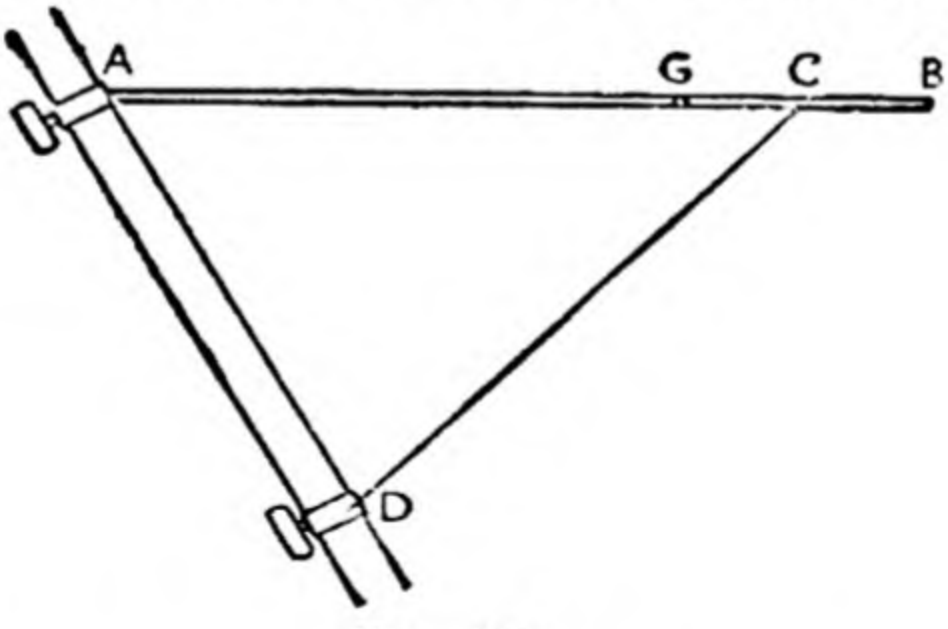

FIG. 277.

CD = 10 in., AD = 9 in. If a load of 28 lb. is imposed on the carrier at G (AG = 8 in.), find graphically the magnitude and direction of the additional stresses caused in CD and at A. (*G.S.*)

18. A uniform rod AB rests with the end A in contact with a smooth vertical wall and on a smooth horizontal peg at C. If the peg is 4 in.

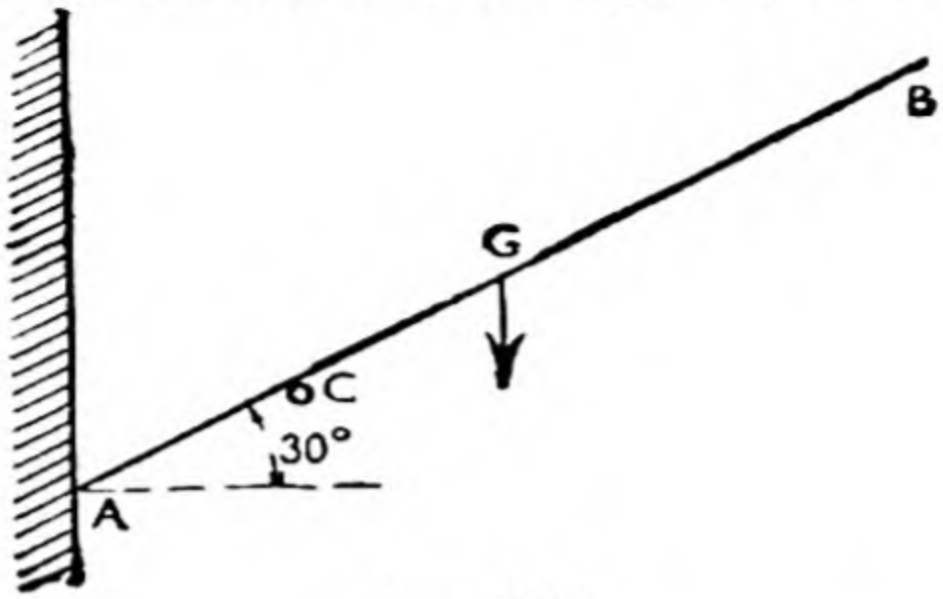

FIG. 278.

from the wall and the rod makes an angle of 30° with the horizontal, find the length of the rod.

19. A uniform bar AB, of weight 20 lb., is smoothly hinged at A. A cord BC is fastened to the end B of the rod, and to a point C at the

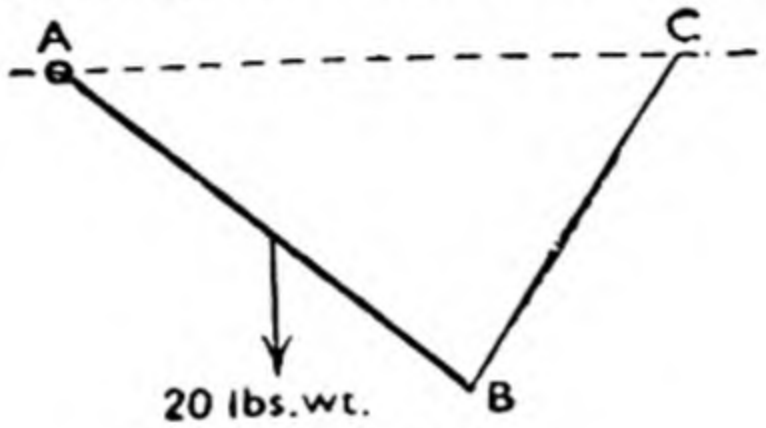

FIG. 279.

same horizontal level as A. If AB = AC = 5 ft. and BC = 6 ft., find the tension in BC and the magnitude and direction of the reaction at the hinge.

20. A uniform rod of weight 2 lb. rests with one end in contact with the inner surface of a smooth fixed hemispherical bowl, whose axis is vertical, and a point of its length resting on the rim of the bowl. If the radius of the bowl is 6 in. and the rod is inclined at 30° to the horizontal, find the length of the rod.

21. A uniform rod rests against a smooth vertical wall with its lower end on a smooth plane inclined at an angle α with the horizontal. Show that the rod makes an angle β with the vertical, where $2 \tan \alpha = \tan \beta$. (*O.C.*)

22. A garden roller of diameter 25 in. and mass 200 lb. is taken from one level to another 5 in. above it by pulling it by the handle up a step. What is the minimum force required to start the movement? What would the minimum force have been if the roller had been pulled up a uniform slope, 25 in. long, from the lower to the higher level? Mark the directions of the forces on a diagram. It may be assumed in each case that no slip occurs. (*G.S.*)

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CHAPTER XV

PARALLEL FORCES

142. The Resultant of Two Parallel Forces.

(*a*) *Like Parallel Forces.*—P and Q are two like parallel forces (*i.e.*, they act in the same direction, *or sense*, along two parallel

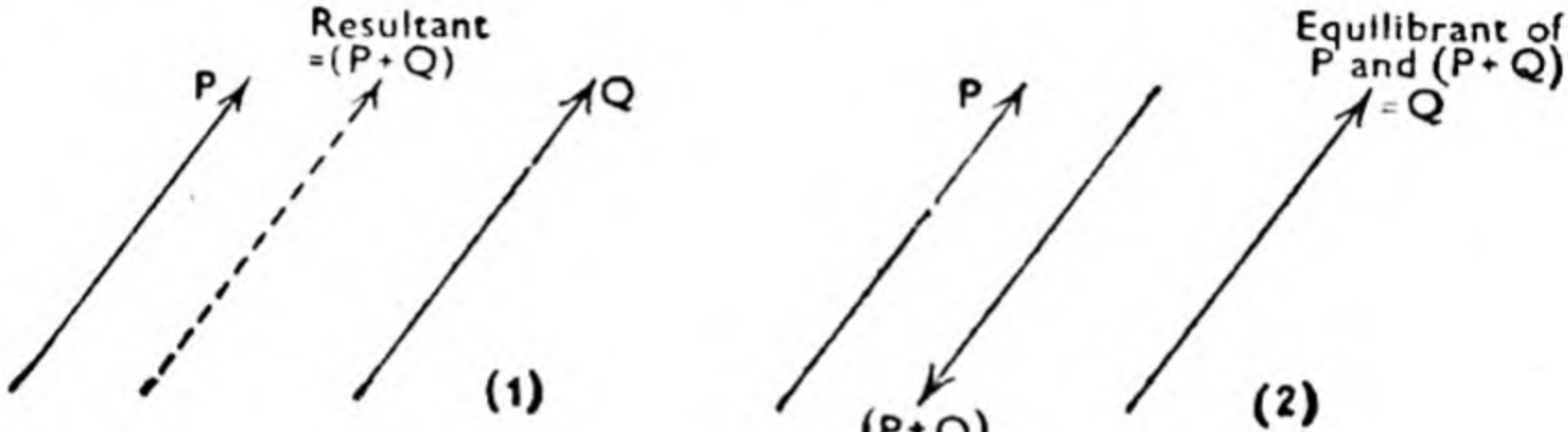

FIG. 280.

lines) acting upon a body. It has been shown experimentally that when a number of parallel forces act in one plane upon a body and keep it at rest,

(1) the algebraic sum of the forces is zero,
(2) the algebraic sum of the moments of the forces about any point in their plane is zero.

Since the equilibrant of P and Q is in equilibrium with P and Q, its magnitude is (P + Q) and it acts in the opposite direction to them. The resultant of P and Q is therefore (P + Q) in magnitude, its line of action is parallel to those of P and Q, and it acts in the same direction in the same plane.

Let AB be the intercept made by the parallel lines of action of P and Q on a line perpendicular to them.

Let the line of action of the resultant of P and Q divide AB internally at C into the two segments p and q.

To find the point C we may either use (2) above or proceed as follows: The moment of the resultant (P + Q) about any

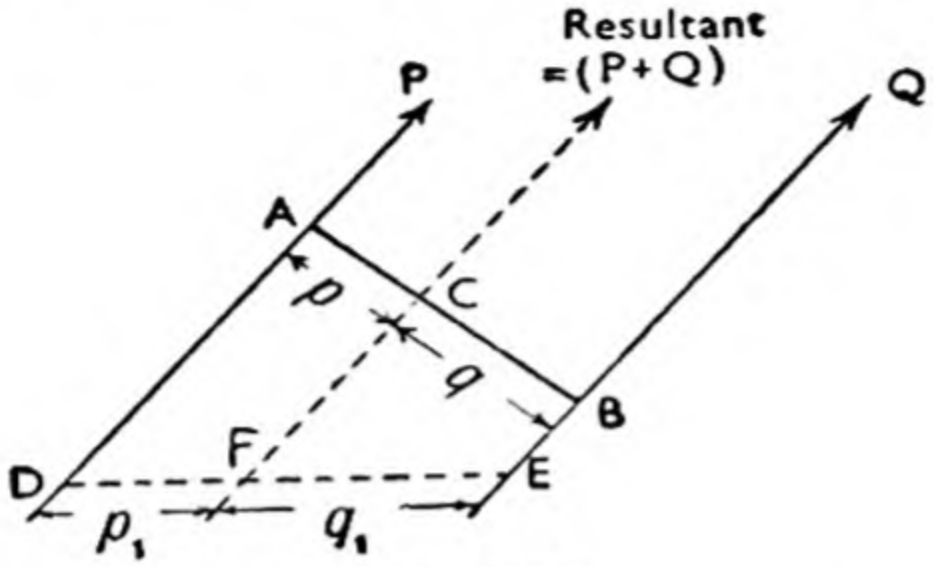

FIG. 281.

point in the plane of the forces is equal to the algebraic sum of the moments of P and Q about the same point, therefore taking moments about C, a point on the line of action of the resultant :

$$Pp - Qq = 0,$$

$$Pp = Qq,$$

$$\frac{P}{Q} = \frac{q}{p}.$$

AB is divided at C in the *inverse ratio* of the forces P and Q. If D and E are any two points on the lines of action of P and Q respectively and the line of action of their resultant divides DE at F into the two segments p_1 and q_1, then, since BE // CF // AD,

$$\frac{p}{q} = \frac{p_1}{q_1}.$$

The resultant of two like parallel forces is equal to their sum in magnitude, is parallel to the original forces and acts in the same direction. Its line of action divides the line joining any two points, one on the line of action of each of the two parallel forces, internally in the inverse ratio of the forces.

(*b*) *Two Unlike Parallel Forces.*—In Fig. 280 (1) we see

the resultant (P + Q) of the forces P and Q. In Fig. 280 (2) we see the force (P + Q) reversed in direction, and it is therefore the equilibrant of the forces P and Q.

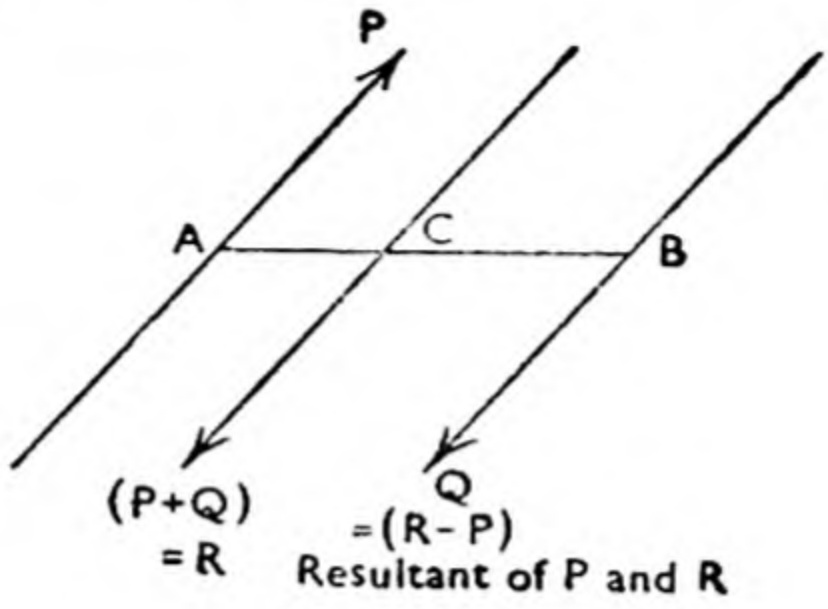

FIG. 282.

Now in Fig. 280 (2) since P, Q and (P + Q) are in equilibrium, the force Q is the equilibrant of the two *unlike* parallel forces P and (P + Q), hence in Fig. 282 the force Q reversed is the resultant of the unlike parallel forces P and (P + Q).

From above,

$$\frac{Q}{P} = \frac{AC}{CB'}$$

$$1 + \frac{Q}{P} = 1 + \frac{AC}{CB'}$$

$$\frac{P+Q}{P} = \frac{CB+AC}{CB} = \frac{AB}{CB}.$$

If we put R in place of (P + Q) then in place of Q we may put (R − P). The last result may be written,

$$\frac{R}{P} = \frac{AB}{CB}.$$

The resultant of two unlike parallel forces R and P (R greater than P) is equal to the difference (R − P) in magnitude, is parallel to the original forces and acts in the direction of the greater force. Its line of action divides the line joining two points, one on the line

of action of each of the forces R and P, externally in the inverse ratio of the forces.

Note.—The resultant acts on the side of the greater of the two given forces opposite to the smaller.

Example.—*Two boys push on the lever of a capstan in parallel-horizontal directions perpendicular to the lever, with forces of 15 and 45 lb. wt. at distances of 6 and 3 ft. from the axis of the drum. Where should a third boy push on the same lever to produce the same effect on the lever,* (a) *when the boys push the lever in the same direction?* (b) *when the boy push the lever in opposite directions?*

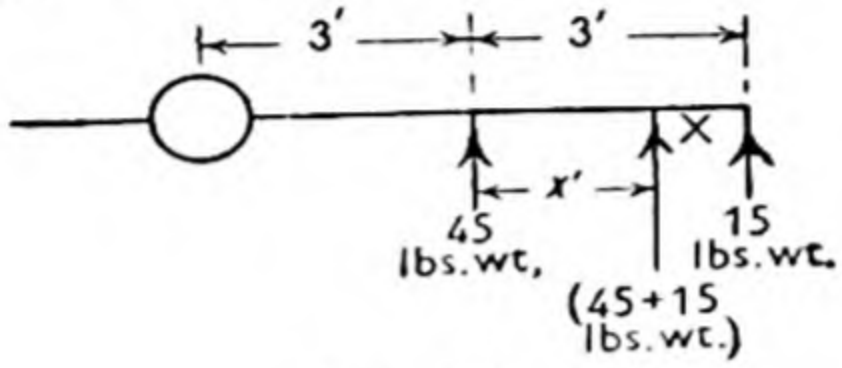

FIG. 283.

(*a*) Referring to the figure, X is the point at which the third boy must push, and B and C are the points at which 15 and 45 lb. wt. forces act.

The magnitude of the resultant of the 15 and 45 lb. wt. forces is 60 lb. wt. Also, taking moments about X,

$$15 \cdot BX = 45 \cdot CX.$$

If CX is x ft.,

$$45x = 15(3 - x).$$
$$60x = 45.$$
$$x = \tfrac{3}{4}.$$

The third boy must apply a force of 60 lb. wt. parallel to the original forces at a distance 3·75 ft. from the axis of the barrel and in the same direction as the original forces.

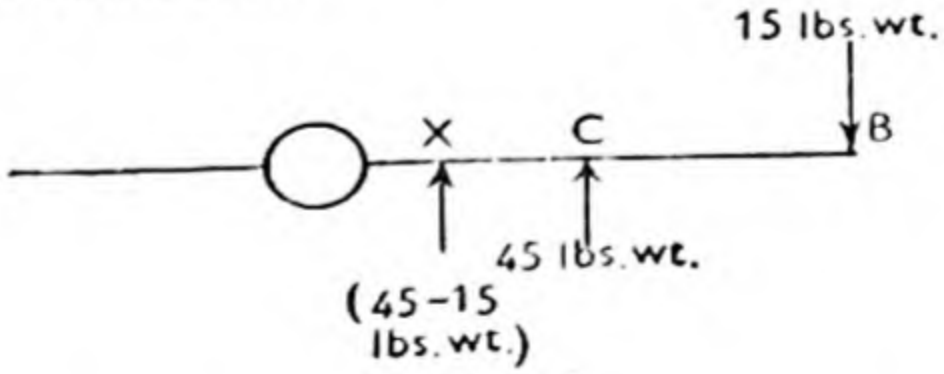

FIG. 284.

(*b*) Referring to the figure, X is the point at which the third boy must push, and B and C are the points at which the 15 and 45 lb. wt. forces act.

The magnitude of the resultant of the two unlike forces at B and C is a force of 30 lb. wt.

Also, if BX is equal to x ft.,

$$15 \cdot BX = 45 \cdot CX.$$
$$15x = 45(x - 3).$$
$$30x = 135.$$
$$x = 4{\cdot}5.$$

The third boy must exert a force of 30 lb. wt. parallel to the original forces at a distance of 1·5 ft. from the axis of the barrel and in the same direction as the 45 lb. wt. force.

EXAMPLES XXXIX

Parallel Forces

1. Parallel forces P and Q act at points A and B, and the line of the resultant meets AB or AB produced at C. Find the magnitude and sense of the resultant and the length of BC.

(*a*) Like force : P = 2 lb. wt., Q = 4 lb. wt., AB = 6 ft.
(*b*) Like forces : P = 30 lb. wt., Q = 40 lb. wt., AB = 7 in.
(*c*) Unlike forces : P = 10 kgm. wt., Q = − 6 kgm. wt., AB = 80 cm.
(*d*) Unlike forces : P = 100 gm. wt., Q = − 180 gm. wt., AB = 14 cm.

2. ABCD is a rectangle in which AB = 4 in. and BC = 3 in. Forces 3, 2, 6 and 3 lb. wt. act along the sides AB, BC, DC, DA respectively and in these directions. Find the resultant of the forces along AB, DC, stating where it cuts AD. Find the resultant of the forces along BC, DA and state where it cuts BA produced. Hence show on a diagram the resultant of all four forces, and find its magnitude and the angle which its line makes with the direction AB.

3. Forces P, 2P, 3P, 4P act along the sides AB, BC, CD, AD of the square ABCD. Find the resultant of these forces and the point at which its line cuts AB. The side of the square is $2a$ units.

143. The Couple.

Two equal, unlike, parallel forces form a couple.

Examples of Couples.—Winding up the spring of a clock. The couple is usually applied by the thumb and forefinger.

Turning a water main on or off by means of a turnkey. The couple is applied at the ends of the horizontal lever.

Using a hand-screw cutting tool for cutting a thread on the end of a metal pipe.

Turning on a household water-tap.

A body is acted upon by two equal, unlike, parallel forces P.

The lines of action of the forces are a distance p apart.

O is any point in the plane of the forces, and OAB drawn perpendicular to the forces meets their lines of action at A and B.

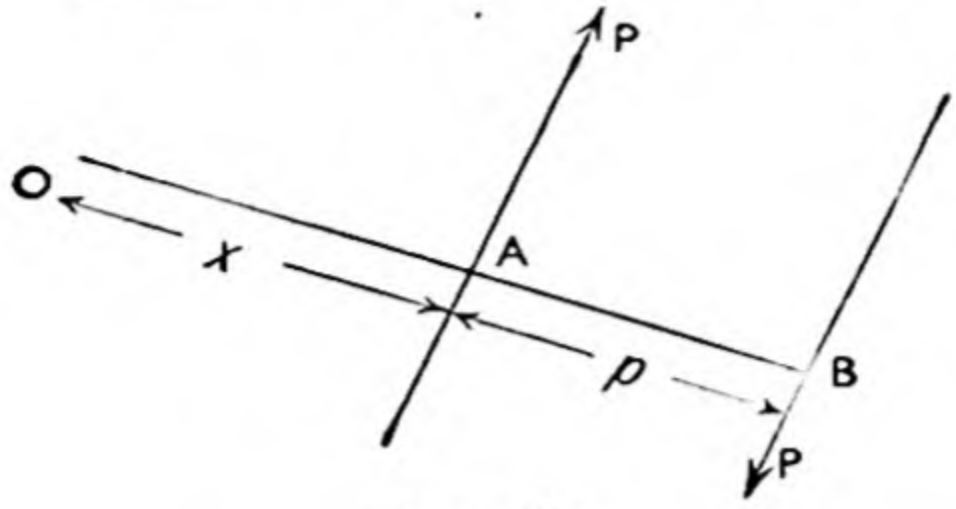

FIG. 285.

OA = x units, OB = $(p + x)$ units.

The moment of the couple about O = $P(p + x) - Px = Pp$.

The perpendicular AB ($= p$) is called the arm of the couple.

The above result may be stated as follows :—

The moment of a couple about any point in its plane is constant and equal to the product of one force of the couple and the arm of the couple.

A singe force cannot have a moment about every point in the plane of the couple equal to that of the couple. It is therefore impossible to balance the effect of a couple by a single force.

Since the turning effect of two equal couples which are opposite in sense (viz., one clockwise and the other anti-clockwise) acting upon the same body are equal and opposite about all points in their plane, the couples are in equilibrium.

A couple can be balanced by an equal and opposite couple acting in the same plane.

144. Graphical Representation of a Moment.

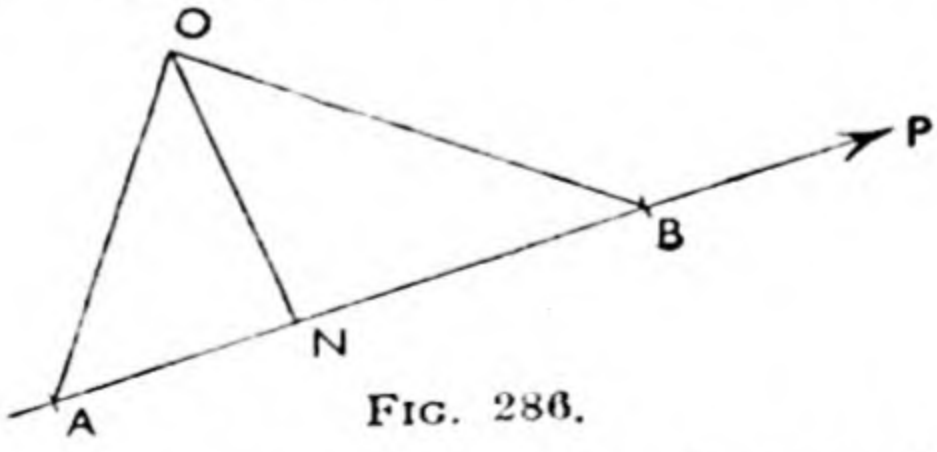

FIG. 286.

Let AB represent a force P marked off along the line of action of the force.

The moment of P about a point O is represented by

$$AB \times ON,$$

where ON is the perpendicular from O to the line of action of P.

The area of the triangle AOB is $\frac{1}{2}$ AB × ON.

Hence twice the area of the triangle AOB represents the moment of P about O.

145. The Algebraic Sum of the Moments of Two Forces about any Point in their Plane is Equal to the Moment of their Resultant about the same Point.

1. *Let the lines of action of the forces meet in a point.*

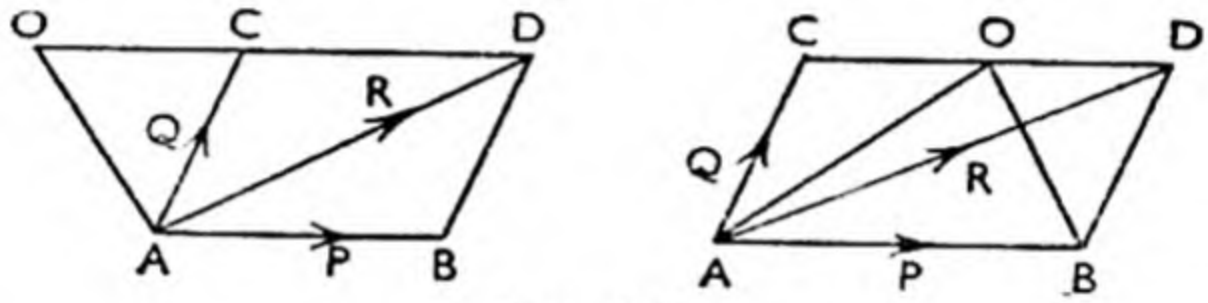

FIG. 287.

P and Q are the forces acting at A, and O is any point in their plane, outside the angle between the forces in one figure and inside this angle in the other.

Draw OC parallel to the direction of P to meet the line of action Q at C.

Take the length of AC to represent the magnitude of Q, and on the same scale let AB represent P.

Complete the parallelogram ABDC and join OA, OB.

AD represents the resultant R of P and Q.

In either figure,

the moment of P about O is represented by 2 △AOB,
" " Q " O " " 2 △AOC,
" " R " O " " 2 △AOD.

Also,

△AOB = △ADB, on the same base and between the same parallels.
= △ADC.

In the figure on the left above, the moments about O of P and Q are both positive, and their algebraic sum is represented by

$$\begin{aligned} 2\triangle AOB + 2\triangle AOC &= 2\triangle ADC + 2\triangle AOC \\ &= 2\triangle OAD \\ &= \text{moment of R about O.} \end{aligned}$$

In the other figure, the moments are in opposite directions and their algebraic sum is represented by

$$\begin{aligned} 2\triangle AOB - 2\triangle AOC &= 2\triangle ADC - 2\triangle AOC \\ &= 2\triangle OAD \\ &= \text{moment of R about O.} \end{aligned}$$

2. *Let the lines of action of the forces be parallel.*

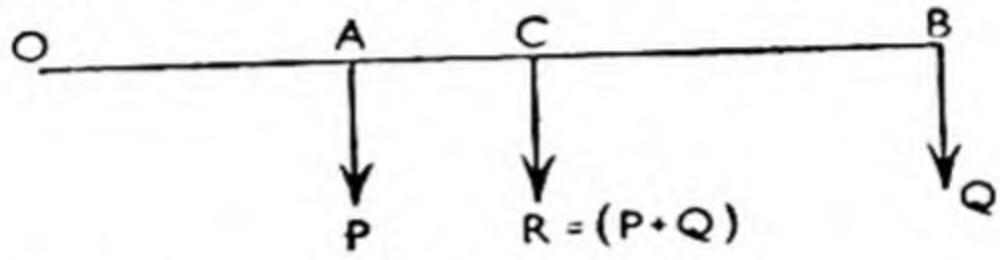

FIG. 288.

P and Q are two parallel forces. O is any point in their plane.

OAB is drawn perpendicular to the lines of action of the forces and meets them at A and B.

OAB meets the line of action of the resultant of P and Q at C such that

$$P \,.\, AC = Q \,.\, CB.$$

In the above figure the sum of the moments of P and Q about O is

$$\begin{aligned} & P \,.\, OA + Q \,.\, OB \\ &= P(OC - AC) + Q(OC + CB) \\ &= (P + Q) \,.\, OC + Q \,.\, CB - P \,.\, AC \\ &= (P + Q) \,.\, OC \\ &= \text{moment of R about O.} \end{aligned}$$

If P and Q are unlike and unequal, then it may be shown as above that the sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

The results 1 and 2 above may be extended to the case of any number of forces acting in one plane, for the resultant R of P and Q may be considered with a third force and the above result applied to these two forces, and so on.

If a number of forces acting on a body in one plane have a single resultant, the moment of this resultant about any point in their plane is equal to the algebraic sum of the moments of the number of forces about the same point.

This theorem is known as Varignon's Theorem. It was first stated by him in the year 1687.

146. The Resultant of any Number of Parallel Forces and the Conditions for Equilibrium.

Suppose that a number of parallel forces act in one plane on a body. The algebraic sum of the equilibrant of the given forces and the forces themselves is zero. Hence if the algebraic sum of the given forces in a direction parallel to their lines of action is S units of force, the equilibrant of the forces is $-S$ and has the same line of action.

Hence the resultant of the given forces is S units, i.e., *the algebraic sum of the forces.*

It has also been shown that for any number of forces acting in one plane *the algebraic sum of the moments of the given forces about any point in their plane is equal to the moment of their resultant about the same point.*

These two statements are sufficient to enable us to find the resultant of any number of parallel forces.

If a number of parallel forces have a zero resultant and the algebraic sum of their moments about any one point in their plane is zero, the forces are in equilibrium.

If the algebraic sum of the forces is zero, but the algebraic sum of their moments about any one point in their plane is not zero, then the forces are equivalent to a couple ; for the resultant of the forces in one direction is equal in magnitude but opposite in direction to the resultant of the remaining forces, these two unlike forces of equal magnitude form a couple which can only keep the body at rest if the lines of action of its two forces coincide,

in which case there is no turning effect about any point in the plane of the forces.

Example.—*A beam is acted upon by forces* 5, 7, 8 *and* −4 *lb. wt. at the positions shown in the figure. If the forces exerted at the supports of the beam are to remain unaffected, find the single force which may be applied to the beam in place of the given forces.*

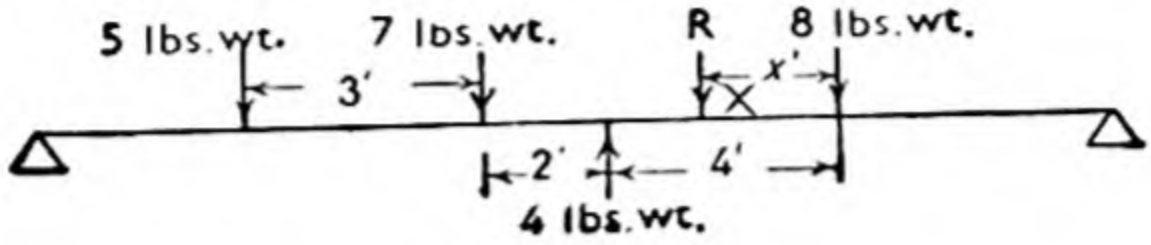

FIG. 289.

The single force which produces the same effect upon the beam as the given forces is the resultant of the forces.

Let the resultant of the 5, 7, 8 and −4 lb. wt. forces act at X, x ft. from the 8 lb. wt. force.

The resultant of the given forces is a parallel force equal in magnitude to their algebraic sum, *i.e.* $(5+7+8-4)$ lb. wt. $=16$ lb. wt. vertically downwards.

To find the position of X, take moments about X.

Since X is a point on the line of action of the resultant of the given forces, the algebraic sum of their moments about X is zero.

$$8x+4(4-x)-7(6-x)-5(9-x)\equiv 0.$$

Therefore $$x=6{\cdot}31.$$

The single force is 16 lb. wt., and acts vertically downwards on the beam at a point 6·31 ft. to the left of the 8 lb. wt. force. This force produces the same effect upon the beam as the given forces, and the reactions on the supports of the beam are unaltered.

EXAMPLES XL

PARALLEL FORCES, COUPLES

1. Determine the resultant of parallel forces 1, 2 and 3 lb. wt. whose lines of action pass through points A, B and C respectively on a straight line ABC. AB $=2$ in., BC $=5$ in. State the distance from A of the point on ABC at which the resultant cuts it.

2. Find the resultant of the parallel forces 5 gm. wt., −6 gm. wt. and 2 gm. wt., the distances between their lines of action being 5 and 12 cm. respectively. Give the distance of the resultant from the line of the 5 gm. wt. force.

3. Parallel forces of 2, 5, 8, 10 lb. wt. act on a body. The distances between the successive lines of action being 3, 4, 7 in. respectively. Find the magnitude of the resultant of this system, stating its distance from the 2 lb. wt. force.

4. Parallel forces 14, −4, 6 and −3 gm. wt. act on a body. The distances between their lines of action are 4, 2 and 5 cm. respectively. Find their resultant and its distances from the 14 and −3 gm. wt. forces.

5. What do you understand by the moment of a force? What factors determine its magnitude?

A mass of 50 lb. is suspended from the end B of a uniform bar AB, 10 ft. long, which is hinged to a vertical wall at A, and projects perpendicularly from the wall. The mass of the bar is 8 lb., and it is kept horizontal by a cord which is fastened to the midpoint and to a point 5 ft. vertically above the hinge. Indicate on a diagram the forces acting on the bar AB, and find the tension in the cord. (*L.M.*)

6. What conditions must be fulfilled if a set of three parallel forces act upon a body and maintain it in equilibrium?

A uniform bar AB, 10 ft. in length and weighing 20 lb., rests on supports placed at its ends. A load of 40 lb. is carried by the bar at a point distant 8 ft. from the end A. Calculate the thrust on each support. (*G.S.*)

7. Show how to find the resultant of a system of parallel forces.

Two weights of 10 and 3 lb., hanging from the extremities of a uniform lever 4 ft. long and weighing 6 lb., keep the lever at rest. If each weight be moved inwards by 6 in., through what distance must the fulcrum be moved to maintain equilibrium? (*L.M.*)

8. What is a couple? Mention three of the chief properties of couples. Find the resultant of three forces represented in magnitude, direction and position by the three sides of a triangle taken in order, these sides being 3, 3 and 5 units long. (*L.M.*)

9. A uniform plank, 8 ft. long and weighing 12 lb., is supported horizontally by two vertical strings at its ends A and B. If the maximum loads which the strings at A and B can support without breaking are 10 and 20 lb. wt. respectively, find where a load of 15 lb. must be placed on the plank so that the string at (*a*) the end A, (*b*) the end B is just on the point of breaking. Where must a load of 18 lb. be placed so as not to break the strings? (*L.M.*)

10. What conditions must be satisfied for a rigid body to be in equilibrium under the action of a number of parallel forces?

A uniform lever AB, 3 ft. long, weighing 500 gm., is suspended by two vertical strings attached 6 in. from each end, and a 200 gm. wt. is hung from the lever 12 in. from A. What is the tension in each string when the lever is horizontal? From what other point on the lever must the 200 gm. wt. be suspended for the tension in the string near the end A to be twice that in the other? (*L.M.*)

11. Show that the moment of a couple about a point in its plane is the same for all positions of the point. A metre rule weighing 200 gm. is pivoted at a point 40 cm. from the left end. It is acted upon by a clockwise couple, consisting of forces of 100 gm. applied at the ends and making 30° with the rule, the forces and rule being in the same vertical plane. What weight hung from a point 30 cm. from one end will keep the rule horizontal? Draw a diagram of the arrangement. (*J.*)

12. A uniform rod 12 ft. long, weighing 9 lb., is placed across two parallel rails 10 ft. apart and at the same height. Show that if a weight of 10 lb. is hung from one end of this rod, equilibrium will be main-

tained however the rod is placed. Find the smallest suspended weight for which this would cease to be true. (*B.*)

13. Explain the method you would use to find the resultant of three parallel forces. Illustrate by finding the resultant of three parallel forces of 4, 6 and 8 dynes which act in the same direction if the distance between the lines of action of the 4 and 6 dynes is 5 cm., and the distance between the 6 and 8 dynes is 10 cm. Find the centre of gravity of 4 masses of 50, 100, 150 and 200 gm. placed in order at the corners of a square of negligible mass and of side 10 cm. (*G.S.*)

14. Show how to find the line of action of the resultant of a system of parallel forces. Prove that the centre of gravity of a uniform triangular lamina coincides with the centre of gravity of three equal masses placed at the angular points. (*L.M.*)

15. State the conditions necessary for the equilibrium of a rigid body acted upon by three parallel forces.

A uniform steel shaft AB is 10 ft. long and weighs 75 lb. It rests on two bearings, one at A and the other 2 ft. from B. A pulley of weight 25 lb. is fixed at the centre of the shaft, and another of weight 20 lb. at the end B. Calculate the force on each bearing. (*G.S.*)

16. State the triangle of forces.

A triangular board has sides AB, BC and CA of lengths 30, 40 and 50 cm. Forces of 600, 800 and 1,000 gm. wt. act at A, B and C in the directions AB, BC, CA respectively. Find the resultant of the system of forces. (*L.M.*)

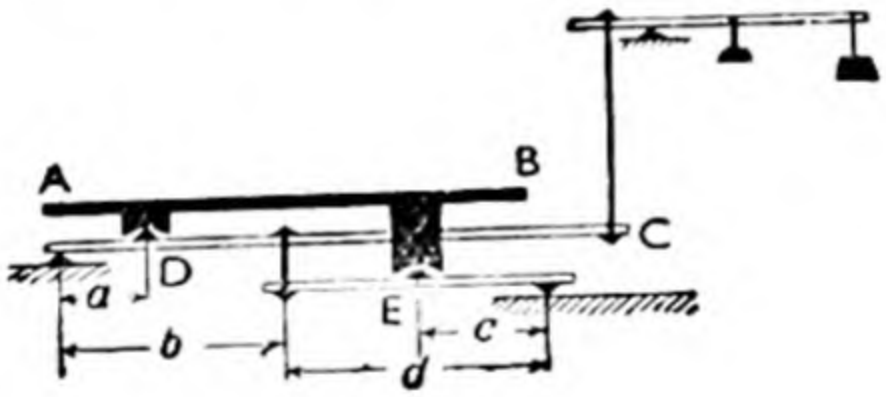

FIG. 290.

17. The diagram shows a luggage-weighing machine. Assuming that the arms of the levers are horizontal and neglecting their weights, prove that if a load L is placed on the platform AB the additional turning moment to be balanced by the upward force at C is L . *a* whatever the forces at D and E may be, and provided that $\frac{a}{b}=\frac{c}{d}$.

18. A uniform bar of wood AB, of mass 8 lb. and length 4 ft., has a piece of lead buried in it. When the bar is pivoted at a point 1 ft. from the end A, a weight of $24\frac{1}{2}$ lb. at that end is needed to balance it, and when pivoted 3 ft. from the same end, a weight of $2\frac{1}{8}$ lb. at this end is needed to balance it. Find the weight of the piece of lead and its position along the bar.

CHAPTER XVI

FRICTION

147. Friction.

When two bodies are in contact, the force which prevents or tends to prevent sliding at their surface of contact is called *The Force of Friction.*

The force of friction acts in a direction parallel to the surface of contact and always *opposes* the motion of the bodies relative to each other at this surface.

It is the force of friction which enables us to walk. When the sole of the shoe on the back foot is about to leave the ground the force of friction prevents it from slipping backwards, and when the heel of the shoe on the foremost foot meets the ground the force of friction prevents the shoe from sliding forwards. Walking is difficult on an icy surface because of the smallness of the force of friction which can be brought into play, and upon a perfectly smooth surface walking would be impossible.

A boy who begins to slide on an icy surface is finally brought to rest (provided he keeps his feet) by the force of friction between his boots and the ice ; if he begins to slide along the pavement he is brought to rest much more quickly.

A ladder which rests with one end against a vertical wall and the other on level ground is prevented from slipping by the forces of friction between the ladder and the wall, and between the ladder and the ground. The first of these forces of friction acts vertically upwards and the second acts horizontally towards the wall, as shown in the opposite figure. It is evident from what has been said that the amount of friction which can be brought into play depends upon the nature of the surfaces in contact ; *e.g.*, consider the same boy attempting to slide on different surfaces wearing different types of footwear. It is for this reason that in carrying out experiments to find out

further facts about the way in which friction acts, we have to be very careful to notice that any change in the nature of the surfaces in contact will affect the amount of friction brought into play.

148. EXPERIMENT.—We shall consider the friction between two plane wooden surfaces. A block of wood rectangular in shape has a staple

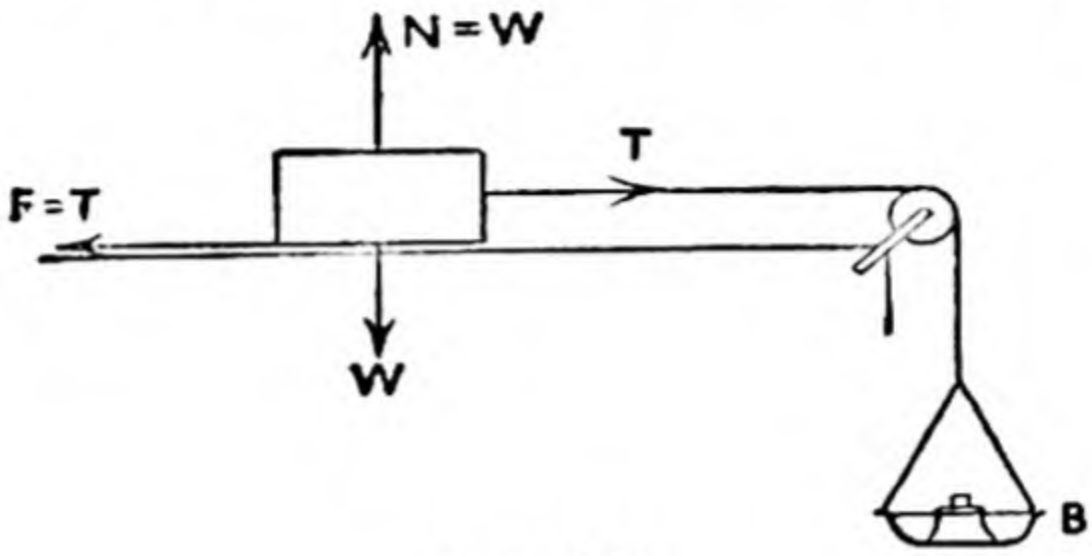

FIG. 291.

fixed in one of its vertical faces. To this staple is attached a string which, after passing over a smooth pulley, supports a scale pan of known weight. Weights are now added to the pan. As long as the block of wood remains at rest the force acting horizontally on the block is balanced by the friction brought into play; it therefore follows that *just enough friction is brought into play to prevent the block from sliding*.

If the weight in the pan is increased, a stage is reached when the block is on the point of moving. At this stage make a note of the weight of the block and the weight producing the tension in the string.

Load the block with further weights, not altering the surface in contact, and find the tension in the string when the block is again just on the point of moving. Repeat the experiment with five different weights of block. The tension in the string is equal to the maximum force of friction brought into play in each case.

Consider the forces acting on the block.

Horizontally, the tension T is balanced by the force of friction F.

Vertically, the normal reaction N, *i.e.*, the force of reaction perpendicular to the surface of contact, balances the weight of the block W.

Thus $F = T$

and $N = W$.

We have, of course, noticed that as the weight on the block is increased the maximum force of friction is increased. If we calculate the ratio F/N, however, we see that the ratio is very nearly the same in each case.

The ratio of the maximum force of friction to the normal reaction is constant.

The maximum force of friction is called the *Limiting* force of friction.

If F denotes the limiting force of friction, then

$$\frac{F}{N} = \mu$$

where μ is a constant number for surfaces of the same kind.

Thus $$F = \mu N.$$

It must be carefully noted, however, that this only applies to *limiting* friction. If the friction is not limiting, F may have any value between zero and μN. To find the best value of μ from the results of our experiment, the most accurate method is to draw the graph of F against N. The points lie approximately upon a straight line and the gradient of the line gives the value of μ.

EXPERIMENT.—For this experiment we require a block of material which has a number of plane surfaces of different areas but *of the same nature*. A block of polished metal or a block of glass is suitable, and the supporting plane surface may be of well-planed wood or, better, a block of well-planed wood covered by a sheet of cartridge paper. The above experiment is repeated in so far as the limiting force of friction is found when the block rests on each of its faces in turn.

The limiting force of friction is found to be the same in each case, i.e., *the limiting force of friction is independent of the areas of the surfaces in contact.*

Kinetic Friction.—The friction brought into play at the surface of contact of two bodies when sliding takes place at this surface is called the force of kinetic friction.

EXPERIMENT.—Returning to the first experiment, let weights be added to the pan until the amount of friction brought into play is nearing limiting friction. Set the block in motion and, if it comes to rest, add weights to the pan until when set in motion the block proceeds with uniform speed as far as may be judged without further measurements.

The friction brought into play under these circumstances is called *Kinetic Friction*, and for present purposes all that we require to learn from our experiment is that *kinetic friction is less than the limiting force of friction*, and, so far as may be judged from the experiment, does not depend upon the speed with which one body slides over the other.

149. Summary of Results.

(1) Friction acts so as to prevent sliding at the surface of contact.

(2) The amount of friction brought into play is just enough to prevent motion, but cannot exceed a certain maximum value called *the force of limiting friction.*

(3) The force of limiting friction is given in magnitude by μN where μ is a constant and N is the normal reaction at the surface of contact.

(4) The force of friction is independent of the area of the surface of contact, but depends upon the nature of the surfaces in contact.

(5) In the case of kinetic friction the force of friction brought into play is less than the limiting force of friction, and may be shown to be

independent of the relative speed of the surfaces in contact between limits $\frac{1}{2}$ to 10 ft./sec. approximately.

It should be noticed that the above results are experimental and are only approximate (as, indeed, we have already discovered). If the nature of the surfaces is altered, as in the case when one of the surfaces is crushed or torn, then the laws no longer hold.

The results given above do not apply to lubricated surfaces.

Some approximate values of the constant μ for certain cases are given below :—

	μ.
Metal on metal	0·15 to 0·3
Metal on metal (greased)	0·1
Wood on wood	0·25 to 0·5
Stone on stone	0·65

The constant μ is often called the Coefficient of Limiting Friction.

150. Example.—*A block of wood of weight 50 lb. can just be drawn along a horizontal table by a horizontal force of 24 lb. wt. If the block is now loaded with another 20 lb. wt., what is the least horizontal force which will move the block?*

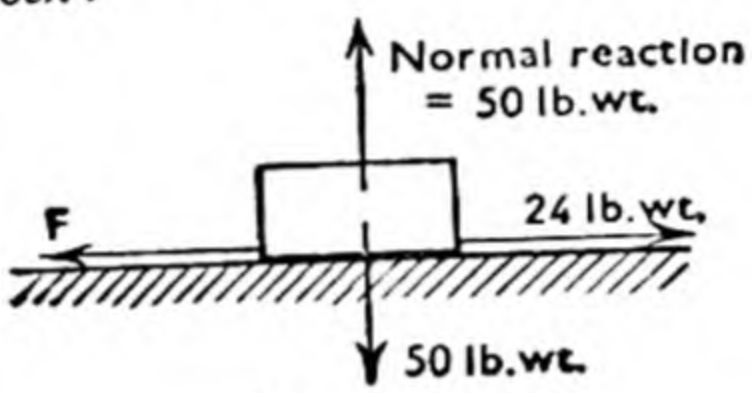

FIG. 292.

(*a*) Since a force of 24 lb. wt. just moves the block, the force of limiting friction is 24 lb. wt.

$$\text{Coefficient of friction} = \frac{\text{Limiting force of friction}}{\text{Normal reaction}} = \frac{24}{50} = 0{\cdot}48.$$

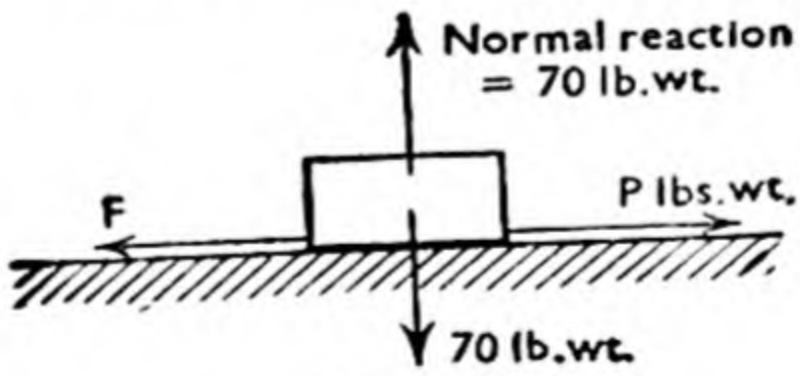

FIG. 293.

(*b*) The normal reaction is 70 lb. wt.

The force P lb. wt. required to cause sliding is equal to the force of limiting friction.

$$\text{Coefficient of friction} = \frac{\text{Limiting force of friction}}{70}.$$

Hence $$0{\cdot}48 = \frac{F}{70}.$$

$$F = 0{\cdot}48 \times 70 = 33{\cdot}6 \text{ lb. wt.}$$

The least horizontal force required to cause the block to slide is 33·6 lb. wt.

EXAMPLES XLI

1. A body of weight 10 lb. is at rest on a horizontal plane. A horizontal force of 7 lb. wt. just causes it to slide. What is the limiting force of friction and the coefficient of friction ?

2. The coefficient of friction between a body of weight 20 lb. and the horizontal plane on which it rests is 0·4. (*a*) Calculate the horizontal force which acting upon the body just causes it to slide. (*b*) If the body is loaded with an additional weight of 10 lb. calculate the least horizontal force which will cause the body to slide.

3. A cart and its load weighs 10 cwt. If the wheels are locked by the brake and the horse must exert a horizontal force of $2\frac{1}{2}$ cwt. to keep the cart moving, find the coefficient of friction between the cart wheels and the ground.

4. The figure shows a pulley of diameter 1 ft. fixed to a shaft of diameter 4 in. A tangential force of 60 lb. wt. acts on the shaft. Calculate

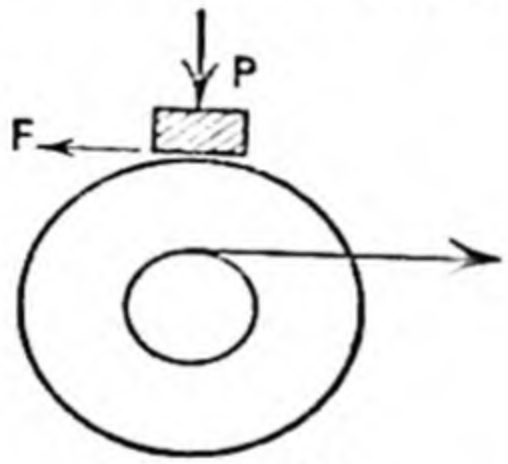

FIG. 294.

the least normal force P which, applied by the brake block on the rim of the pulley, will prevent the pulley from turning. The coefficient of friction between brake block and pulley is 0·24.

5. A horizontal force of 20 lb. wt. acting on a body which rests on a horizontal table just causes the body to slide. What is the weight of the body if the coefficient of friction between block and table is 0·32.

6. What is meant by the moment of a couple ? If the coefficient of friction between my finger and thumb and the head of a screw $\frac{1}{2}$ in. in diameter is 0·6, with what force must I pinch the screw in order that I may turn it with a couple of 2 lb.-ft. units ? (*O.C.*)

7. A body resting on a horizontal plane is pulled by two horizontal ropes attached to it, which are inclined to each other at 90°, and in which the tensions are 30 and 40 lb. wt. Calculate the resultant force on the body and the force of friction brought into play, stating the direction in which the latter force acts.

8. In a loom brake the coefficient of friction between the brake block and the brake wheel is 0·5, and the pressure of the block against the wheel is 20 lb. The brake stops the loom normally in a quarter of a revolution

of the brake wheel, but as a result of oil dropping on the brake wheel it is found that it makes two revolutions before stopping. What is the new coefficient of friction between block and wheel ?

151. The Tractive Force of a Locomotive.

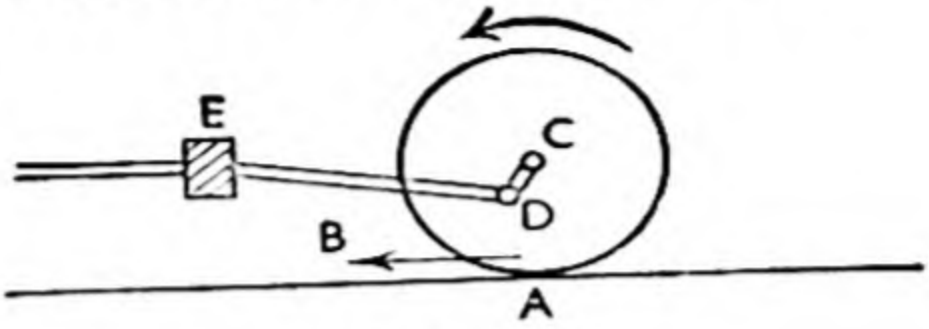

FIG. 295

In the above figure C is the centre of the driving wheel of a locomotive. The force exerted by the connecting rod ED upon the crank CD turns the crank and hence the driving wheel to which the crank is fixed. If there were no force of friction between the driving wheel and the rail at the point of contact A the driving wheel would spin uselessly round in an anti-clockwise direction. In this case no force is brought into play to drive the locomotive forward. At A, however, the force of friction brought into play (static friction if the wheel does not slip and kinetic friction if slipping occurs) acts in the direction AB upon the wheel, and it is this force alone which drives the engine forward. From our previous work we know that the amount of friction which can be brought into play depends only upon the normal reaction and the nature of the surfaces in contact, and we are therefore able to say that the greatest driving force which a locomotive can exert is about 0·17 times the total normal reaction on the driving wheels. However great the steam pressure available the driving force can never exceed this value ; indeed, if the steam pressure is too great the wheels begin to slip and the driving force is decreased, for in this case only kinetic friction is brought into play.

The free wheels of the engine are moved forward by the force exerted upon them at their axles, and if there were no friction at the point of contact of such a wheel with the rail the wheel would simply slide along. It is therefore readily observed that the force of friction acts on a free wheel in the direction opposite to that in which the wheel moves. At the driving wheels the

friction acts on the wheels in the direction of motion of the train and at the driven wheels in the opposite direction. This is only true when the train is being driven, for when the brakes are applied and steam shut off friction at all wheels opposes the motion of the train.

It is usual to express the resistance to motion of a train caused by such forces as friction at the bearings of wheels and at points of contact of the wheels with the rails as so many lb. wt. per ton wt. of the train.

In the two types of wheel referred to we have examples of cases in which friction is essential and useful on the one hand and a drawback on the other. We are so familiar with the precautions taken to reduce the effects of friction that we sometimes forget the essential part played by friction in assisting the comfort of our everyday life. As you sit at your desk imagine frictional forces entirely removed and consider what would happen to you and the objects in the room. Consider also how you would be able to walk, play football, cricket or knit a jumper.

152. Rolling Wheels.

FIG. 296.

Consider a perfectly rigid wheel rolling along a perfectly rigid horizontal plane, and suppose that the resistance of the air be neglected. The weight of the wheel acts vertically downwards through its centre of gravity and its line of action passes through the point of contact A with the horizontal plane (figure on the left above). The reaction of the ground upon the wheel also acts through A, and there is consequently no force acting upon the wheel which has any moment about A. Since there is no moment tending to prevent the turning of the wheel about its point of contact with the ground, it continues to roll with uniform motion.

In practice no wheel or plane is perfectly rigid, and both wheel and plane are slightly deformed, as shown in the figure on the right. As the wheel rolls along the plane the material of both the wheel and plane is squeezed, and there is a puckering up of the plane in front of the surface of contact. The point about which the wheel may be considered to turn is now slightly in front of the vertical through its centre of gravity, and therefore the weight of the wheel has a moment about the point of contact which opposes its turning. Again, the reaction of the plane on the wheel has a horizontal component which opposes its horizontal motion. In this case the forces acting on the wheel are not in equilibrium and the wheel is steadily reduced to rest.

The harder the surfaces in contact the smaller will be the force opposing rolling. Ball and roller bearings are specially case-hardened to render them more efficient. Most of us are familiar with their use in bicycles and roller skates.

The Angle of Friction.—A block of weight W rests on a

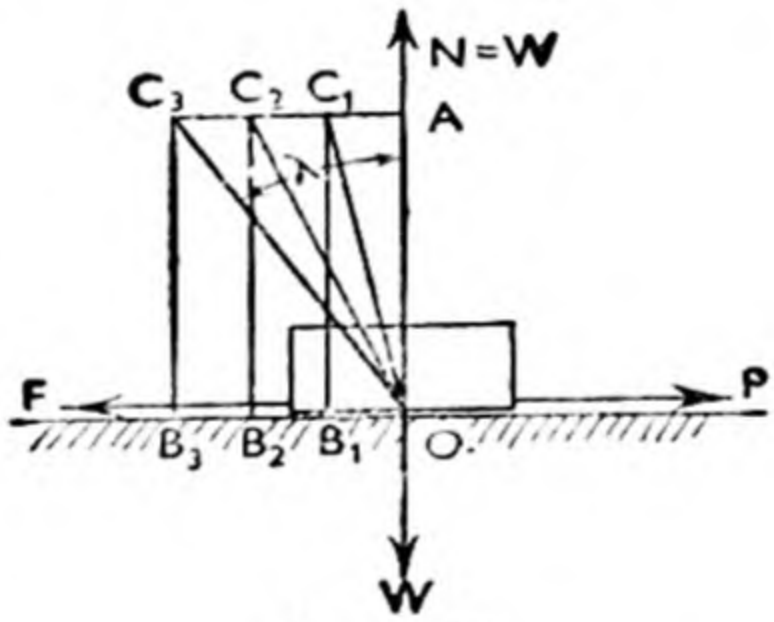

FIG. 297.

horizontal plane and a force P is applied to it parallel to the surface of contact, as shown in the above figure.

The normal reaction N is equal to W, the weight of the block.

The force of friction F brought into play is equal to P as long as the block remains at rest.

Let OA represent the normal reaction and OB_1 the force of friction F.

Then OC_1 represents their resultant R.

If the force P is increased, more friction is brought into play as represented by OB_2 and OB_3, and the angle between R and N increases.

When the limiting force of friction is exerted, F has its maximum value for the given normal reaction, and therefore the angle between the resultant reaction R and the normal reaction N has also its greatest value (Fig. 298).

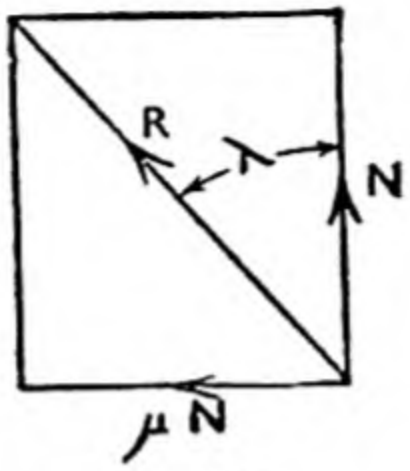

FIG. 298.

Let OB_3 represent the limiting force of friction, then,

$$\mu = \frac{F}{N} = \frac{AC_3}{OA} = \tan \angle AOC_3.$$

$\angle AOC_3$ is called the *angle of friction* and it is usually denoted by λ.

$$\mu = \tan \lambda.$$

The value of μ and therefore of λ is independent of the value of the normal reaction.

The resultant R of the force of friction and the normal reaction can never act at a greater angle with the normal than the angle of friction.

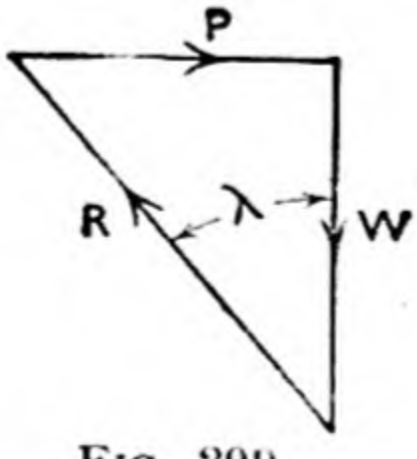

FIG. 299.

If N and F are replaced by their resultant R when limiting

friction is being exerted, the block is at rest under the action of the three forces R, P and W, as shown in Fig. 299.

153. A Body on a Rough Inclined Plane.

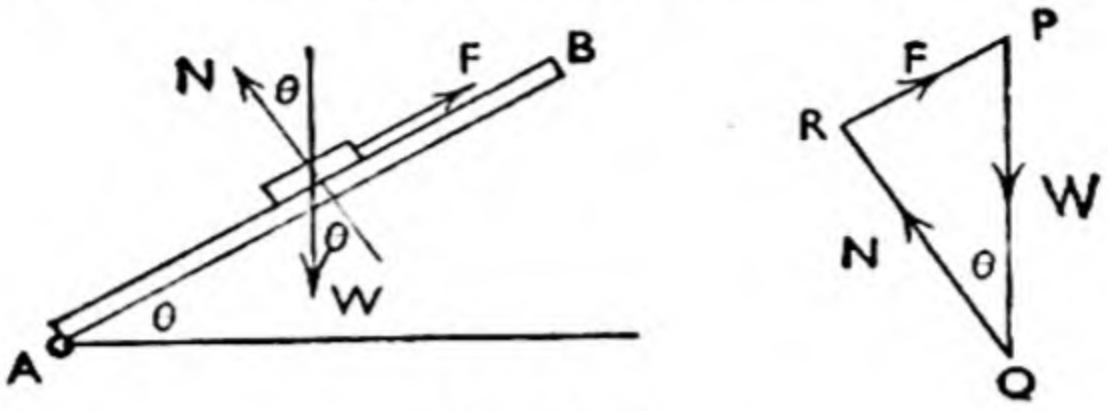

FIG. 300.

A body rests upon the inclined plane AB. The inclination of the plane to the horizontal can be varied.

Let the plane be tilted at the angle θ to the horizontal.

The body is acted upon by the following three forces which are in equilibrium :—

(1) The weight of the body W.

(2) The normal reaction N of the plane on the body.

(3) The force of friction F parallel to the line of greatest slope of the plane.

Draw PQR, a triangle of forces, then

$$\frac{F}{W} = \sin\ \theta,$$

$$F = W \sin\ \theta.$$

$$\frac{N}{W} = \cos\ \theta,$$

$$N = W \cos\ \theta.$$

These two results may also be found by resolving the forces acting on the body parallel and perpendicular to the plane AB. Let us now suppose that the inclination of AB to the horizontal is gradually increased until the body just begins to slide. It will be noticed that once sliding begins the body gains speed. This is in agreement with previous experience, for when sliding begins the friction brought into play is *kinetic* friction and is therefore less than the force of limiting friction.

If θ is the angle of inclination of the plane to the horizontal when sliding is just beginning, from the two above results,

$$\frac{F}{N} = \frac{W \sin \theta}{W \cos \theta} = \tan \theta.$$

But when the force of limiting friction is brought into play,

$$\frac{F}{N} = \tan \lambda,$$

therefore $$\tan \theta = \tan \lambda, \textit{ i.e.}, \ \theta = \lambda.$$

The angle of inclination of the plane to the horizontal when sliding is about to take place is equal to the angle of friction.

The above method is convenient for finding the coefficient of friction between two surfaces. The angle at which sliding takes place should be found a number of times and the average of the values taken as the angle of friction.

If the angle PRQ is a right angle in Fig. 300,

$$\mu = \tan \lambda = \tan \theta = \frac{PR}{RQ}.$$

154. The Angle of Rest.

When sand, coal, earth or similar material is built into a sloping embankment, the angle with the horizontal at which the material remains more or less permanently at rest is called the angle of rest of the material. If an embankment is piled at a greater angle than this there is danger of the material slipping away.

155. Example.—*A rectangular block of wood rests on a plane inclined to the horizontal at* 10°. *The smallest force acting parallel to the line of greatest slope which will just move the block down the plane is* 4 *lb. wt. If the coefficient of friction between the block and the plane is* 0·32, *find the weight of the block and the smallest force acting parallel to the line of greatest slope which will just move the block up the plane.*

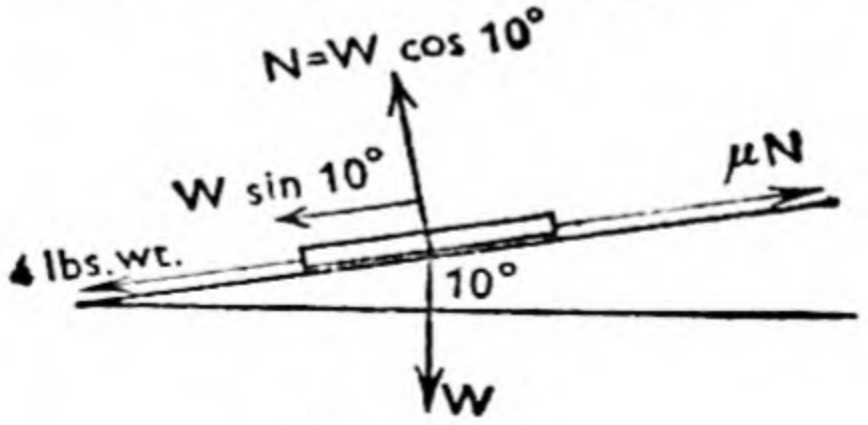

FIG. 301.

(*a*) The block is in limiting equilibrium and on the point of sliding down the plane.

Let N lb. wt. be the normal reaction of the plane on the block,

$$\mu = 0{\cdot}32.$$

The forces acting on the block are :

(1) The weight of the block, resolved into two components : W cos 10°, perpendicular to the inclined plane, and W sin 10° // to the inclined plane.

(2) The normal reaction N.

(3) The force of friction acting up the piane and equal in magnitude to 0·32N.

(4) 4 lb. wt. acting down the plane.

Consider the components of the forces acting on the block parallel to the inclined plane,

$$0{\cdot}32\text{N} = \text{W} \sin 10° + 4 = 0{\cdot}1736\text{W} + 4.$$

Consider the components of the forces acting on the block perpendicular to the inclined plane,

$$\text{N} = \text{W} \cos 10° = 0{\cdot}9848\text{W}.$$

Put 0·9848W for N in the first equation,

$$0{\cdot}32 \times 0{\cdot}9848\text{W} = 0{\cdot}1736\text{W} + 4.$$

$$0{\cdot}3151\text{W} = 0{\cdot}1736\text{W} + 4.$$

$$0{\cdot}1415\text{W} = 4.$$

$$\text{W} = 28{\cdot}27.$$

The weight of the block is 28·3 lb. wt.

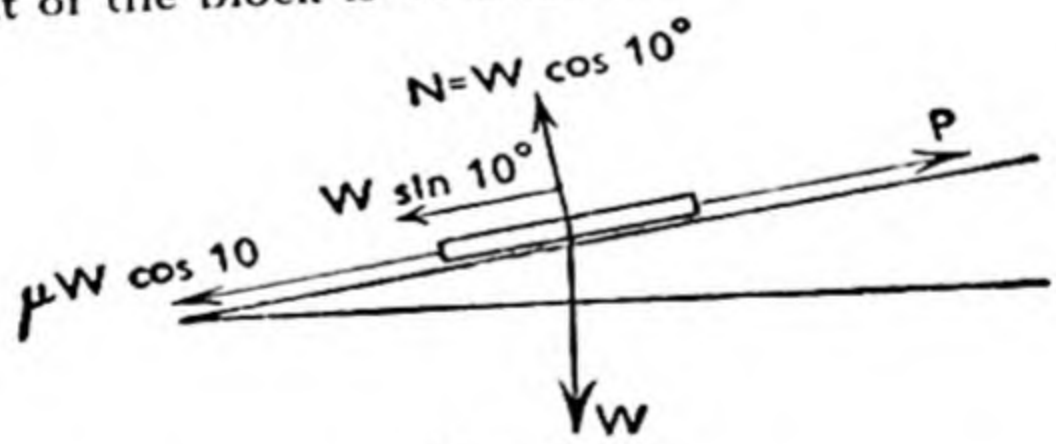

FIG. 302.

(*b*) Let P lb. wt. be the smallest force acting parallel to the plane which will just move the block up the plane.

The limiting force of friction acts down the plane.

Resolving perpendicular to the inclined plane,

$$\text{N} = 0{\cdot}9848\text{W}.$$

Resolving parallel to the inclined plane,

$$\begin{aligned}\text{P} &= \text{W} \sin 10° + 0{\cdot}32\text{N}.\\ &= 28{\cdot}27 \times 0{\cdot}1736 + 0{\cdot}32 \times 0{\cdot}9848 \times 28{\cdot}27\\ &= 28{\cdot}27(0{\cdot}1736 + 0{\cdot}3151)\\ &= 13{\cdot}8.\end{aligned}$$

The smallest force required to move the block is 13·8 lb. wt.

EXAMPLES XLII

1. A body of mass 10 lb. rests on a plane which is gradually tilted. The body begins to slide when the angle of the plane with the horizontal is 25°. What is the coefficient of friction between the body and the plane, and what force of friction is called into play just before slipping begins ?

2. A roller weighing 200 lb. is on a slope of inclination 10° to the horizontal. What pull up the slope is necessary to support the roller ? If, owing to friction, it is found that the pull can be diminished to 25 lb. wt. before the roller begins to roll down the slope, what pull will be necessary to make it begin to move up the slope ? Point out any assumption that you make in your calculation. (*O.C.*)

3. A load of 50 lb. resting on a plane begins to slip when the plane is tilted to an angle of 30° with the horizontal. What force up the plane will be necessary to keep the load from slipping if the angle of slope is increased to 45° ? (*O.C.*)

4. A mass of 50 lb. rests on a rough slope inclined at 20° with the horizontal. The least force directed up the plane which will move the mass is 56 lb. wt. Find, graphically or otherwise, (1) the least force directed down the plane which will move the mass, (2) the angle of friction. (*O.C.*)

5. A body weighing 30 lb. can just be drawn along a horizontal plane by a force of 6 lb. wt. parallel to the plane. What force inclined at 30° to the plane is required to move the body ?

156. The Cone of Friction.

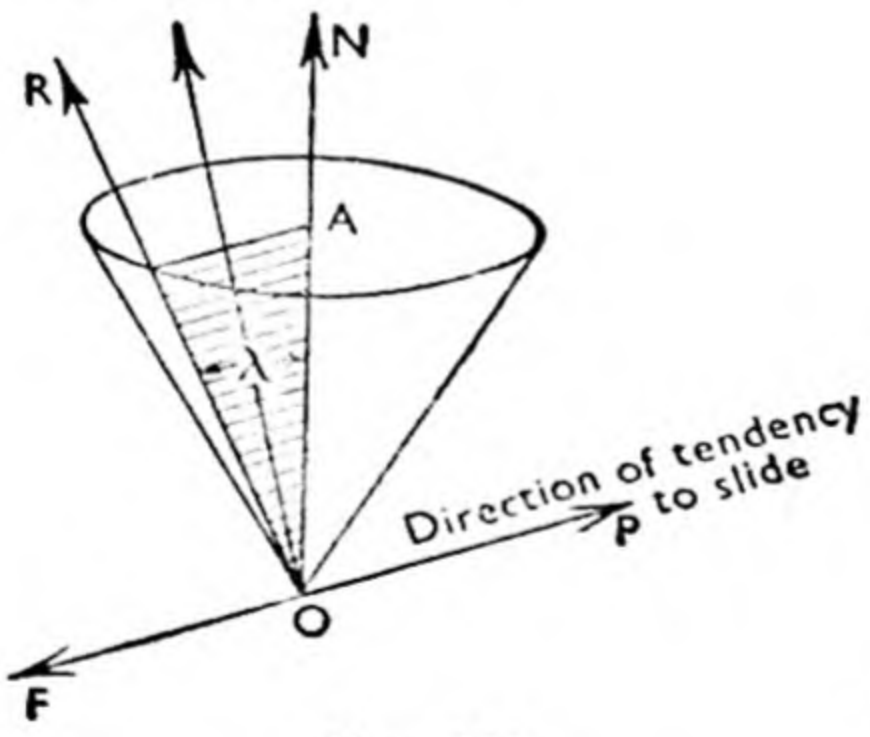

FIG. 303.

Suppose that OA represents the normal reaction of a surface on a body. If the body tends to slide along this surface in the direction OP then the force of friction acts in a direction opposite to this. In the figure the force of friction acts along OF. If limiting friction is exerted to keep the body at rest, the resultant

R of the normal reaction and the limiting force of friction makes an angle λ with the normal reaction.

If the body is on the point of sliding in another direction, it is readily observed that the line of action of R again makes an angle λ with OA.

When limiting friction is exerted the line of action of R always lies in the surface of an inverted right circular cone of semi-vertical angle λ. This cone is called the cone of friction.

When limiting friction is not exerted, the line of action of R lies *within* the cone of friction.

The force R will be referred to as the resultant reaction.

157. A Body on a Rough Plane acted on by a Force Inclined to the Plane.

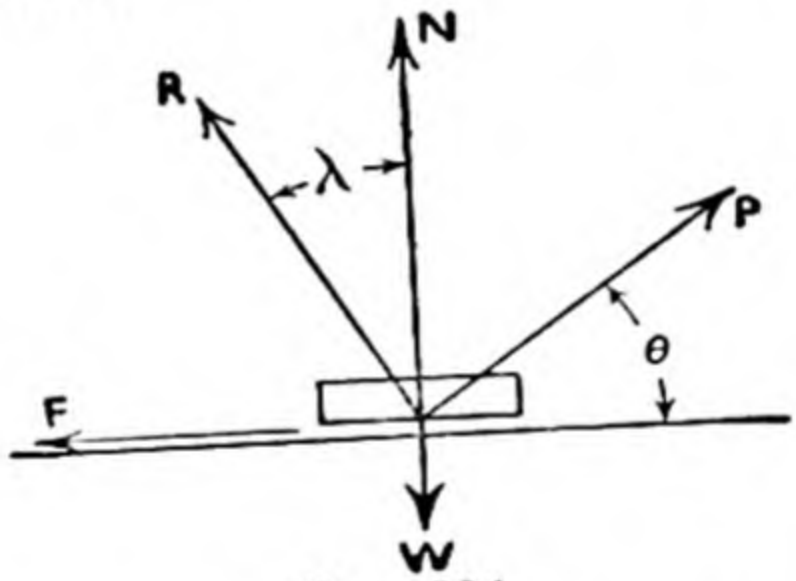

FIG. 304.

Let P be the force, W the weight of the body and λ the angle of friction.

N is the normal reaction, F the force of friction and R is their resultant.

Suppose that the force P acts at an angle θ to the horizontal.

The block is at rest and the magnitude of P is such that limiting friction is being exerted.

The three forces acting on the block are :

(1) R, acting at an angle λ with the normal reaction.

(2) P.

(3) W acting vertically downwards.

Since these forces are in equilibrium, we may draw the triangle of forces as follows :—

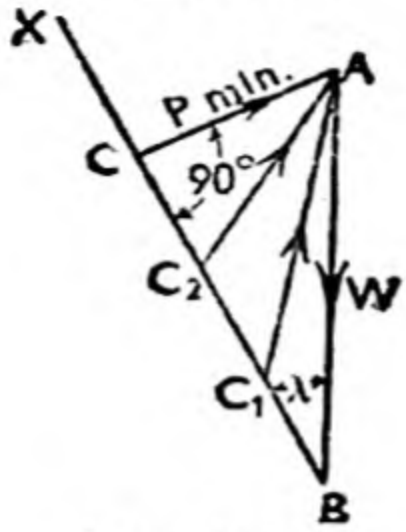

FIG. 305.

Draw AB to represent W.

Draw BX // to the line of action of R, *i.e.*, making $\angle ABX = \lambda$.

The force P which must be the equilibrant of W and R may now be drawn and the triangle of forces completed.

C_1A, C_2A and CA are the values of P corresponding to different directions in which P is applied which are just sufficient to cause limiting friction to be brought into play in each case.

The least value of P which will just begin to move the body is readily seen to be that given by CA which is perpendicular to BX.

In this case P is given by

$$\frac{P}{W} = \frac{CA}{AB} = \sin \lambda$$

$$P = W \,.\, \sin \lambda.$$

From the figure we see that $\theta = \lambda$.

The smallest value required to move the body is W sin λ, and it acts at an angle λ to the horizontal.

158. Example.—*A friction block weighing 50 lb. can be just moved on a horizontal table by a horizontal force of 20 lb. wt. What force will be needed to move the block if the direction of the force makes an angle of 30° to the horizontal* (a) *upwards*, (b) *downwards?*

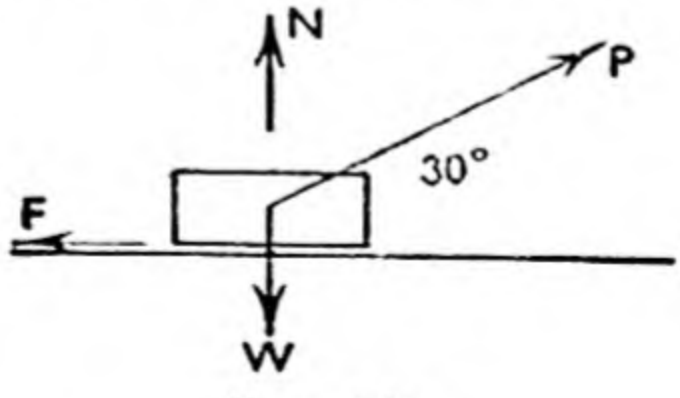

FIG. 306.

Since a force of 20 lb. wt. horizontally just moves the block, the limiting force of friction is 20 lb. wt.

Hence $$\frac{\text{Limiting force of friction}}{\text{Normal reaction}} = \frac{20}{50} = 0{\cdot}4.$$

(*a*) The forces acting on the block are :

Horizontally—
The force of limiting friction F.
The component of the force P horizontally, P . cos 30°.

Since these are the only horizontal components of force acting on the block,

$$F = P \cos 30°.$$

Vertically—
The weight of the block, 50 lb. downwards.
The normal reaction on the block, N upwards.
The component of P upwards, P sin 30°.

Since these are the only vertical components of force,

$$N + P \,.\, \sin 30° - 50 = 0.$$

$$N = 50 - 0{\cdot}5P.$$

But $$\frac{F}{N} = 0{\cdot}4.$$

$$F = 0{\cdot}4N.$$

Hence $$0{\cdot}4N = P \,.\, \cos 30°.$$

$$0{\cdot}4N = 0{\cdot}8660P.$$

Also $$0{\cdot}4N = 0{\cdot}4(50 - 0{\cdot}5P).$$

Therefore $$0{\cdot}8660P = 20 - 0{\cdot}2P.$$

$$1{\cdot}0660P = 20.$$

$$P = \frac{20}{1{\cdot}066} = 18{\cdot}8.$$

Hence the least force which will move the block is 18·8 lb. wt

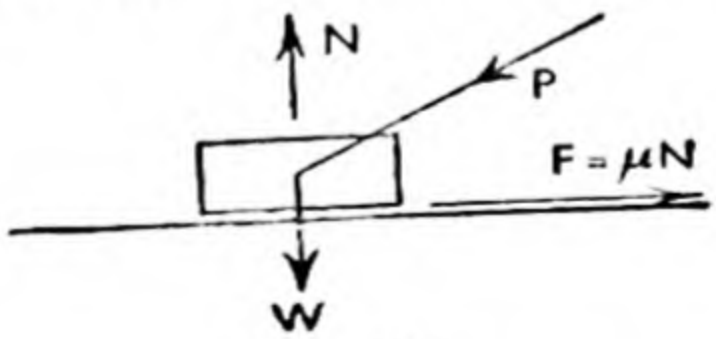

FIG. 307.

(*b*) With the same notation as above:

Resolving horizontally, $F = P \cos 30°.$

Resolving vertically,

$$N - P \cdot \sin 30° - 50 = 0.$$

$$N = P \sin 30° + 50.$$

Again,
$$\frac{F}{N} = 0{\cdot}4.$$

$$0{\cdot}4N = P \cdot \cos 30°.$$

Also
$$0{\cdot}4N = 0{\cdot}4(0{\cdot}5P + 50).$$

$$0{\cdot}8660P = 0{\cdot}4(0{\cdot}5P + 50).$$

$$0{\cdot}6660P = 20.$$

$$P = \frac{20}{0{\cdot}6660}.$$

The least force which will just move the block is 30·0 lb. wt.

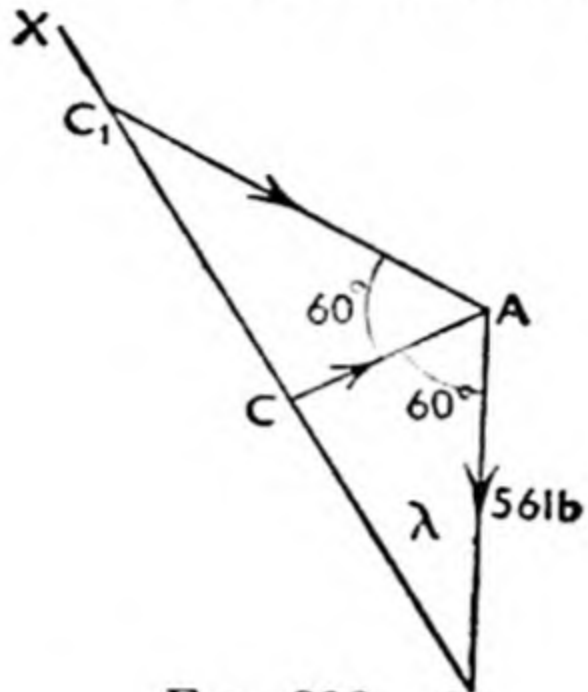

FIG. 308.

Graphical Solution.—Suppose that A is the angle of friction,

$$\tan 22° = 0{\cdot}4.$$

This value is as accurate as we require for our purpose.

Draw AB to represent the weight 56 lb. Draw BX, making $\angle ABX$ equal to 22°, the angle of friction.

(*a*) Through A draw AC making $\angle CAB = 90° - 30°$ to meet BX at C. CA represents the force P, which will just cause the block to move.

By measurement $P = 18{\cdot}8$ lb. wt.

(*b*) Through A draw AC_1 making $\angle C_1AB = 90° + 30°$ to meet BX at C_1. C_1A represents the force which will just move the block.

By measurement $P = 30{\cdot}0$ lb. wt.

Both triangles of forces are shown on the same figure.

EXAMPLES XLIII

1. Explain the term coefficient of friction.

A uniform ladder rests at an angle of 45° against a smooth vertical wall. Show that the coefficient of friction between the foot of the ladder and the ground (supposed horizontal) must be at least $\frac{1}{2}$. (*O.C.*)

2. A man is standing three-quarters the way up a ladder of negligible weight which is resting in limiting equilibrium at an angle of 30° with the vertical against a smooth vertical wall. The lower end rests on a rough horizontal pavement. Find, graphically or otherwise, the angle of friction for the ladder and the pavement. (*O.C.*)

3. A friction block weighing 56 lb. can be just moved on a horizontal table by a horizontal force of 20 lb. What force will be needed to move the block if the direction of the force makes an angle of 30° with the horizontal (*a*) upwards, (*b*) downwards ? (*O.C.*)

4. A block weighing 40 lb. is pulled very slowly along a horizontal plane, the coefficient of friction being 0·2. Find the magnitude of the pull if its line of action is (i) horizontal, (ii) at 45°, (iii) such that the pull is a minimum. (*O.C.*)

5. A sledge is pulled along a horizontal snow surface by a rope inclined at 30° to the horizontal. If the coefficient of kinetic friction between the runners of the sledge and the snow is 0·1 and the tension in the rope is 4 lb. wt., what is the weight of the sledge ?

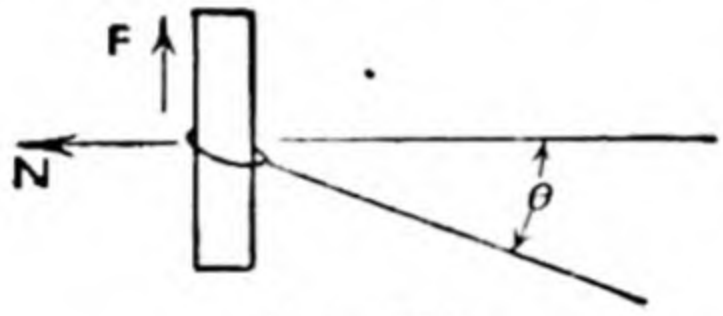

FIG. 309.

6. A light ring is attached to a string which is pulled taut and makes an angle θ with the horizontal. The ring is threaded on a vertical rod. Show that the angle θ cannot be greater than λ, the angle of friction between the ring and the rod.

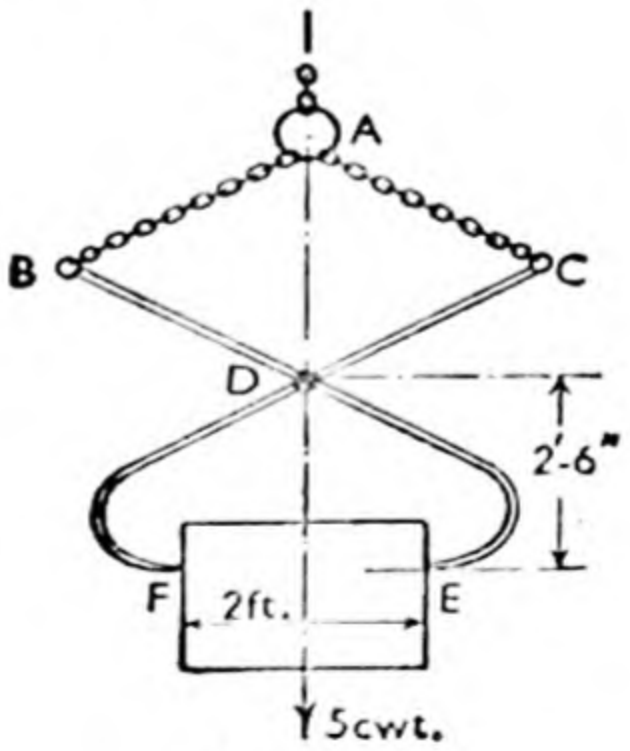

FIG. 310.

7. Fig. 310 shows a grab for lifting stones. $AB = BD = DC = CA = 2$ ft., and $\angle BAC = 120°$. Show that the tensions in each of the chains AB, AC are $\overline{5}$ cwt. when the grab is lifting a stone block of weight 5 cwt. (Neglect the weight of the grab.) Show also that the forces of friction (supposed equal) brought into play at E and F are each $2\frac{1}{2}$ cwt., and that the coefficient of friction between the stone and the grab must at least be equal to 0·56. [Hint: Consider the equilibrium of BDE and take moments about D.]

8. The diameters of the wheels of a car are 2 ft. 6 in., and the centre of gravity of the car is 6 in. above the line joining the front and rear axles (which are 8 ft. apart) and symmetrically placed between them. The car rests on an incline facing uphill of 1 in 8 with the rear wheels locked. Draw a diagram showing how the reactions at the wheels and weight of the car are in equilibrium, and use your diagram to show that the car will slide downhill if the coefficient of friction between the back wheels and the ground is less than $\frac{32}{135}$.

CHAPTER XVII

THE FIRST LAW OF MOTION

WORK AND POWER

159. The First Law of Motion.

If a piece of string attached to a hook fixed in a wall is pulled tight, a small piece of the string is at rest under the action of equal and opposite tensions in the string on either side of it. A book at rest upon a table is acted upon by its weight and the reaction of the table upon it. These two vertical forces are in equilibrium. A brick built into a wall is at rest under the action of its weight and the forces exerted upon it by the surrounding bricks and mortar. The forces acting on the brick are in equilibrium. In the foregoing pages we have, in fact, made it a condition that for a body to remain at rest the forces acting upon it must be in equilibrium. Since all bodies are acted upon by force, all bodies in a state of rest are acted upon by forces which are in equilibrium.

The bodies which we have referred to as being in a state of rest are, however, actually moving with the earth in its orbit round the sun. The so-called state of rest is therefore only rest relative to the earth. It appears that the motion of the earth does not affect the behaviour of a body at its surface.

We have described force by stating its effect upon a body; thus force changes or tends to change the state of rest or of uniform motion in a straight line of a body. The changing motion of a body allowed to fall freely from rest indicates that it is acted upon by a resultant force, in this case its weight. The speed of a parachutist vertically downwards increases at first, then less rapidly as his parachute opens and finally reaches a maximum value when his weight is balanced by the resistance of the air; beyond this point his speed remains constant, and he is finally

brought to rest by the reaction of the ground upon his body. (The maximum speed of a parachutist is called his *terminal speed* and is about 16 m.p.h.)

Let us suppose we are sitting in the carriage of a train facing the direction in which the train is about to move. The starting of the train appears to throw the body backwards against the cushions, whereas, in fact, the train begins to leave us behind. When our backs are brought into contact with the cushions these exert upon us the force necessary to change our motion so that we may travel with the carriage. When the train is moving with uniform velocity and the brakes are suddenly applied, we are apparently thrown forward; this is not so, however, for relative to the track we simply tend to continue our uniform motion in a straight line, and in order that we may remain on our seats a force of friction must be brought into play, or if we try to push or pull the carriage forward we may achieve the same object, that is, to remain at rest relative to the carriage.

We are now in a position to state *Newton's First Law of Motion. Every body continues in its state of rest or of uniform motion in a straight line unless it be compelled to change this state by resultant force.*

160. Example (1).—*A cyclist free-wheels down an incline of 1 in 28 with uniform velocity. Neglecting all resistance except that due to the wind, find the latter resistance given that the weight of the rider and bicycle is 200 lb.*

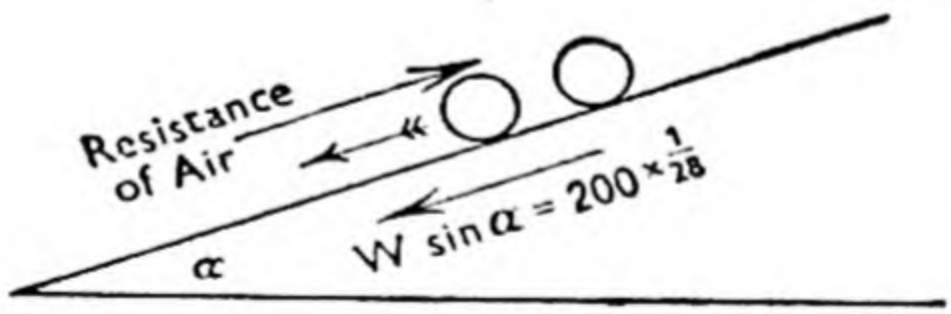

FIG. 311.

Since the cyclist is travelling with uniform velocity, the forces acting parallel to the inclined plane are in equilibrium.

The component of the weight of the cyclist and cycle parallel to the incline is balanced by the resistance of the air, hence

$$\text{Resistance of the air} - 200 \times \frac{1}{28} = 0.$$

$$\text{Resistance of the air} = 7{\cdot}14 \text{ lb. wt.}$$

Example (2).—*A train ascends an incline of* 1 *in* 100 *with uniform speed. The weight of the train is* 160 *tons, and the resistance to motion (due to friction, etc.) parallel to the track is* 12 *lb. wt. per ton. What is the force exerted at the driving wheels of the engine parallel to the track?*

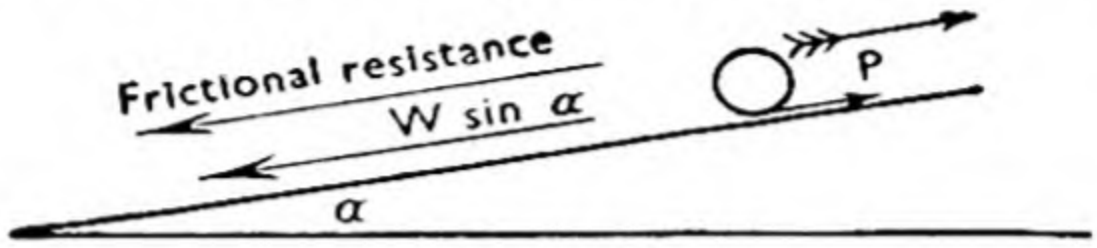

FIG. 312.

Since the speed of the train is constant the force exerted at the driving wheels parallel to the track is equal to the total resistance to motion parallel to the track.

Resistance to motion due to friction, etc. $= 12 \times 160 = 1920$ lb. wt.

Component of the weight of the train parallel to the track

$$= 160 \times 2240 \times \frac{1}{100} = 3584 \text{ lb. wt.}$$

Force exerted at the driving wheels $= 3584 + 1920 = 5504$ lb. wt.

Example (3).—*A weight of* 2,000 *lb. is dragged at a uniform speed up an incline of* 1 *in* 10 *by a rope which passes over a smooth pulley at the top of the plane, and then supports a* 250 *lb. wt. vertically. Find the resistance to motion parallel to the inclined plane.*

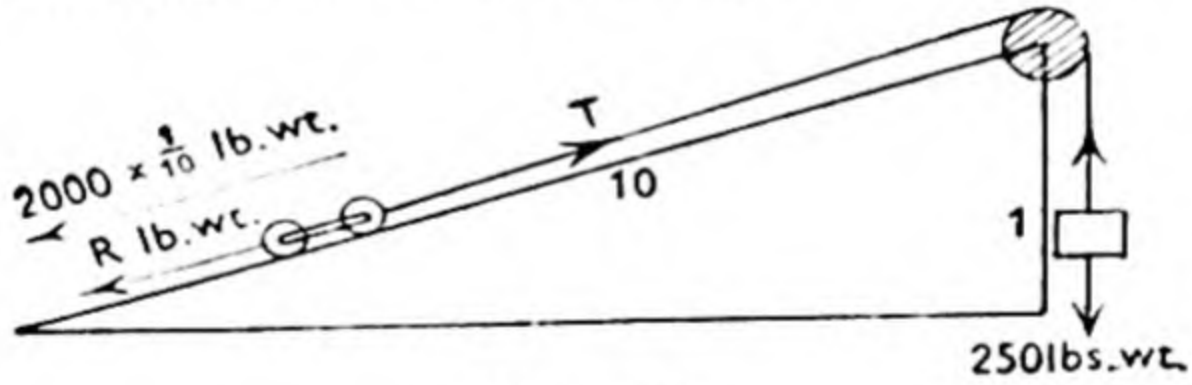

FIG. 313.

Let R lb. wt. be the resistance to motion parallel to the plane. This force acts down the plane upon the body.

Since the vertical motion of the weight of 250 lb. is one of *uniform velocity* the forces acting on it are in equilibrium, hence the tension in the string is equal to the weight of 250 lb.

Consider the motion of the body on the inclined plane.

Since its motion is one of uniform velocity parallel to the plane the components of force parallel to the plane are in equilibrium, and therefore

Tension in string = R + Resolved part of weight of body parallel to the plane.

$$250 = R + 2000 \times \frac{1}{10}.$$

$$R = 50.$$

The resistance to motion parallel to the plane is 50 lb. wt.

EXAMPLES XLIV

1. A cyclist free-wheels at uniform speed down an incline of 1 in 20. If his weight and that of his machine is 200 lb., what is the resistance to his motion parallel to the incline ?

2. A train weighing 250 tons travels along a horizontal track at uniform speed against resistances amounting to 10 lb. wt. per ton. What is the total force exerted parallel to the track at the driving wheels ?

3. A motor car weighing 25 cwt. is travelling up an incline of 1 in 16 at uniform speed. If the resistance parallel to the track due to frictional effects is 12 lb. wt. per ton, what is the total force exerted at the driving wheels parallel to the track ?

4. A motor car weighing 32 cwt. is free-wheeling down an incline of 1 in 20 at constant speed. If the resistance parallel to the track due to friction is 30 lb. wt., find to the nearest lb. the force parallel to the track due to wind resistance.

5. A train ascends an incline of 1 in 100 with uniform speed. The weight of the train is 160 tons and the resistance due to friction is 10 lb. wt. per ton. What is the force exerted at the driving wheels parallel to the track ?

6. A weight of 1,500 lb. is dragged at a uniform speed up an incline of 1 in 15 by a rope which passes over a smooth pulley at the top of the plane and then supports a 150 lb. wt. Find the resistance to motion parallel to the inclined plane.

7. Two equal weights of 10 lb. hang at the end of a string which passes over a pulley. A 1-lb. weight added to one of the 10-lb. weights just keeps the weights moving with uniform velocity. What is the tension in each of the two vertical parts of the string, and what moment is necessary just to turn the pulley if its diameter is 2 in. ?

8. A parachutist, whose weight is 14 st., descends with constant speed vertically downwards. What is the force exerted on him by the parachute ?

9. A body of weight 8 lb. slides down a plane with uniform speed when its angle of inclination is 30° to the horizontal. What force of friction is brought into play, and what is the coefficient of kinetic friction ?

10. A block of wood weighing 4 lb. is pulled along a horizontal table by a string which passes over a smooth pulley and then supports a weight of 1 lb. If the block is kept moving at uniform speed, what is the coefficient of kinetic friction between it and the table ?

11. A block of wood weighing 5 lb. is pulled down a plane inclined at 20° to the horizontal by a string which passes over a smooth pulley at the bottom of the plane, and there supports a weight W lb. Find the value of W which will keep the block moving with uniform speed. The coefficient of kinetic friction between block and plane is 0·5.

12. A pulley wheel is kept moving at constant angular speed by a belt. The difference in the tension of the tight and slack side of the belt where it leaves the pulley is 2 lb., and the diameter of the pulley is 2 ft. 6 in. Find the moment of the frictional forces acting on the pulley about its axis.

161. Work and Power.

In 1825 George Stephenson's steam engine, " The Rocket," drew a train from Stockton to Darlington and at times reached a speed of 12 m.p.h. If we were asked to compare a modern main-line locomotive with " The Rocket " we should probably say, among other things, that the modern locomotive is the more powerful of the two. The word " powerful " may have a number of different meanings when used in ordinary language, but in the subject of mechanics it has a special meaning and must never be used in any other sense. If in the above comparison we wish to convey the idea that the modern locomotive could draw the same train as " The Rocket " from Stockton to Darlington in a shorter time, then we should be giving the word " powerful " the meaning which is always assigned to it in mechanics.

Power is rate of working and is measured by the work done in unit time.

A machine moves a 4-ton wt. load at 1 ft./sec.
The power output of the machine = 4 ft. tons/sec.
A machine moves a 1000-lb. wt. load at 30 m.p.h.
The power output of the machine = 1000×44 ft. lb./sec.
= 19·6 ft. tons/sec.

Notice that the *force* exerted by the first of these two machines is greater than that exerted by the second, but that the smaller force *does more work in the same unit of time.* The second machine is the more powerful of the two.

The Practical Unit of Power.—The unit of power used in mechanics was devised by James Watt in 1784. In Watt's time coal mines were being sunk to increasing depths, and thousands of horses were in constant use to keep these mines free of water. He constructed and developed steam-driven engines to take the place of these horses, and from 1769 until 1785 his engines were used solely for the single purpose for which they were designed, viz., pumping water from mines. In 1785 the engines began to be used for other purposes, such as driving the machinery in cotton factories. It became necessary for

Watt to be able to advise those who wished to replace horses by his machines how many horses these machines would replace. To this end in 1784 he conducted experiments at Barclay & Perkins' brewery, in which he made use of the dray horses owned by the brewery. He arranged for a horse to draw a 100-lb. wt. out of a deep well by means of a rope attached to the weight and which, after passing over a pulley at the top of the well, was pulled horizontally at a steady pace by the horse.

He came to the conclusion that on an average a horse might be expected to raise a 100-lb. wt. vertically at $2\frac{1}{2}$ m.p.h.

A horse might therefore be expected to do

$$\frac{5}{2} \times \frac{88}{60} \times 100 \text{ ft. lb. of work per sec.} = \frac{1100}{3} \text{ ft. lb./sec.}$$

Not wishing to overestimate the capabilities of his engines, Watt added 50 per cent. to the above amount to allow for the work done against friction, and his final estimate was

$$\frac{1100}{3} \times \frac{3}{2} = 550 \text{ ft. lb./sec.}$$

This unit remains the practical unit of power in use to-day.

$$1 \text{ H.P.} = 550 \text{ ft. lb./sec.} = 33{,}000 \text{ ft. lb./min.}$$

It has been estimated since the time of Watt that a horse may be expected to work continuously (for some hours) at about three-quarters of this rate.

162. In the examples and problems which follow it will be assumed that work is being done at a constant rate by a constant force. This assumption is justified when the motion which is being considered is not changing in any way. In this case if we consider a machine, the input of work in unit time is constant, the output of work in unit time is constant, and therefore the work done against friction and in moving parts of the machine in unit time is also constant. In order to see the way in which power is affected if the above assumption is *not* justified, consider a locomotive which is drawing a train from rest. The force at the driving wheels remains very nearly constant for a modern

locomotive while the train is started from rest and its speed is increased to, say, 30 ft./sec. We shall then assume that the driving force falls off at higher speeds, due to the fact that the steam-raising capacity of the engine cannot supply sufficient steam to maintain the pressure equal to the pressure in the cylinder at smaller speeds.

Force at Driving Wheels.	Speed of Train.	Power Output.
Tons wt.	Ft./sec.	Ft. tons/sec.
5·0	0	0
5·0	10	50
5·0	20	100
5·0	30	150
4·5	40	180
3·0	50	150
1·0	60	60
1·0	60	60

By examining the above figures we see that the power output has its greatest value at a speed of the train about 40 ft./sec. After this speed the power output falls off. Now this falling off in power output is not entirely a disadvantage, for when the speed of the train has reached a value which it is required to maintain for some time the power output must just be sufficient to keep the train travelling at uniform speed against constant resistance to motion, and this power may very well be less than that required to accelerate the train and overcome resistance at a somewhat lower speed.

Example (1).—*Find the horse-power developed by the engine of a train travelling at a uniform speed of* 30 *m.p.h. against resistances amounting to* 2,000 *lb. wt.*

Since the speed of the train is uniform, the force exerted parallel to the track at the driving wheels is equal to the total resistance to motion.

$$\text{Force at the driving wheels} = 2000 \text{ lb. wt.}$$

$$\text{Work done per second} = 2000 \times 44 \text{ ft. lb.}$$

$$1 \text{ H.P.} = 550 \text{ ft. lb./sec.}$$

$$\text{Therefore Power} = 88000 \text{ ft. lb./sec.} = \frac{88000 \text{ H.P.}}{550} = 160 \text{ H.P.}$$

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Example (2).—*A car, whose engine is working at the rate of* 10 *H.P., travels on a level road at a uniform speed of* 30 *m.p.h. Find the resistance to motion in lb. wt.*

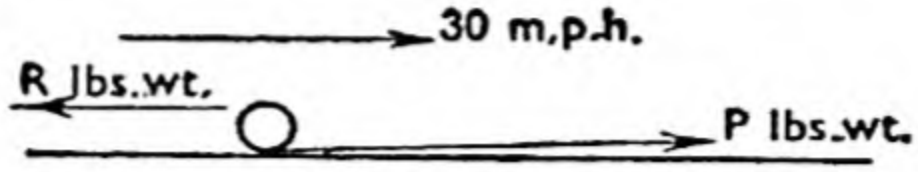

FIG. 314.

Since the speed of the car is uniform, the force exerted by the car at the driving wheels is equal to the resistance to motion.

Let R lb. wt. be the resistance to motion.

Work done per second against the resistance to motion

$$= R \text{ . } 44 \text{ ft. lb.}$$

Now,
$$10 \text{ H.P.} = 10 \times 550 \text{ ft. lb./sec.}$$
$$44R = 5500$$
$$R = \frac{5500}{44} = 125 \text{ lb. wt.}$$

Example (3).—*A cyclist free-wheels down a slope of* 1 *in* 70 *at a constant speed of* 7 *m.p.h. If the mass of the cyclist and his machine is* 210 *lb., what must be his rate of working for him to travel at the same uniform speed up the same slope?* (*L.M.*)

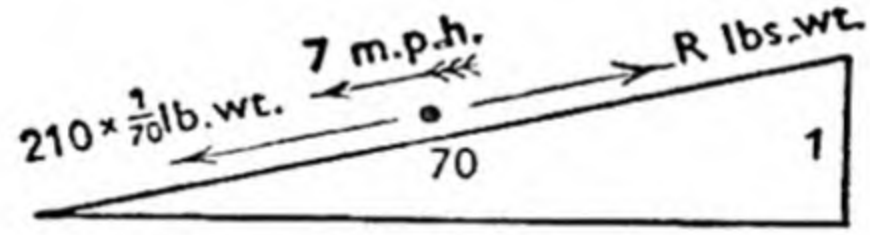

FIG. 315.

Since the cyclist free-wheels down the slope with uniform speed, the resistance to motion balances the component of his weight parallel to the plane.

Let R lb. wt. be the resistance to motion parallel to the plane.

$$R = 210 \times \frac{1}{70} = 3 \text{ lb. wt.}$$

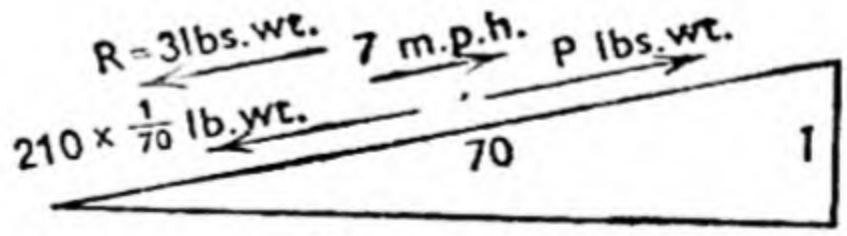

FIG. 316.

When the cyclist rides up the same slope at uniform speed the force at the driving wheel parallel to the plane acting up the plane must be equal to the resistance to motion (which now acts down the plane) plus the component of the total weight parallel to the plane.

Let P lb. wt. be the force at the driving wheel.

$$P = 3 + 210 \times \frac{1}{70} = 6.$$

Work done per second by the cyclist $= 6 \times \frac{7 \times 88}{60}$ ft. lb.

Rate of working of cyclist $= 61{\cdot}6$ ft. lb./sec. $= \frac{61{\cdot}6}{550} = 0{\cdot}112$ H.P.

Example (4).—*A train of weight* 150 *tons is ascending an incline of* 1 *in* 100 *with a uniform velocity of* 30 *m.p.h. against resistance to motion parallel to the track amounting to* 10 *lb. wt./ton. What is the horse-power transmitted by the engine?*

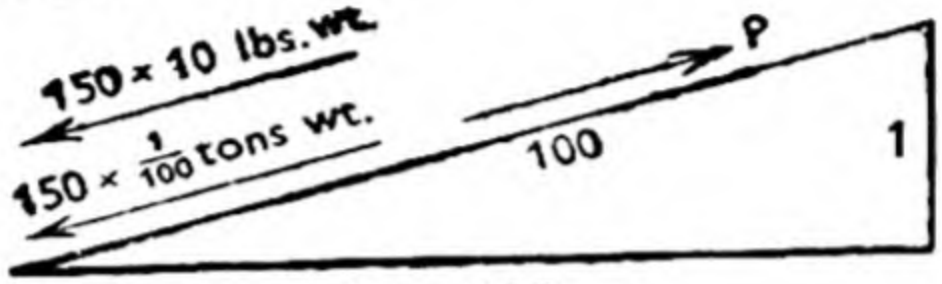

FIG. 317.

Since the velocity of the train is uniform there is no resultant force acting upon it parallel to the track.

The forces acting on the train down the plane are :

(1) The resistance to motion $= 150 \times 10 = 1500$ lb. wt.

(2) The component of the weight of the train

$$= 150 \times \frac{1}{100} \text{ tons wt.} = 3360 \text{ lb. wt.}$$

The force P lb. wt. acting upon the train up the plane is the force at the driving wheels parallel to the track.

These three forces are in equilibrium.

Hence the force exerted by the engine $= 3360 + 1500 = 4860$ lb. wt.

The work done per second by this force

$$= 4860 \times 44 \text{ ft. lb.} = \frac{4860 \times 44}{550} \text{ H.P.}$$

Power transmitted by the engine $= 388{\cdot}8$ H.P.

Example (5).—*An engine pumps water from a well to the ground level, which is* 40 *ft. above the mean level of the water surface. If* 2 *cu. ft. of water is raised per second and two-thirds of the work of the engine is used in lifting the water, what is the horse-power developed by the engine?* (*L.M.*)

Volume of water raised per second $= 2$ cu. ft.

Weight of water raised per second

$= 2 \times 62{\cdot}5$ lb. (1 cu. ft. weighs $62{\cdot}5$ lb.)

Work done per second in raising 125 lb. of water through 40 ft.

$$= 125 \times 40 \text{ ft. lb.}$$

Since this is only two-thirds of the work done per second by the engine,

$$\text{Rate of working of the engine} = 125 \times 40 \times \frac{3}{2}$$
$$= 7500 \text{ ft. lb./sec.}$$
$$= \frac{7500}{550} \text{ H.P.}$$
$$= 13\tfrac{7}{11} \text{ H.P.}$$

163. Belt Transmission of Power.

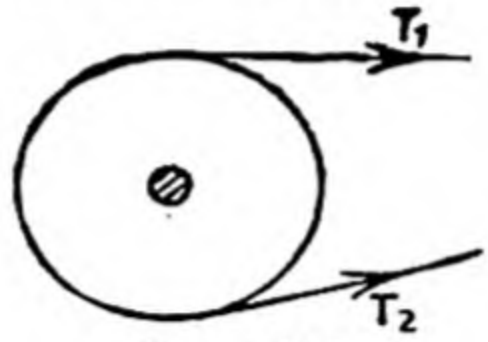

FIG. 318.

Consider the pulley in the above figure which is driven by a belt passing round the rim of the pulley. If the tensions in the two parts of the belt where it leaves the pulley are equal there will be no resultant turning effect upon the pulley. If, however, the tensions T_1 and T_2 are different and T_1 is greater than T_2, then $(T_1 - T_2)$ is called the *Effective Tension* in the belt and the pulley will turn as if only the single tension $(T_1 - T_2)$ acted tangentially upon its rim. The power transmitted by the belt is equal to the effective tension multiplied by the distance through which a point on the rim of the pulley moves in unit time. If the horse-power transmitted by the belt is known and the speed of the pulley, then the effective tension may be found but *not* the actual tension on either the tight or slack side.

Example.—*The effective pull in a belt is* 100 *lb. wt., and it drives a pulley of diameter* 14 *in. at* 180 *revs. per min. What horse-power is it transmitting? If the tension in the tight side is twice that in the slack side, find these tensions and the width of the belt if it must not take more than* 50 *lb. wt. per in. of width.*

Distance moved through per minute by the belt

$$= 2 \times \frac{22}{7} \times \frac{14}{12} \times 180 = 1320 \text{ ft.}$$

$$\text{Work transmitted per minute} = 100 \times 1320 \text{ ft. lb.}$$
$$1 \text{ H.P.} = 33000 \text{ ft. lb./min.}$$
$$\text{Power transmitted} = \frac{132000}{33000} = 4 \text{ H.P.}$$

If T lb. wt. is the tension on the slack side of the belt,

$$\text{Effective tension} = 2T - T = T \text{ lb. wt.}$$

Hence $T = 100$, and the tension on the tight side is 200 lb. wt. The width of the belt must be at least 4 in.

EXAMPLES XLV

Express in horse-power the rate of doing work in each of the first four examples :—

1. A horse drawing a load of 1½ tons on the level at the rate of 3 m.p.h., the resistance being 60 lb. wt./ton.

2. Five tons of coal raised vertically at the rate of 55 ft./sec.

3. A belt exerting an effective pull of 200 lb. wt. tangentially to the rim of a pulley 3 ft. in diameter, when the pulley is rotating at 140 revs. per min.

4. A boy of weight 8 st. runs up 30 stairs each of height 9 in. in 8 sec.

5. An engine working at 240 H.P. moves a train of weight 220 tons against resistances amounting to 15 lb. wt. per ton along a horizontal track at constant speed. What is this constant speed ?

6. A motor working at 12 H.P. travels at a uniform speed of 60 m.p.h. along a level road. What is the resistance to motion ?

7. A train of weight 200 tons travels down a slope of 1 in 100 at a uniform speed when steam is shut off. What must be the horse-power transmitted at the engine driving wheels if the train is to ascend the same gradient at 30 m.p.h. ?

8. A cyclist working at 1/10th H.P. can just maintain a speed of 15 m.p.h. along a level road. What is the resistance to motion ? What is the total weight of the cyclist and his machine if he can maintain the same speed down a gradient of 1 in 100 working at the rate of 1/20th H.P. (Assume the same resistance parallel to the track.)

9. Explain the meaning of work and rate of working, and state the units of each in the c.g.s. and British system of units.

A mass of 10 tons is supported on an endless chain, which is made to lift the load by means of an engine. Calculate the average horse-power required to raise the load, neglecting friction, etc., (*a*) if the load moves upwards at a mean speed of 1 in./sec., (*b*) if the load is raised 6 ft. in half a minute. (*L.M.*)

10. Define power, horse-power.

If the maximum power a man weighing 12 st. can develop is ½ H.P., find (*a*) the minimum time in which he can run up stairs to a room 40 ft. above his starting point, (*b*) the maximum speed at which he can ride a bicycle weighing 30 lb. up a 1 in 12 gradient. (*G.S.*)

11. Name and define the units of work in absolute c.g.s. units and in gravitational f.p.s. units.

A boy weighing 8 st. and riding a bicycle weighing 21 lb. rides up a hill with a gradient of 1 in 21 at 9 m.p.h. Assuming that friction is equivalent to a force of 2 lb. wt. resisting his motion up the hill, find how much work he is doing per second.

12. Define power.

A waggon weighing 5 tons is pulled up an incline of 1 in 15 at 6 m.p.h. If the frictional resistance is 25 lb. wt./ton, what horse-power is being employed ? (*G.S.*)

13. Distinguish between work and power.

A water turbine is driven by a waterfall in which 16,000 gals. of water per minute fall through 16 ft. and 50 H.P. is developed. What fraction of the available energy is utilised? (1 gal. of water weighs 10 lb.) (*L.M.*)

14. An engine working at 2 H.P. pumps water through a height of 30 ft., the efficiency of the pumping machinery being 60 per cent. How much water is pumped in an hour? (1 gal. of water weighs 10 lb.) (*O.C.*)

15. Assuming the resistance to motion of a train on the level is $0{\cdot}009V^2$ lb./ton where V is the speed in miles per hour, find the horse-power required to draw a train of 300 tons up an incline of 1 in 150 at a uniform speed of 20 m.p.h. (*O.C.*)

16. If a motor cycle can travel on a level road at 50 m.p.h. when the engine is working at 5 H.P., find the total resistance to motion. (*O.C.*)

17. A cyclist is riding on a level road at 12 m.p.h. and working at 0·1 H.P. Find the greatest slope on the same road which he can ascend at 8 m.p.h., working at the same rate, assuming the resistance, apart from gravity, to be unaltered and that the total weight of the cyclist and his bicycle is 200 lb. (*O.C.*)

18. An engine driving pulley is 18 in. diameter and runs at 160 revs. per min. Calculate the speed of the belt in feet per minute, ignoring slippage. What horse-power is the belt transmitting if the effective driving tension is 50 lb. wt.?

19. A belt passes over a pulley 35 in. diameter and transmits 26 H.P. when the pulley is revolving 193 times per minute. Determine the difference between the tensions in the two sides of the belt.

20. Calculate the horse-power required to drive an engine if the driving pulley is 15 in. diameter, makes 160 revs. per min., and the 3-in. belt has an effective driving tension of 44 lb. wt./in. of width.

21. In the arrangement shown, details of the diameters of a system of pulleys together with the tensions on the tight and slack sides of the belt A are given.

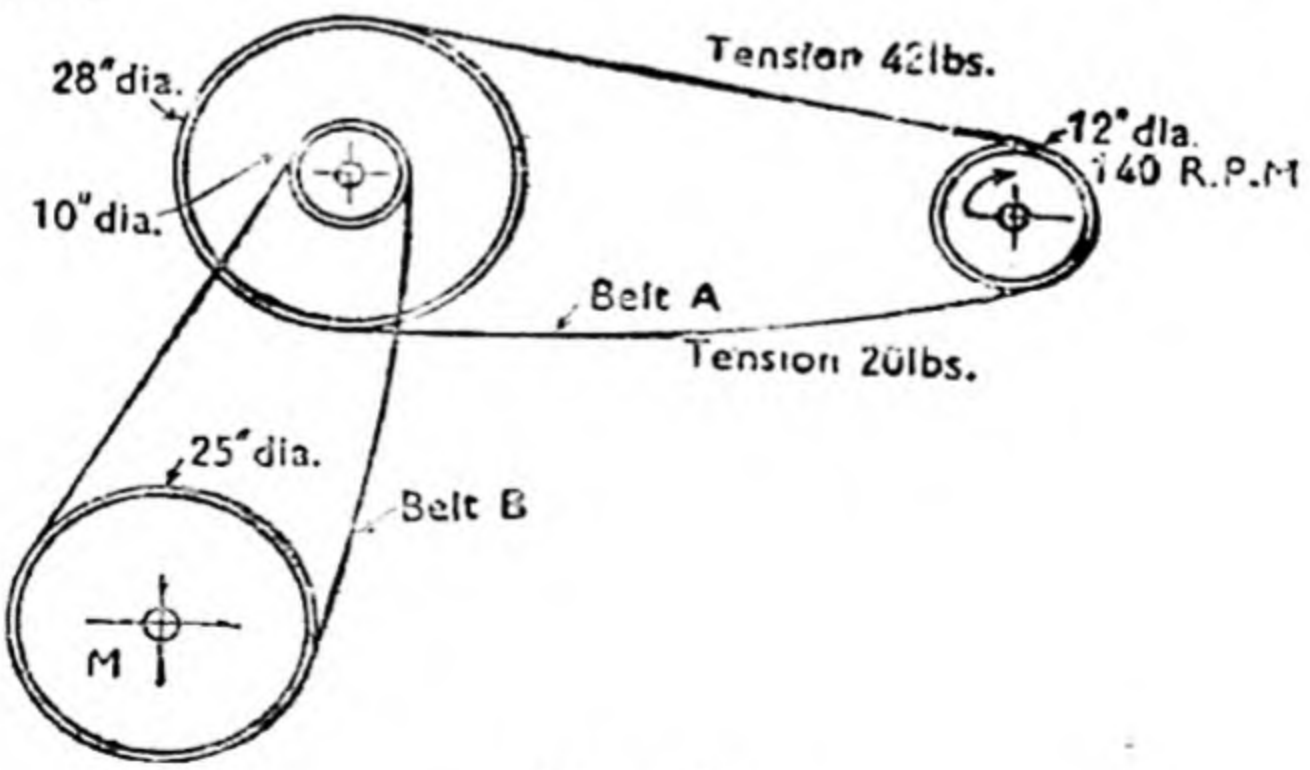

FIG. 319.

Calculate:

(*a*) The horse-power transmitted by the belts.
(*b*) The difference of the tensions on the tight and slack sides of the belt B.
(*c*) The turning moment applied to the machine shaft M.

Note.—Intermediate transmission losses between the belt A and the shaft M may be neglected.

22. A belt transmits 5 H.P. to a pulley of diameter 14 in. running at 250 revs. per min. What is the difference in the tensions on the tight and slack sides ?

23. The speed of a belt is 700 ft./min. and the tensions on the tight and slack sides are 400 and 114 lb. wt. respectively. Calculate the horse-power transmitted by the belt.

24. A cruiser is propelled at a speed of 30 m.p.h. by engines whose transmitted horse-power is 40,000. Calculate the resistance to the motion of the ship, and, assuming that the resistance varies as the square of the speed, what horse-power (to the nearest 100) would be required at a speed of 24 m.p.h. ?

25. (*a*) The cutting tool of a lathe exerts on the work a force of 200 lb. wt., the cut being made at 33 ft./min. What horse-power is being consumed ?

(*b*) The horse-power supplied is 0·25. What work per minute is being wasted ? (In foot-pounds.)

164. Brake Horse-power.

Let us suppose that an engine is being used to turn a shaft. The output of power at the shaft may readily be found for any engine speed by allowing the shaft to do work against a friction brake instead of delivering its power to the machinery to which the shaft is usually coupled. For small engines the two friction brakes described below give a sufficiently accurate measure of the horse-power output, but it is readily appreciated that the brakes must get very hot if the power is at all great, and in this case precautions have to be taken to cool the brake.

165. The Prony Brake.

A pair of wooden blocks are fitted to a pulley carried by the shaft. These blocks are held together by bolts which may be adjusted to any desired tightness. Secured to the blocks is a long arm which carries a weight W lb. This arm is only free to rotate through a small angle, for its motion is restricted by two pegs which prevent the weight from whirling round in a dangerous manner. The blocks are made to grip the pulley just a sufficient

amount to cause the weight to " float " between the two pegs. When this adjustment has been made the frictional moment

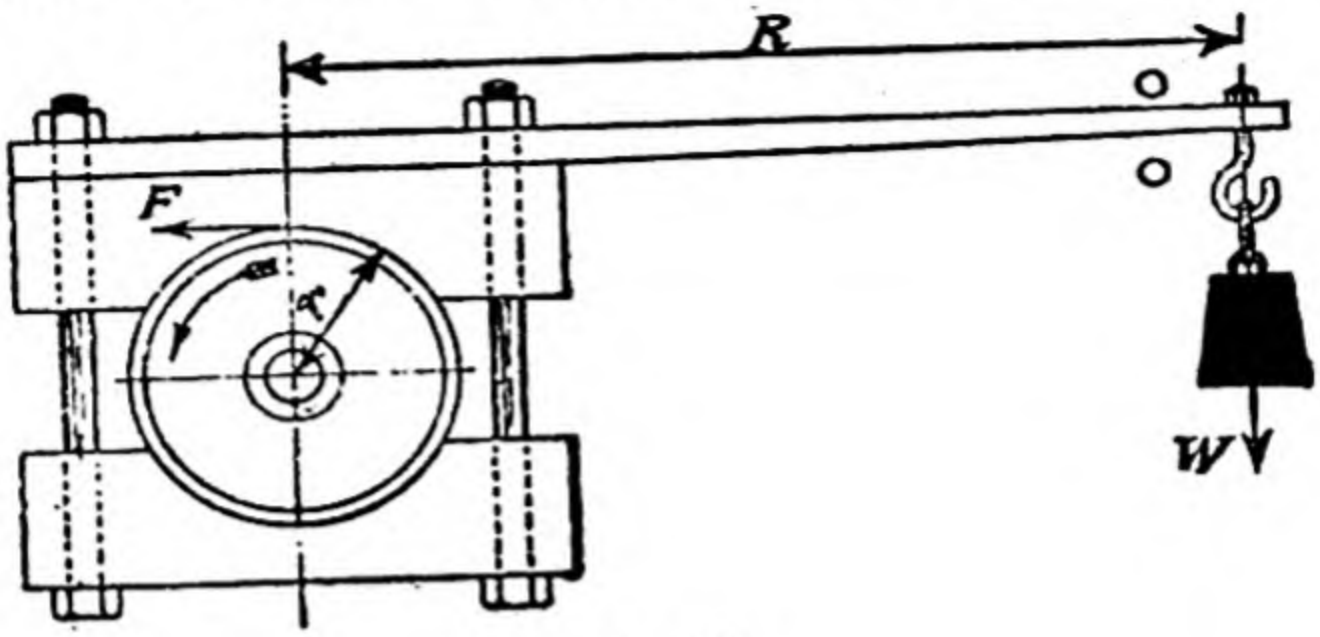

FIG. 320.

exerted by the pulley upon the blocks is balanced by the moment of the weight W about the axis of the shaft.

Referring to the above figure,

F lb. wt. represents the tangential force of friction between the blocks and the pulley, and r is the radius of the pulley.

$$Fr = WR.$$

Work done against friction in 1 rev. $= 2\pi rF = 2\pi RW.$

If n revs. of the pulley are made per minute,

$$\textbf{Horse-power output} = \frac{2\pi RWn}{33000}.$$

166. The Rope Brake.

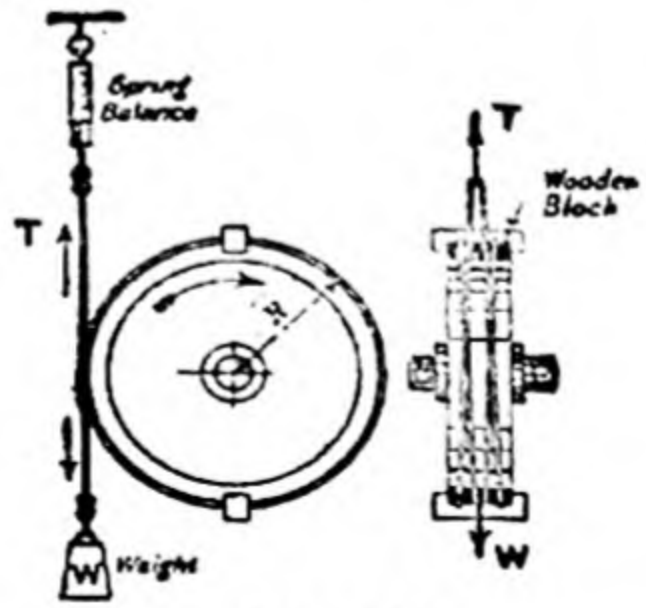

FIG. 321.

The principle of the rope brake is the same as that of the Prony brake, but the friction takes place between a rope and the surface of a flywheel secured to the shaft. In this case the engine works against the effective tension in the rope passing round the rim of the flywheel. If the reading of the spring balance (shown in the figure) is subtracted from the weight W and this is substituted for W in the formula of the last section, the horse-power output is again obtained.

167. Graphical Representation of Work.

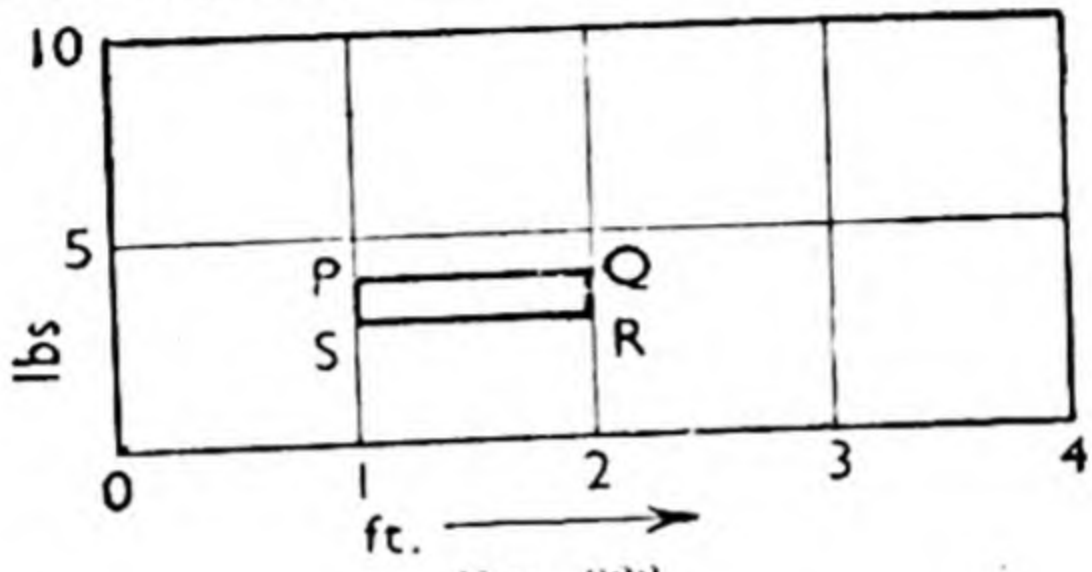

FIG. 322.

Suppose that a constant force of 10 lb. wt. acting upon a body moves the body a distance of 4 ft. in the direction of the force. The graph showing the force acting against a distance base is shown in the above figure and is the straight line AB. An area such as PQRS, in which PQ represents 1 lb. wt. and QR represents 1 ft., represents 1 ft. lb. of work. It therefore follows that the number of units of area each equal to PQRS

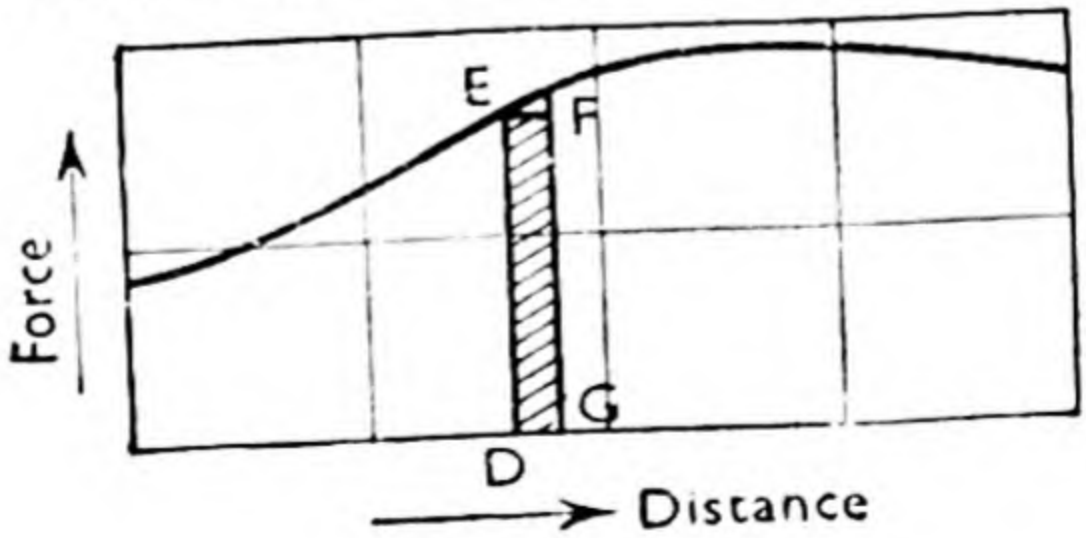

FIG. 323.

which are contained in the area OABC is a measure of the work done by the force. The area beneath the force-distance graph represents the work done by the force.

If a variable force acts upon a body the force-distance graph is a curved line, as shown in Fig. 323. If the area beneath the graph is divided into a large number of very thin rectangular strips parallel to the force axis, such as DEFG, then the sum of the areas of these rectangular strips, to a first approximation, represents the work done by the variable force. The greater the number of strips into which the area is divided, the more accurately will the sum of their areas represent the work done by the force.

The area beneath the force-distance graph represents the work done by the force.

The methods of finding the area of irregular figures described on pp. 166-67 are suitable for determining the areas beneath force-distance graphs.

168. Indicated Horse-power.

The power which is being developed by a steam or petrol engine, *i.e.*, the rate at which work is being done by the steam (or other gas) in the cylinder on the piston, can only be obtained if the force acting on the cylinder is known during the whole of its motion. This force is variable, but its value can be automatically registered during the stroke by an instrument such as the Crosby steam-engine indicator. In the figure below this instrument is shown.

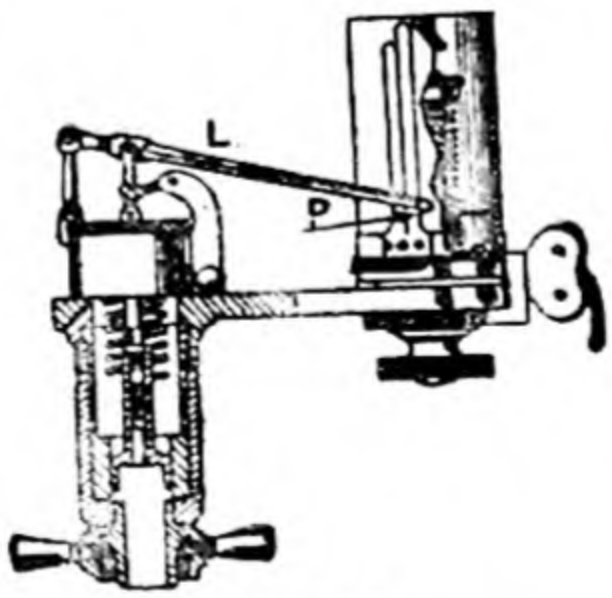

FIG. 324,

The steam in the engine cylinder is admitted to the indicator where it compresses a very strong spiral spring by an amount proportional to the pressure in the cylinder. The compression of the spring causes the lever L to rise and fall and the pencil P marks on the drum, which rotates in a manner governed by the motion of the piston a record of the pressure in the cylinder during the whole of the stroke.

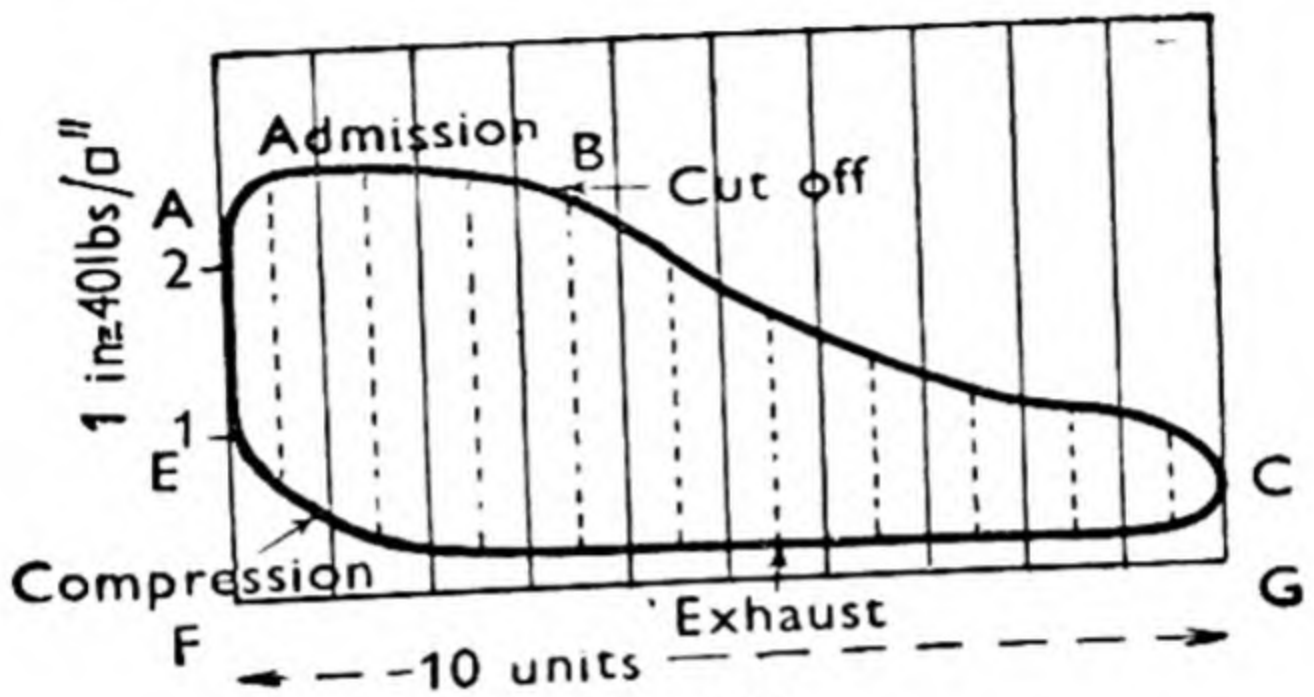

FIG. 325.

A typical indicator diagram for a steam engine is shown in the above figure. It will be noticed that the pressure diagram is closed and that the pressure in the cylinder does not fall below 20 lb. wt. per sq. in. ABC represents the part of the stroke during which the piston is being driven by the steam and CE the part of the stroke in which the piston is moving back into position ready for the next stroke. Along CE the motion of the piston is being opposed by the steam in the cylinder, and therefore the work available for driving the locomotive is represented by the difference of the areas ABCGF and CEFG, *i.e.*, the area ABCEA. To find this area the distance between the tangents to the curve at A and C should be divided into a number of equal parts, say ten, each being called one unit. Divide the area into ten strips of equal width and draw the mid-ordinates of these strips.

Area ABCE = (sum of lengths of mid-ordinates) × width of strip.

Using the above indicator diagram, suppose that the diameter of the piston is 28 in.,

Length of the stroke = 4 ft.,
Number of strokes per minute = 75.

The pressure scale is in inches.

Area ABCE = (1·50 + 1·90 + 2·00 + 1·72 + 1·44 + 1·10 + 0·88 + 0·75 + 0·60 + 0·4) units
= 12·29 units.

1 in. on the pressure scale = 40 lb. wt./sq. in.

10 units on the base = 4 ft.

1 unit of area represents 16 ft. lb./sq. in. of piston.

Work done per stroke $= 12{\cdot}29 \times 16 \times \frac{22}{7} \times \frac{28 \times 28}{4}$

$= 12{\cdot}29 \times 16 \times 22 \times 28$ ft. lb.

Power output per minute $= 12{\cdot}29 \times 16 \times 22 \times 28 \times 75$ ft. lb.

Indicated horse-power $= \frac{12{\cdot}29 \times 16 \times 22 \times 28 \times 75}{33000} = 275.$

If P represents the average pressure in the cylinder during the whole of a stroke, then,

P = Average of the mid-ordinates × value of unit on pressure scale.

Let L be the length of the stroke,
A the area of the piston,
N the number of strokes per minute.

Work done per stroke = P . A . L.
" " " minute = P . A . L . N.

Hence the I.H.P. is given by the expression,

$$\frac{P \cdot A \cdot L \cdot N}{33000}.$$

Referring to the previous indicator diagram,

P = 1·229 × 40 = 49·16 lb. wt./sq. in

169. Mechanical Efficiency of an Engine.

It has now been shown how the power output and the power developed may be found for an engine. It is readily understood that the power developed is the power put into the machine which, if the machinery were perfect, would all be transformed into useful work. The mechanical losses in the machine result in the power output being less than the power developed, and it is this power output which is given by the brake horse-power. Hence what is called the *Mechanical Efficiency* of the machine is given by the ratio,

$$\frac{\text{Power output}}{\text{Power developed}} = \frac{\text{Brake horse-power}}{\text{Indicated horse-power}}.$$

The mechanical efficiency of an engine usually lies between the values 0·75 and 0·95 (*i.e.*, 75 and 95 per cent.).

EXAMPLES XLVI

1. A student performed an experiment on a spring and obtained the following data in his notebook :—

Load.	Length of Spring.	Load.	Length of Spring.
Lb.	In.	Lb.	In.
2	6·25	12	7·50
4	6·49	14	7·75
6	6·75	16	8·00
8	7·00	18	8·24
10	7·05	21	8·50

Plot a graph showing the behaviour of the spring, and from your graph check the accuracy of the student's observations. How many foot-pounds of work have been done in stretching the spring ?

2. The force exerted in pulling a truck is given at intervals of 10 yd. in the following table :—

Force.	Distance.	Force.	Distance.
Lb. wt.	Yd.	Lb. wt.	Yd.
50	0	18	50
38	10	17	60
23	20	10	70
16	30	0	80
14	40		

Draw a graph showing the force against a distance base. Calculate the total work done.

3. A lath, or beam, is supported horizontally at two points 42 in. apart. It is loaded at the centre with a gradually increasing load, and the loads and corresponding deflections are as follows :—

Load.	Deflection.	Load.	Deflection
Lb.	In.	Lb.	In.
2	0·013	20	0·128
6	0·039	22	0·145
10	0·065	24	0·166
14	0·088	26	0·317
18	0·113		

Plot the load deflection graph and explain its important features.

How many inch-pounds of energy are expended in deflecting the lath 0·1 in. ?

Use scales—1 in. to 10 lb., and 1 in. to 0·1 in. deflection.

4. From the force-distance diagram shown below, find the average

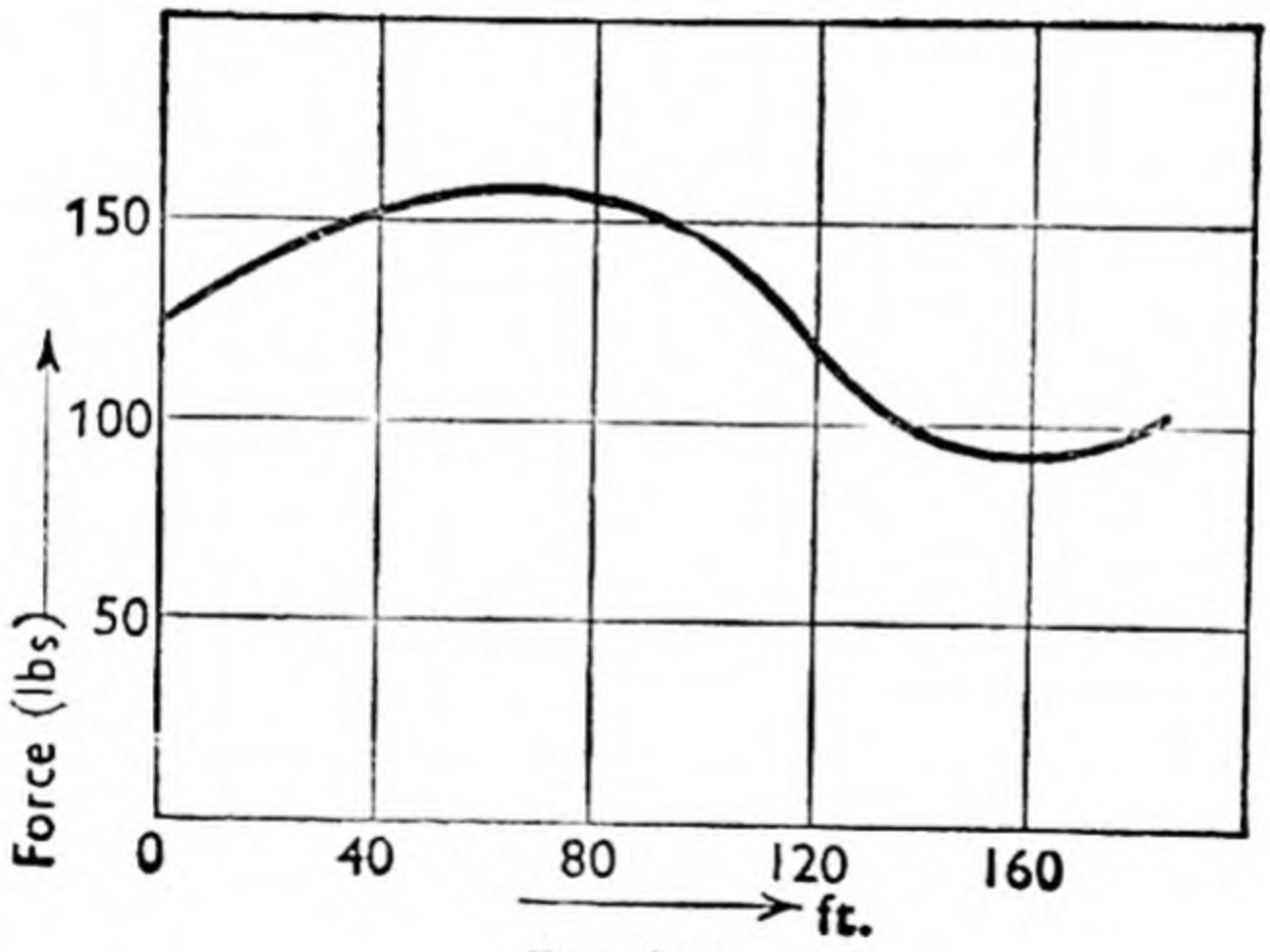

Fig. 326.

force exerted (*a*) in the first 80 ft., (*b*) in the first 160 ft. Calculate the total work done during the first 160 ft.

5. In the diagram the pressure in a steam-engine cylinder is shown. The stroke of the piston is 4 ft. and its diameter is 28 in. Find the mean pressure during the stroke and calculate the work done per stroke. If

the number of strokes is 120 per min., calculate the indicated horse-power.

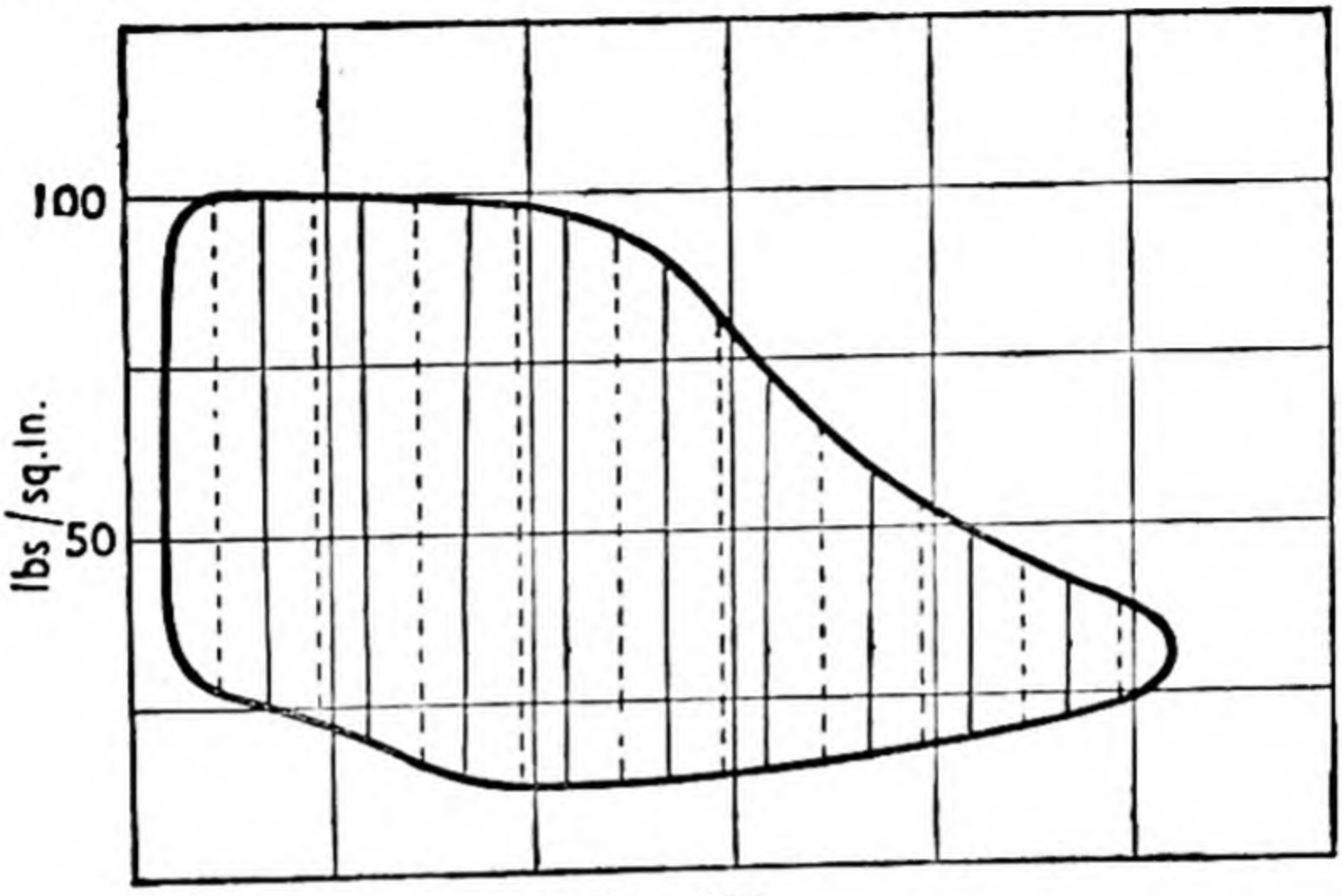

FIG. 327.

6. Make a sketch of a suitable brake for finding the brake horse-power of an engine of about 7 H.P.

Determine the brake horse-power of an engine which makes 300 revs. per min. when the brake drum diameter is 36 in. and the effective load on the brake is 95 lb. If the mechanical efficiency of the engine is 0·8, what is the indicated horse-power ?

7. P is the pulley of diameter d' fixed to shaft of an electric motor

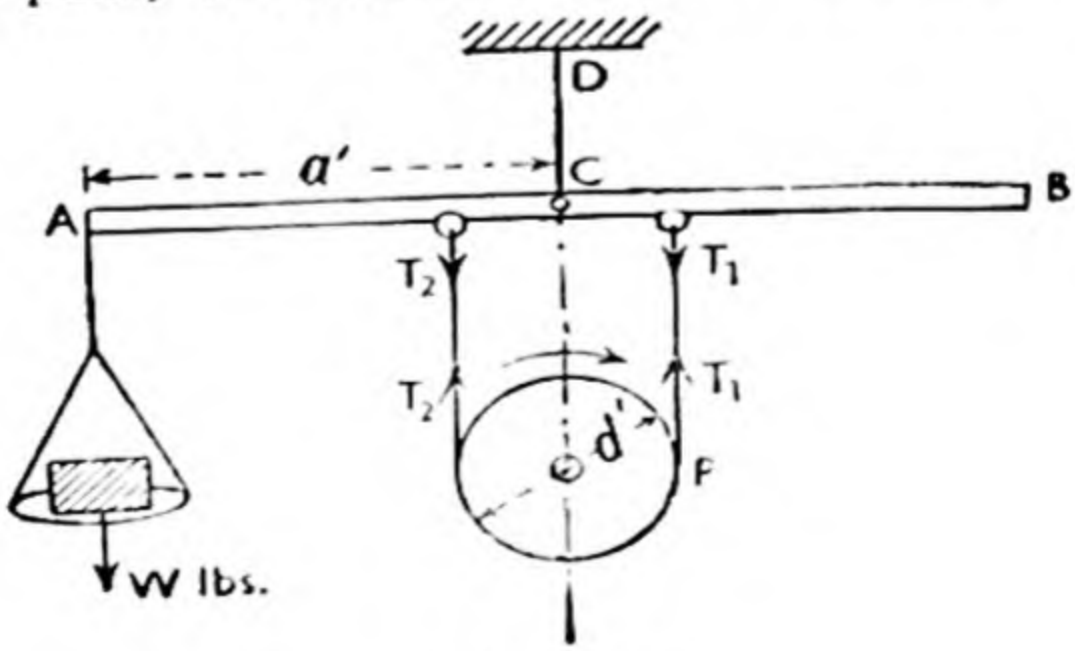

FIG. 328.

whose brake horse-power is being measured. AB is a steel bar whose centre of gravity is at C, and CD is tightened by a screw so that the belt which passes round the pulley is taut on both sides when the pulley is rotating. Show that if a weight W lb. is suspended at the end A of the steel bar,

$$W \cdot a = \frac{(T_1 - T_2)d}{2}.$$

Show also that if the pulley makes n revs. per min. the brake horse-power of the motor is $\frac{2\pi W \cdot a \cdot n}{33000}$.

8. What should be the length of stroke of a double-acting steam engine in order to generate 220 H.P. when running at a speed of 120 revs. per min., the mean effective steam pressure being 40 lb. wt./sq. in. and the diameter of the cylinder 21 in. ?

9. The average pressure in a steam cylinder during a stroke is 120 lb. wt./sq. in. The diameter of the piston is 21 in. and the horse-power generated is 210. Find the length of the stroke if the number of strokes in 1 min. is 90.

10. Calculate the average effective pressure in an engine cylinder given that the diameter of the piston is 14 in., the length of the stroke is 18 in., the number of strokes per minute is 60 and the horse-power developed is 66.

11. Calculate the horse-power developed by an engine given the following details :—

Average effective pressure . . .	200 lb. wt./sq. in.
Diameter of piston	14 in.
Number of strokes per minute . .	110
Length of stroke	2 ft.

12. An engine developing 12 H.P. drives a motor car along a horizontal track at 40 m.p.h. against resistances amounting to 60 lb. wt. Calculate the mechanical efficiency of the engine.

13. A work diagram for a gas engine is shown. Find the average

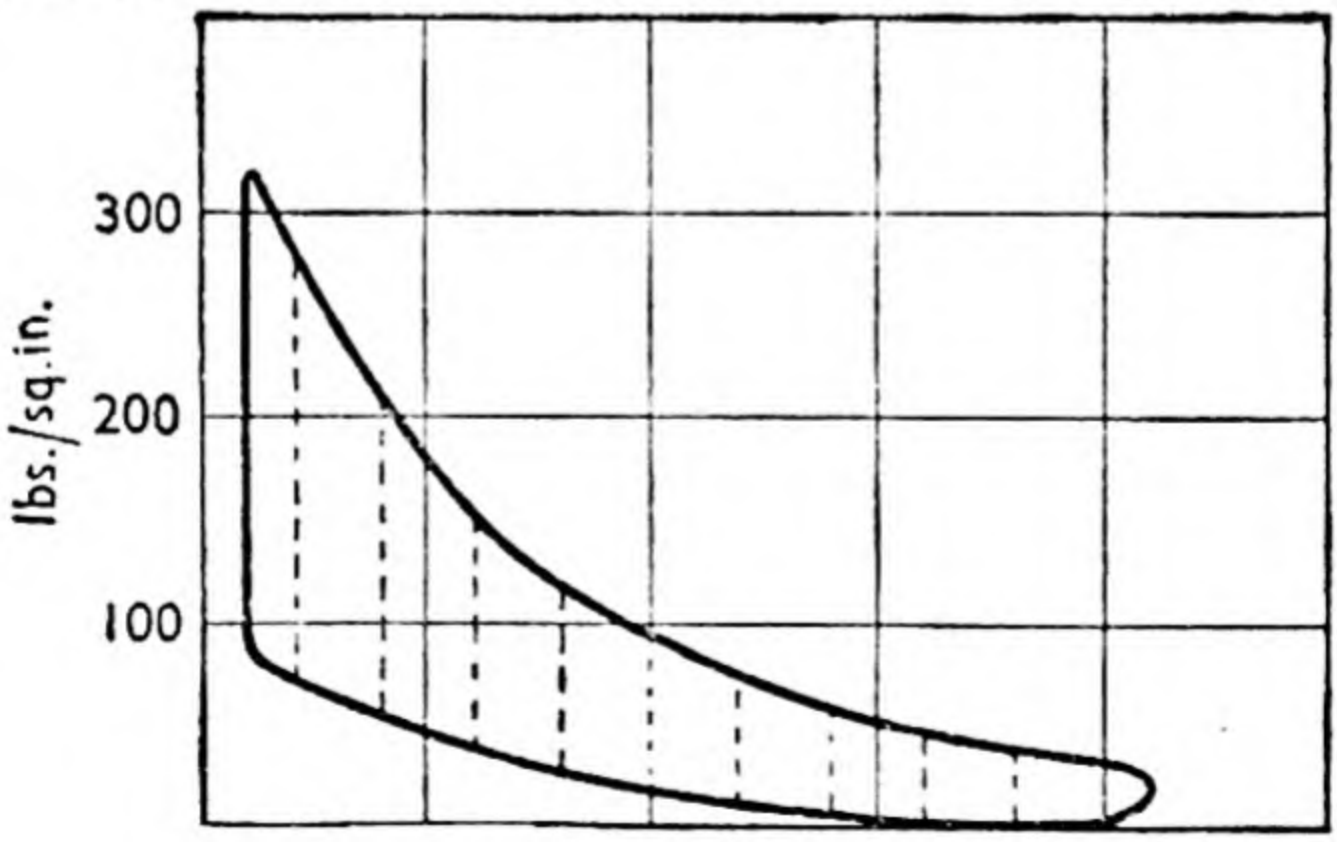

FIG. 329.

resultant pressure during a stroke in pounds per square inch and the work done in foot-pounds in each stroke. The piston diameter is 6·3 in. and the length of the stroke is 14 in.

14. In question 13, if the number of strokes per minute is 120, calculate the horse-power developed and the brake horse-power if the mechanical efficiency is 0·75.

CHAPTER XVIII

COMPOSITION OF VELOCITIES

170. Displacement.

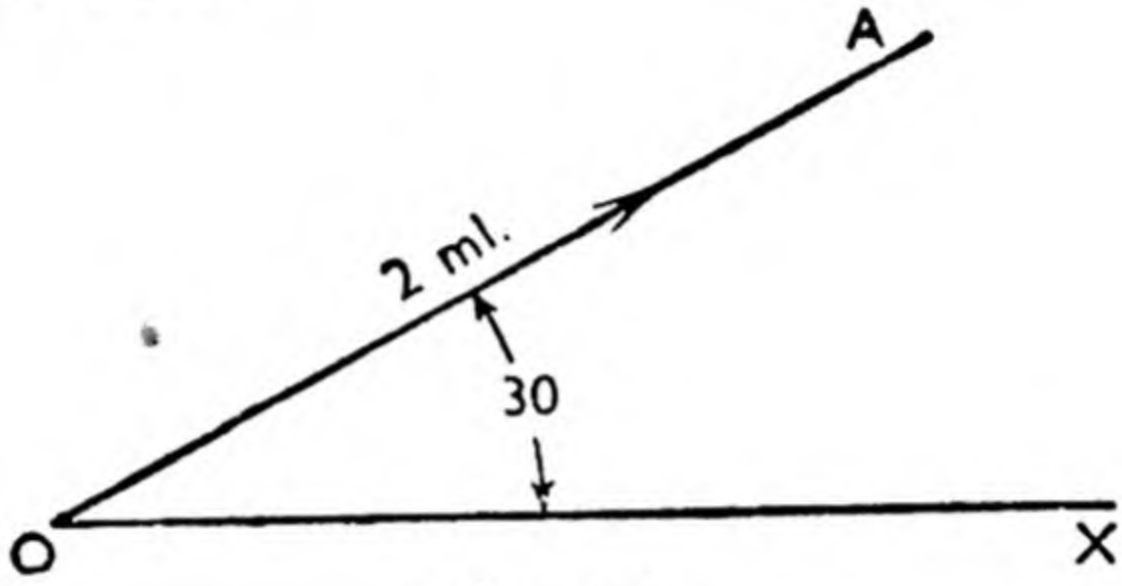

FIG. 330.

Suppose that O represents the position of a cyclist and OX the easterly direction. If the cyclist rides along a road to the new position A, his change of position may be described fully by stating the distance travelled and the direction of the line OA.

Displacement of cyclist = 2 mile in a direction 30° N. of E.

Displacement is change of position and is completely described by stating the magnitude and direction of this change.

It is usual to represent the displacement from O to A as $\overrightarrow{OA}$ to distinguish it from the symbol OA which represents the displacement in magnitude alone.

171. The Resultant of Successive Displacements.

If the cyclist now proceeds to the position B, as shown in Fig. 331, the additional displacement is AB, 1 mile in a direction 75° N. of E. The final displacement from the original position O is $\overrightarrow{OB}$, and this displacement is the resultant of

the first two displacements. The last statement may be written,

$$\vec{OB} = \vec{OA} + \vec{AB}.$$

It should be noticed that the addition on the right-hand side of this equation is the same as that used to find the resultant of

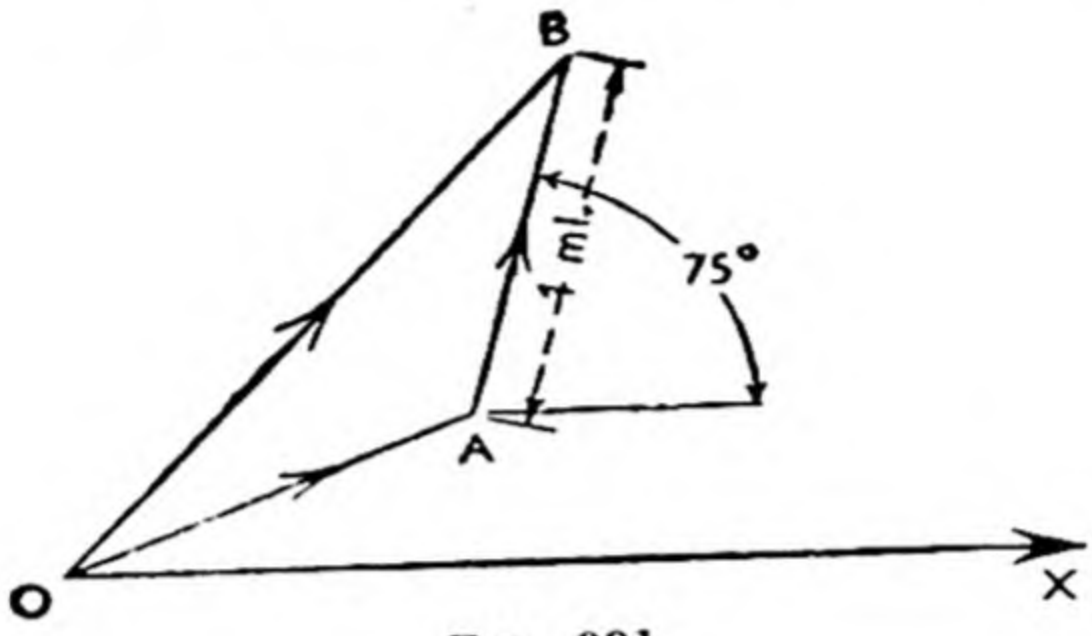

FIG. 331.

any number of forces acting at a point by the polygon of forces. Quantities such as forces, velocities and accelerations which have both magnitude and direction are called *Vector Quantities*, and when they are combined to give a single resultant the process is called *Vector Addition*.

172. The Resultant of Two Velocities by the Parallelogram Law.

In Fig. 332 XY and ZW are the parallel banks of a river which flows in the direction XY.

In the discussion which follows it will be assumed that the whole stream moves downstream *as if it were a solid body* so far as its motion is concerned.

A is the position of a boat which is rowed in the direction AD ; *i.e.*, the boat would proceed along AD but for the current which carries it downstream.

The current alone would have carried the boat in the direction AB. The boat therefore has two velocities simultaneously,

one produced by the occupants of the boat and the other by the river current.

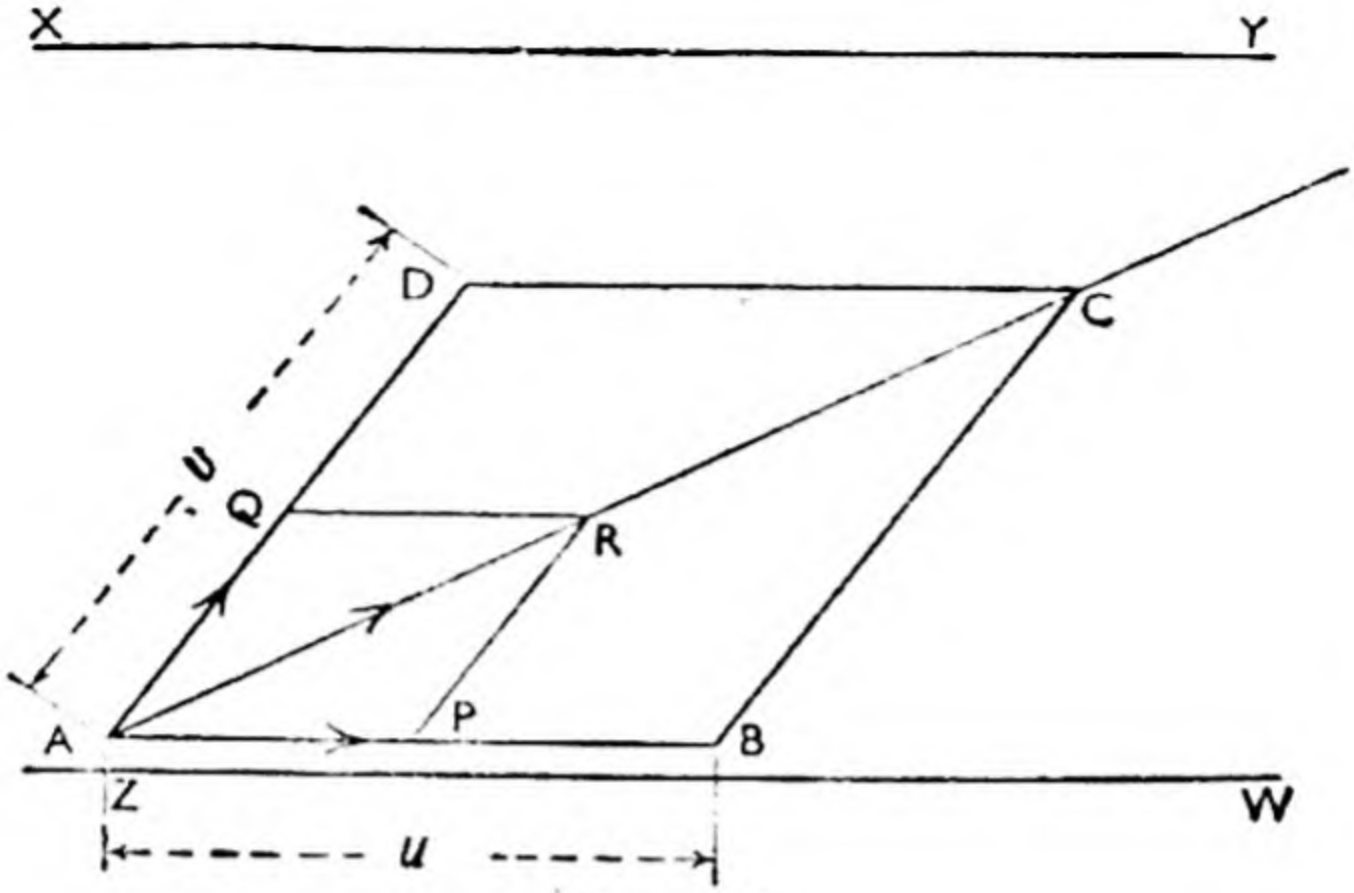

FIG. 332.

To find the resultant path of the boat consider the position of the boat after 1 sec.

Let u ft./sec. be the velocity of the stream,

v ft./sec. be the velocity with which the boat is rowed.

In 1 sec. the river current alone would have carried the boat to B where $AB = u$ ft.

In 1 sec. the boat would have travelled to D where $AD = v$ ft. if there had been no current.

If we now complete the parallelogram ABCD the final position of the boat after 1 sec. is clearly at C and is the same as if the boat had received two consecutive displacements represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

AB represents the velocity of the stream in magnitude and direction.

AD represents the velocity with which the boat is rowed, and it therefore appears that AC represents the resultant velocity of the boat. Now all we have shown is that AC represents the displacement of the boat in 1 sec., and we must therefore show

first that the boat travels along the diagonal AC continuously, and second, that the velocity of the boat along AC is uniform like those of the two components of which it is the resultant.

In the figure let R be a point on the diagonal OC and RP be drawn to meet AB at P. Complete the parallelogram APRQ.

Because QR // DC,

$$\frac{AR}{AC} = \frac{AQ}{AD} = \frac{\text{Time for AQ}}{\text{Time for AD}} = \frac{t_2}{1} = t_2.$$

Because PR // BC,

$$\frac{AR}{AC} = \frac{AP}{AB} = \frac{\text{Time for AP}}{\text{Time for AB}} = \frac{t_1}{1} = t_1.$$

Hence $$t_1 = t_2 = \frac{AR}{AC} = t, \text{ say.}$$

Since the time for AQ is equal to the time for AP, R is on the path of the boat, and because R is any point either on AC or AC produced, AC is the path of the boat. Moreover, because $AR = AC \cdot t$, and therefore $\frac{AR}{t} = AC$, the velocity of the boat along AC is uniform and is represented by AC in magnitude and direction.

If a body has two velocities simultaneously and these velocities are represented in magnitude and direction by the sides AB, AD of a parallelogram ABCD, the resultant velocity of the body is represented in magnitude and direction by the diagonal AC.

173. Assuming the opposite banks of the river to be parallel, in what direction should the boat be headed in order that it may cross the river in the shortest possible time ?

Consider the effect of the current alone. Since the displacement of the boat due to the current is parallel to the banks, *the current cannot have any influence upon the time taken to travel in a direction perpendicular to the banks.*

We have therefore to row the boat in a direction which will enable it to cross the river in the shortest possible time exactly

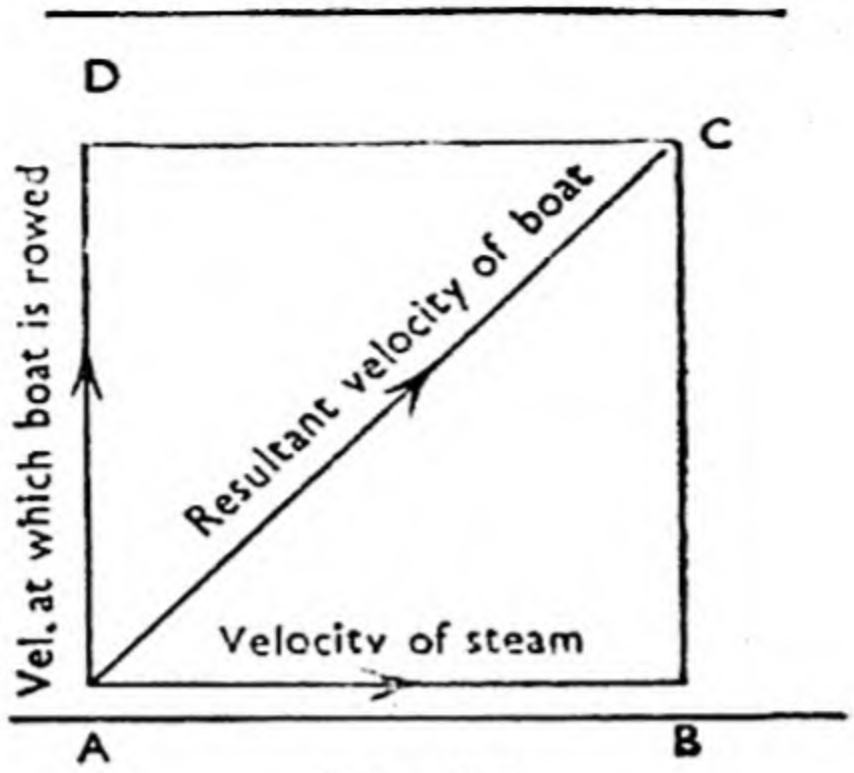

FIG. 333.

as if there were no current. Hence the boat must be headed in a direction perpendicular to the banks. In the figure,

AB represents the velocity of the river,
AD " " " with which the boat is rowed,
and AC " " resultant velocity of the boat.

The time taken by the boat to cross the river is equal to the width of the river divided by the speed with which the boat is rowed.

174. Example (1).—*Suppose that the velocity with which the boat can be rowed is 4 ft./sec. and that the velocity of the current is 3 ft./sec. In what direction should the boat be headed in order that it may cross the stream in a path perpendicular to the banks?* (i.e., *the distance travelled by the boat is to be the shortest possible.*)

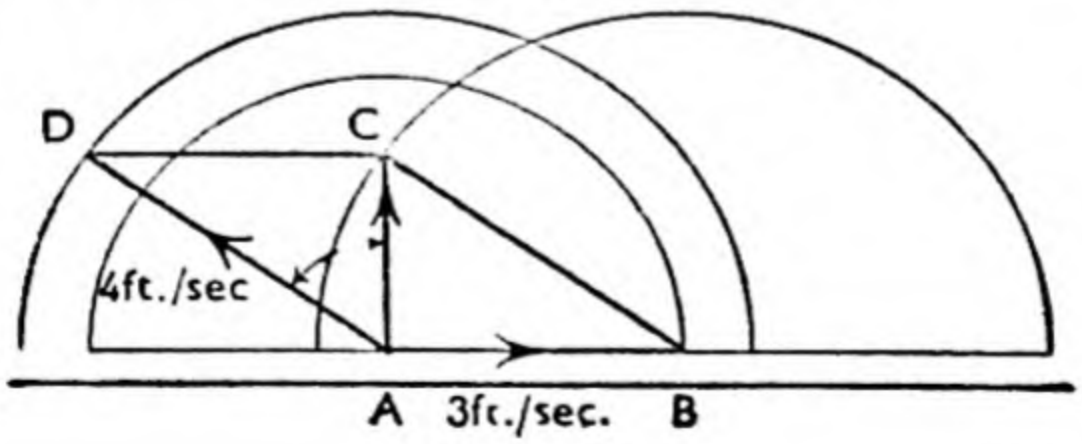

FIG. 334.

Let A be the original position of the boat.
Consider the effect of the rowing alone.

In 1 sec. the boat could be at any point on the circumference of a semicircle, centre A and radius 4 ft.

The current alone would take the boat to B, where AB = 3 ft.

The boat will cross by a path perpendicular to the banks when the resultant velocity is in this direction.

If we find the resultant of the two velocities 3 ft./sec. and 4 ft./sec., we find that in 1 sec. the boat may be anywhere on the circumference of a semicircle, centre B and radius 4 ft. Suppose the figure is drawn to scale.

Let the semicircle, centre B and radius to represent 4 ft., cut AX drawn perpendicular to AB at C. Complete the parallelogram ABCD, then AD represents the velocity with which the boat must be rowed. From the right-angled triangle ACD,

$$\sin \mathrm{DAC} = \frac{\mathrm{DC}}{\mathrm{DA}} = 0{\cdot}75,$$

$$\angle \mathrm{DAC} = 48^\circ\ 35'.$$

The boat must be rowed at an angle 41° 25′ to the bank in an upstream direction.

To find the time taken by the boat to cross the stream we must divide the width of the river by the resultant velocity of the boat represented by AC.

Is the above construction possible if the velocity of the current is greater than that with which the boat can be rowed ?

Example (2).—*An aeroplane is timed to arrive at a town* 100 *miles N.E. of its starting point in* 1 *hr. If there is a* 40 *m.p.h. wind from the west, in what direction should the aeroplane be headed and what should be its speed in still air ?*

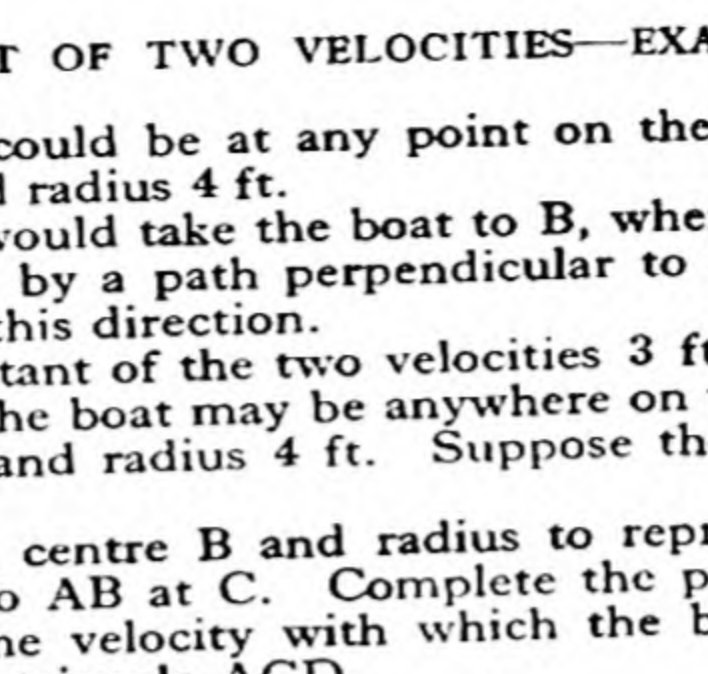

FIG. 335.

It is assumed that so far as the motion of the wind is concerned, it may be considered to move *as if it were a solid body*.

The resultant velocity of the plane is 100 m.p.h. in a north-easterly direction.

Draw AC to represent the resultant velocity of the plane.

This velocity is the resultant of the velocity of the plane in still air and the velocity of the wind.

Draw AB to represent the velocity of the wind.

Join BC and complete the parallelogram ABCD.

AD represents the velocity of the plane in still air.

175. Component Velocities and Resolution of Velocities.

Just as it has proved convenient to consider a force as the resultant of two components, we may by means of the parallelogram law consider a velocity as the resultant of two components.

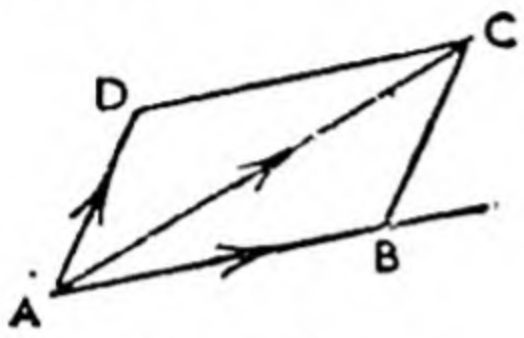

FIG. 336.

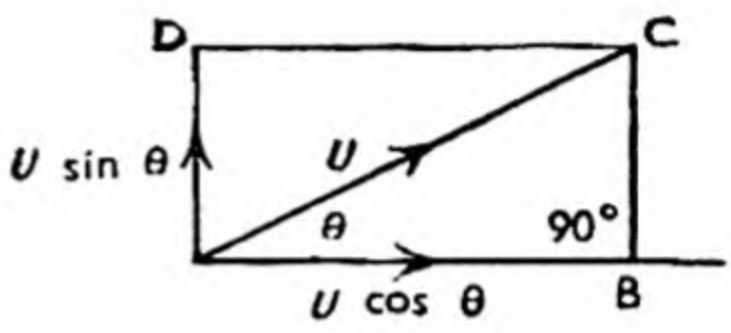

FIG. 337.

Thus if AC represents the velocity of a body, *any* parallelogram having AC as its diagonal enables us to consider the velocity as made up of two simultaneous component velocities, such as AB and AD in Fig. 336. It is usually desirable to choose the two components at right angles to each other. Referring to Fig. 337, the components of the velocity v may be written as $v \cos \theta$ and $v \sin \theta$.

We shall find the results of this paragraph of immediate practical value in the next section of our work.

EXAMPLES XLVII

1. A man walks at 3 m.p.h. across the floor of a carriage perpendicular to the direction in which it is travelling. If the speed of the carriage is 4 m.p.h. due east, what is the direction and magnitude of the resultant velocity of the man ?

2. A river flows due north at 2 m.p.h. A boat is headed due west across the river perpendicular to the direction in which it flows and is rowed at 2 m.p.h. Show that the resultant direction in which the boat travels is north-west, and calculate the magnitude of this resultant velocity.

3. A bird can fly at 20 m.p.h. in still air. If it heads due north when there is a wind of 30 m.p.h. blowing from the south-west, find the magnitude and direction of the resultant velocity of the bird.

4. An aeroplane heads due south. The speed of the plane in still air is 100 m.p.h. If a south-west wind (*i.e.*, from the south-west) is blowing

at 50 m.p.h., find the magnitude and direction of the resultant velocity of the plane.

5. The speed of the current of a river is 1 m.p.h. A man rows a boat so as to cross the river perpendicular to the direction of the current. If the man can row the boat in still water at 2 m.p.h., in what direction is the boat headed and what is the velocity with which he crosses the stream?

6. A steamer heads due south at 10 knots, but it is found that owing to the presence of a sea current it actually travels in a direction 10° west of south at 12 knots. Find the direction and magnitude of the sea current.

7. A crane travelling horizontally lowers a load at 15 ft./sec. If the load travels in a direction inclined at 30° to the vertical, calculate the horizontal velocity of the crane.

8. A stone is travelling at 80 ft./sec. in a direction inclined at 20° to the horizontal. Calculate the horizontal and vertical components of the velocity of the stone at the instant under consideration.

9. The banks of a stream are parallel and a current of $1\frac{1}{2}$ m.p.h. is flowing. A boat is headed in a direction of 60° with the downstream direction. If the speed at which the boat can be rowed in still water is 2 m.p.h., calculate the time taken to cross the stream, its width being 50 yd.

10. An aeroplane is to be flown from A to B, a distance of 200 miles, B being in a direction 30° north of east of A. If there is a wind of 40 m.p.h. blowing due east and the plane takes 2 hr. to complete the journey (not including landing and taking off), find the direction in which it was headed and the speed with which it would have travelled had there been no wind blowing.

11. A car is travelling at 50 m.p.h. in a direction 60° north of east. What is the component of its velocity in an easterly direction? How long elapses before the car is 20 miles east of its original position? How far is it then north of its original position?

12. Explain the terms component velocity, resultant velocity, and show how velocities may be compounded.

A river 2 miles wide flows with a speed of 1 m.p.h. Find the time required for a man, rowing with a speed of 2 m.p.h., to cross the river at right angles to the bank. (*C.*)

13. What is the difference between a vector quantity and a scalar quantity? Give two examples of each.

The pilot of an aeroplane, which flies at 100 m.p.h. when there is no wind, wishes to travel due north when a 20 m.p.h. wind is blowing towards the east. Find graphically the direction in which he must point the nose of the machine and his actual speed relative to the ground. (*J.*)

14. What is meant by the parallelogram of velocities? Explain why distant objects seen from a moving train remain in view much longer than nearer objects.

The smoke from a vessel steaming north at 5 m.p.h. points 30° south of east, the wind blowing from the west. Find the velocity of the wind. [Hint: When the smoke leaves the vessel its velocity is that of the wind.] (*O.*)

15. State and explain the theorem of the parallelogram of velocities.

A river is $\frac{1}{4}$ mile wide and flows at the rate of $1\frac{1}{2}$ m.p.h. A man who can swim at the rate of 2 m.p.h. in still water wishes to cross directly to a point 110 yd. downstream. Find the time of crossing.

CHAPTER XIX

PROJECTILES

176. Projectiles.

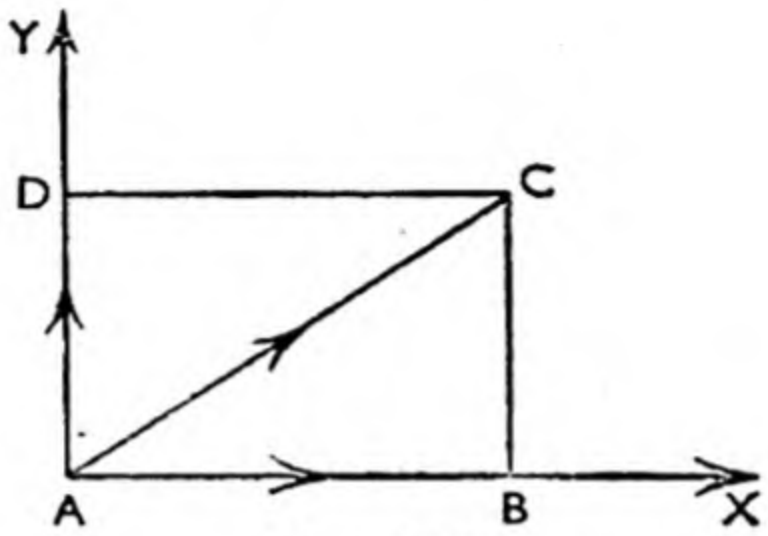

FIG. 338.

If AX and AY are at right angles to each other, a velocity represented by AC has been shown to be equivalent to the two component velocities represented by AB and AD. The position of a body moving along AC with uniform velocity represented by AC may therefore be found by adding the displacements given to the body by the two components of its velocity.

The component of the displacement of the body parallel to AD is unaffected by the component of velocity parallel to AB, and similarly the component of the displacement of the body parallel to AB is unaffected by the component of velocity parallel to AD. Hence the position of the body at any time may be found by adding the displacements due to each of the components of velocity, these displacements being found *independently of each other*.

We are most of us familiar with the shape of the curved

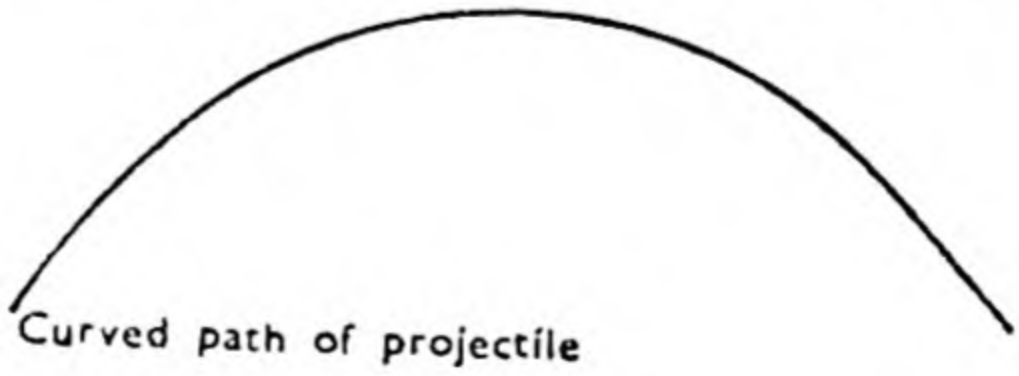

FIG. 339.

path * of a ball thrown into the air. The motion of such a ball is not one of uniform velocity. The motion can, however, at any instant be considered as made up of a horizontal motion

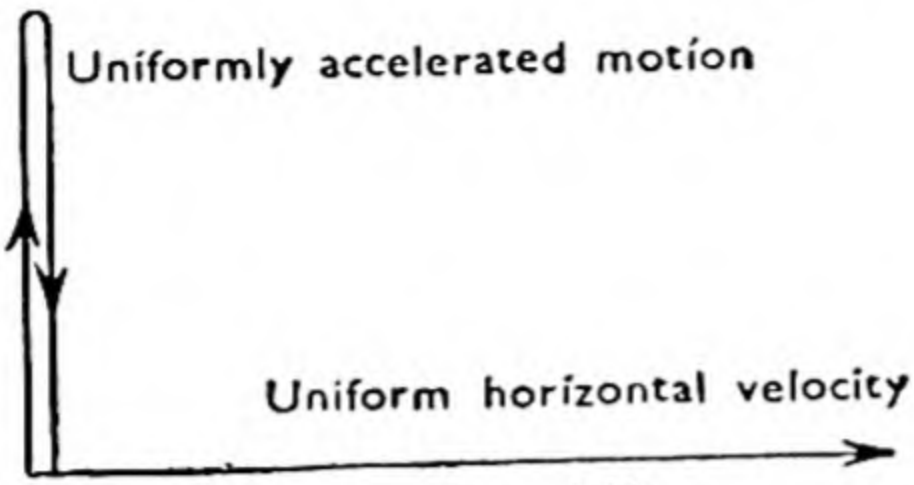

FIG. 340.

and a vertical motion. Now a body moving horizontally travels with uniform velocity if there is no resistance to its motion, and a body moving freely in a vertical direction travels with uniformly accelerated motion. Can the position of a body describing a curved path freely in the air be found by considering its motion as made up of the two motions, one of uniform velocity horizontally and the other a uniformly accelerated motion vertically? Again, can the position of the body at any time be determined by finding *independently* the displacements produced by each of these two motions (*i.e.*, exactly as if the other motion were not present)?

We may remark that Galileo knew the correct answer to these questions, whereas his immediate predecessors certainly did not.

177. EXPERIMENT (1).—Place a smooth piece of wood so that it projects a small distance over the edge of a table, and on the part which projects place a brass weight of, say, 50 gm. Arrange the wood so that when it is struck a smart blow the 50 gm. wt. falls vertically to the ground, whilst at the same time a second weight, which is placed as close as possible to the edge of the table adjacent to that first mentioned, is struck a horizontal blow by the wood and describes a curved path through the air before it hits the ground. It will be found that the two weights strike the ground together as far as may be judged by the sounds of their impacts with it. Moreover this remains true whatever the horizontal distance travelled by the second weight may be before it hits the ground.

* It is surprising to find that Santbach in 1561 was of the opinion that a cannon ball sped onwards in a straight line until its velocity was exhausted and then dropped vertically to the ground!

GIRLS COLLE
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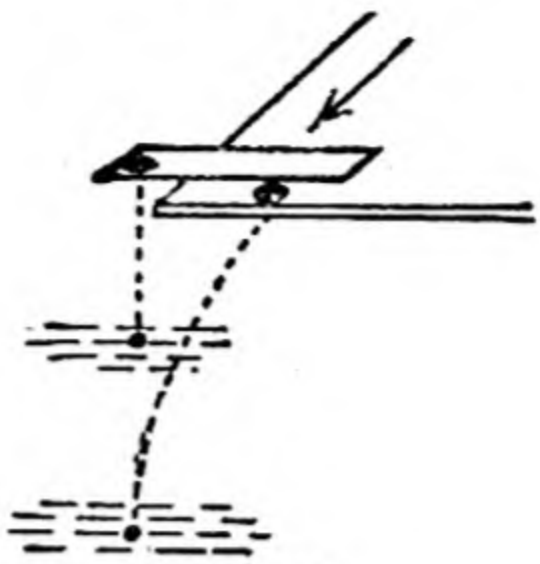


FIG. 341.

It therefore appears that the time taken by the second weight to reach the ground is the same as it would be if it were allowed to fall vertically, and we see therefore that the vertical motion of the weight is unaffected by the horizontal motion

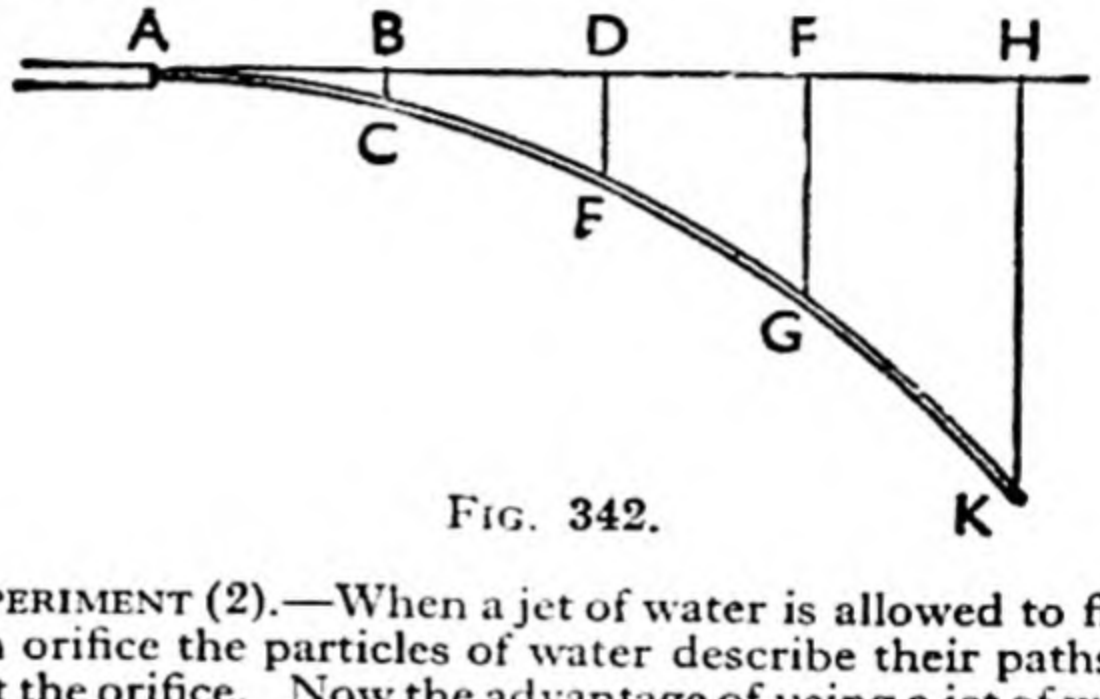


FIG. 342.

178. EXPERIMENT (2).—When a jet of water is allowed to flow horizontally from an orifice the particles of water describe their paths freely once they have left the orifice. Now the advantage of using a jet of water to study the path of a projectile rather than a solid body is obviously that we can arrange the jet to show *continuously* the path traced out by the particles of water. We have shown that the vertical motion of a body describing a curved path freely in the air is not affected by its horizontal motion, and we must now try to show that the horizontal motion of the particles is the same as it would be if the vertical motion were not present. Now if air resistance is neglected since there is no horizontal force acting on the particles of water, they should move with a constant horizontal component of velocity. If this were true the horizontal displacements in equal times would be equal, and conversely the times for equal horizontal displacements would be equal. Let A be the position of the orifice and K a point on the curved path of the jet. H is a point vertically above K, and AH is horizontal. AH is divided into, say, four equal parts, AB, BD, DF, FH, and the vertical displacements BC, DE, FG, HK correspond to the horizontal displacements AB, AD, AF, AH. We *know* that the vertical motion is given by

$$s = \tfrac{1}{2}gt^2, \text{ i.e., } \frac{s}{t^2} = \text{constant.}$$

If the times for AB, AD, AF, AH are in the proportion 1 : 2 : 3 : 4,

$$\frac{BC}{1^2}=\frac{DE}{2^2}=\frac{FG}{3^2}=\frac{HK}{4^2}=\text{Constant.}$$

Now it is quite a simple matter to measure BC, DE . . ., and hence to test the truth of the last equations. It will be found that the results of the experiment show that these ratios are equal, and therefore we may infer that the horizontal motion of the particles is one of uniform velocity.

Hence the horizontal motion of a body describing a curved path freely in the air is not affected by the vertical motion.

179. Example (1).—*An aeroplane travelling horizontally with velocity 60 m.p.h. at a height of 400 ft. drops mails into a field. If the resistance of the air is neglected, how long do the mails take to reach the ground and what horizontal distance have they travelled before they strike the ground.*

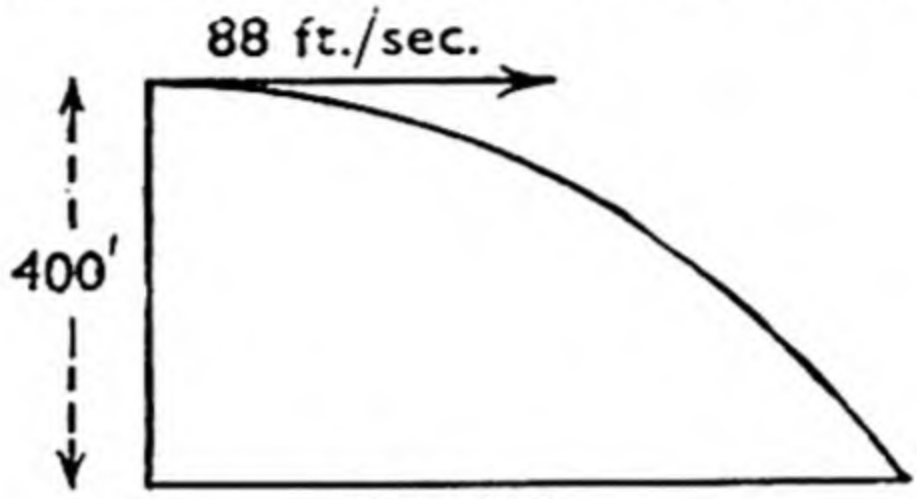


FIG. 343.

The time taken by the mails to reach the ground is the same as if they had fallen 400 ft. from rest.

Let t sec. be the required time. Using

$$s=\tfrac{1}{2}gt^2,$$
$$400=16t^2,$$
$$t^2=\frac{400}{16}, \text{ and } t=5.$$

In 5 sec. the horizontal component of the plane's velocity, which is constant and equal to 88 ft./sec., carries the mails a horizontal distance of

$$88\times 5=440 \text{ ft.}$$

Example (2).—*A tennis ball is struck horizontally at a height of 6 ft. above the ground and hits the ground when it has travelled 60 ft. in a horizontal direction (this being the distance from the base line to the service line). With what velocity did it leave the tennis racket?*

The time taken by the ball to travel vertically a distance of 6 ft. is given by

$$s=16t^2. \quad (g=32)$$

Therefore $6=16t^2$, and $t=\sqrt{0\cdot375}=0\cdot608$ sec.

The constant horizontal component of velocity is equal to the velocity of projection.

$$\text{Velocity of projection}=\frac{60}{0\cdot608}=98 \text{ ft./sec.}$$

EXAMPLES XLVIII

1. A stone is thrown with a velocity of 50 ft./sec. horizontally from the top of a high tower. Calculate the horizontal and vertical distances which it travels (*a*) in 1 sec., (*b*) in 2 sec.

2. A rifle bullet is fired horizontally with a velocity of 1,000 ft./sec. If air resistance is neglected, how far has it travelled horizontally when it has fallen 16 ft. below its initial position ?

3. An aeroplane travelling horizontally at 75 m.p.h. drops mails which land 3 sec. after they are released. How high above the ground was the aeroplane when the mails were released and how far do the mails travel horizontally after they are released before they strike the ground ?

4. A stone is projected horizontally from the top of a tower 121 ft. high. How long does it take the stone to reach the ground (supposed horizontal) and how far from the foot of the tower does it strike the ground if the velocity of projection is 100 ft./sec. ?

5. A stone is projected horizontally at a height of 64 ft. above the ground and strikes the horizontal plane through the foot of the tower at a distance of 220 ft. from the point on the ground vertically below the point of projection. Calculate the time taken by the stone to reach the ground and its velocity of projection.

6. A jet of water flows from a small orifice and issues in a horizontal direction. When the jet has descended 9·72 in. it has travelled horizontally 3 ft. Calculate the speed at which the water issues from the orifice.

7. An aeroplane flying horizontally at a height of 10,000 ft. with a northward air speed of 100 m.p.h. at a place where there is a 50 m.p.h. west wind blowing drops a heavy bomb. Show as a diagram the relative positions of the point on the ground directly below the aeroplane at the moment of release and the point where the bomb strikes. Calculate in feet the distance between the two points. (Neglect the action of the air on the bomb.) (*G.S.*)

8. A golf ball is hit horizontally (without spin) from the top of a vertical cliff 256 ft. high with an initial velocity of 100 ft./sec. Determine the position of the ball at the end of each second of its motion and hence plot its path, making the following assumptions :—

(*a*) That the ball obeys the laws of falling bodies ($g = 32$).
(*b*) That the horizontal velocity remains unaltered.
(*c*) Neglect air resistance.

Find, by any method, the resultant velocity of the ball $1\frac{1}{2}$ sec. after it is hit. (*G.S.*)

9. A ball is held at a height of $2\frac{1}{4}$ ft. above the floor of a carriage which is travelling at 15 m.p.h. along a horizontal track. Find the vertical and horizontal components of the velocity of the ball when it strikes the floor (relative to the track), and calculate the angle made by the direction of the resultant velocity of the ball with the horizontal.

10. A stone is thrown in a horizontal direction with a velocity of 50 ft./sec. from the edge of a vertical cliff 80 ft. high. Determine the time which elapses before it reaches the ground and the distance of the point of impact from the point of projection. (*L.M.*)

11. An aeroplane travelling at a height of 1,600 yd. above the ground, at a uniform speed of 100 m.p.h., drops a bomb. Find (*a*) the time taken for the bomb to reach the ground, (*b*) the horizontal distance between the vertical line through the point at which the bomb was released and the point at which it struck the ground, (*c*) the magnitude and direction of the velocity of the bomb on striking the ground. (Neglect air resistance.) (*L.M.*)

12. A stone is projected horizontally from the top of a cliff with a velocity of 10 ft./sec. and strikes the water 5·3 sec. afterwards. Draw a space-time graph showing the position of the body after each second of its motion. Determine the height of the cliff and the distance from the foot of the cliff to the point where the stone strikes the water. (*L.M.*)

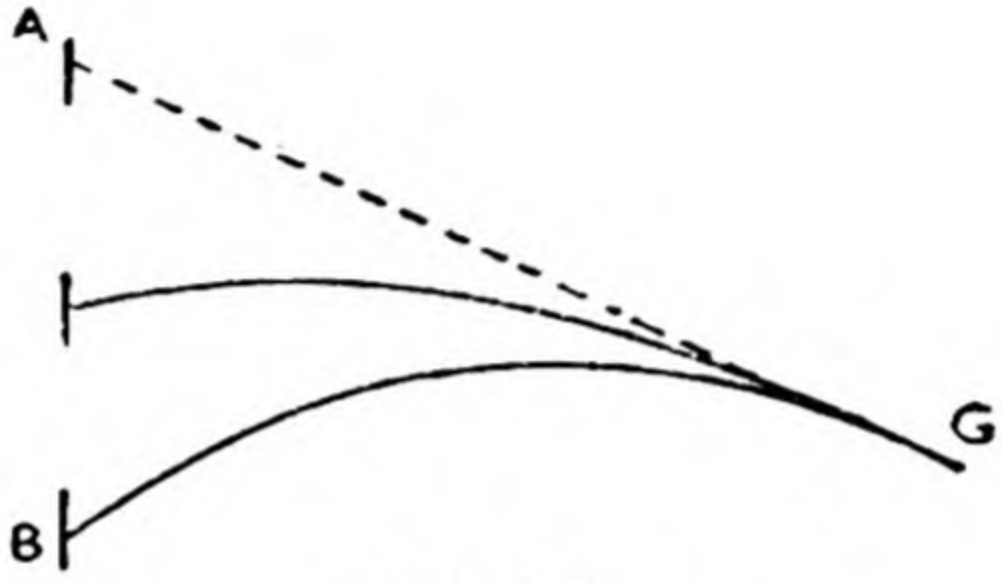


FIG. 344.

180. EXPERIMENT.—If a body is projected in a direction inclined to the horizontal, we can show by the following experiment that the effect of gravity upon the motion of the body is to give to it a vertical displacement equal to that of a body falling vertically from rest for the same time.

A target in the form of a square is supported by a small electromagnet. A spring gun is directed so that a projectile leaving the gun would strike the target but for the effect of gravity. At the instant the gun is fired the circuit of the electromagnet is broken and the target is released. With a little care it will be found that the projectile strikes the target whatever the velocity of projection may be.

Referring to the figure, but for the effect of gravity the path of the projectile would be along GA, now the target has fallen freely from rest, the height AB, and since the target is struck by the projectile the projectile receives the *additional* downward displacement AB, due to the effect of gravity. Hence the effect of gravity is to give an additional vertical displacement to the projectile equal to the displacement it would acquire if it fell freely for the same time from rest.

181. Oblique Projection.

Suppose that a body is projected with a velocity v ft./sec., as shown in the figure, from a point A. We can find the position

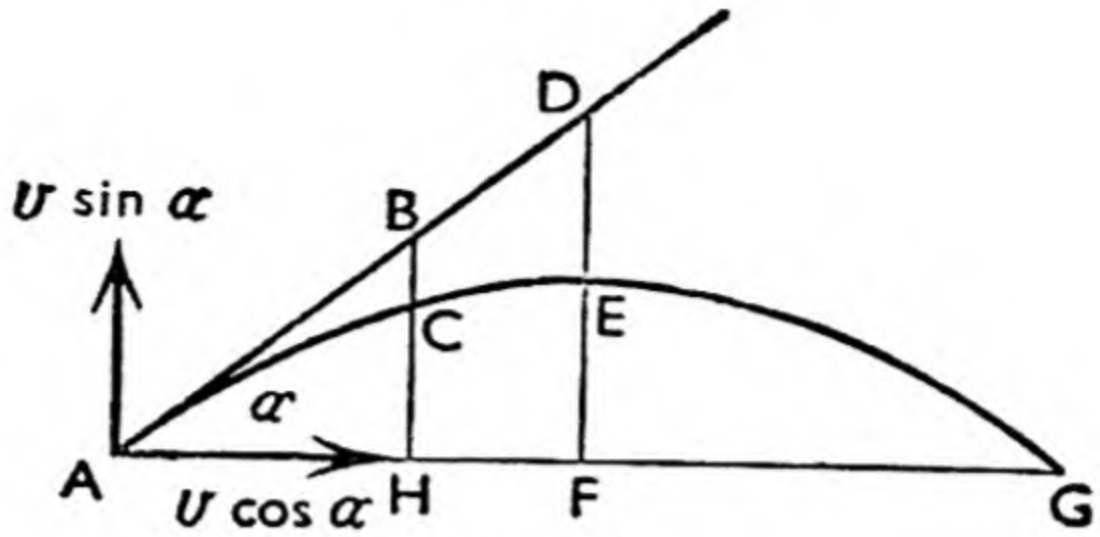


FIG. 345.

of the body at any time by making use of the result of the above experiment.

In t sec. but for the effect of gravity the body would be at B, such that

$$AB = vt.$$

The effect of gravity is to add a vertical displacement of $\frac{1}{2}gt^2$ to the body, thus bringing it into the position C where

$$BC = \tfrac{1}{2}gt^2.$$

In particular, if T is the time required for the body to reach its greatest height (E in the figure) the vertical distance $DE = \frac{1}{2}gT^2$.

Now the body would take T sec. to fall from the position E in which it is travelling horizontally to the ground, and therefore its height above the ground is also $\frac{1}{2}gT^2$.

Hence $$DE = EF.$$

182. To Find the Position of the Projectile after t Sec.

It is convenient to regard the velocity of a projectile at any instant as made up of two components, the one a constant horizontal velocity and the other a variable vertical velocity. At the moment of projection these components, as shown in the figure, are $v \cos \alpha$ and $v \sin \alpha$.

In t sec. the vertical component alone causes the displacement BH which is equal to $v \sin \alpha \,.\, t$, but the effect of gravity

reduces this displacement by an amount BC, which we have seen to be equal to $\frac{1}{2}gt^2$.

Thus $$CH = v \sin \alpha \, . \, t - \tfrac{1}{2}gt^2,$$

and $$AH = v \cos \alpha \, . \, t.$$

183. To Find the Velocity and Direction of Motion after *t* Sec.

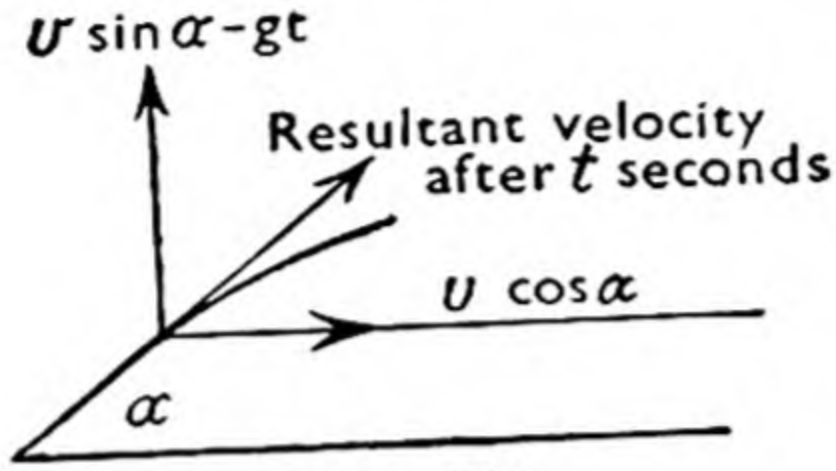


FIG. 346.

In a vertical direction the initial velocity is v sin a ft./sec. and the acceleration measured in a vertically upward direction is $-g$ ft./sec.^{2}. Hence in t sec. the vertical component of velocity is

$$v \sin \alpha - gt.$$

The horizontal component is

$$v \cos \alpha.$$

By combining these two components of velocity their resultant velocity is obtained, and this gives the velocity of the projectile and its direction of motion after t sec.

184. Example (1).—*A stone is thrown with velocity* 100 *ft./sec. at an angle of* 30° *to the horizontal. Find the greatest height reached and the horizontal range on a plane through the point of projection.*

Initial component of velocity vertically upwards $= 100 \sin 30° = 50$ ft./sec.

Let t sec. be the time to reach maximum height.

After t sec. the vertical component of velocity $= 50 - gt$ ft./sec.

When the projectile reaches maximum height,

$$50 - gt = 0,$$

$$t = \frac{50}{32} = 1\tfrac{9}{16} \text{ sec.}$$

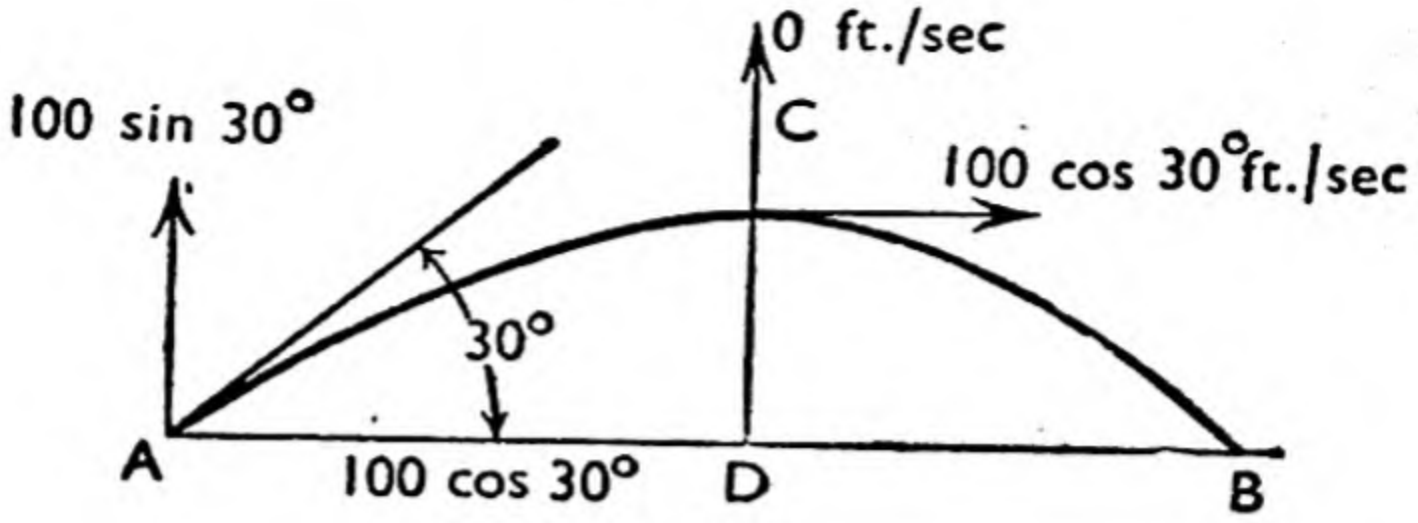


FIG. 347.

Maximum height reached,

$$CD = \tfrac{1}{2}gt^2 = 16 \cdot \left(\frac{25}{16}\right)^2 = 39{\cdot}1 \text{ ft.}$$

Horizontal component of velocity $= 100 \cdot \cos 30° = \frac{100\sqrt{3}}{2}$ ft./sec.

Horizontal range AB = Horizontal component of velocity × Total time of flight

$$= \frac{100\sqrt{3}}{2} \times \frac{25}{16} \times 2 = 271 \text{ ft.}$$

Example (2).—*A body is projected at an angle of 60° to the horizontal with a velocity of 120 ft./sec. Find its velocity in magnitude and direction after 2 sec.*

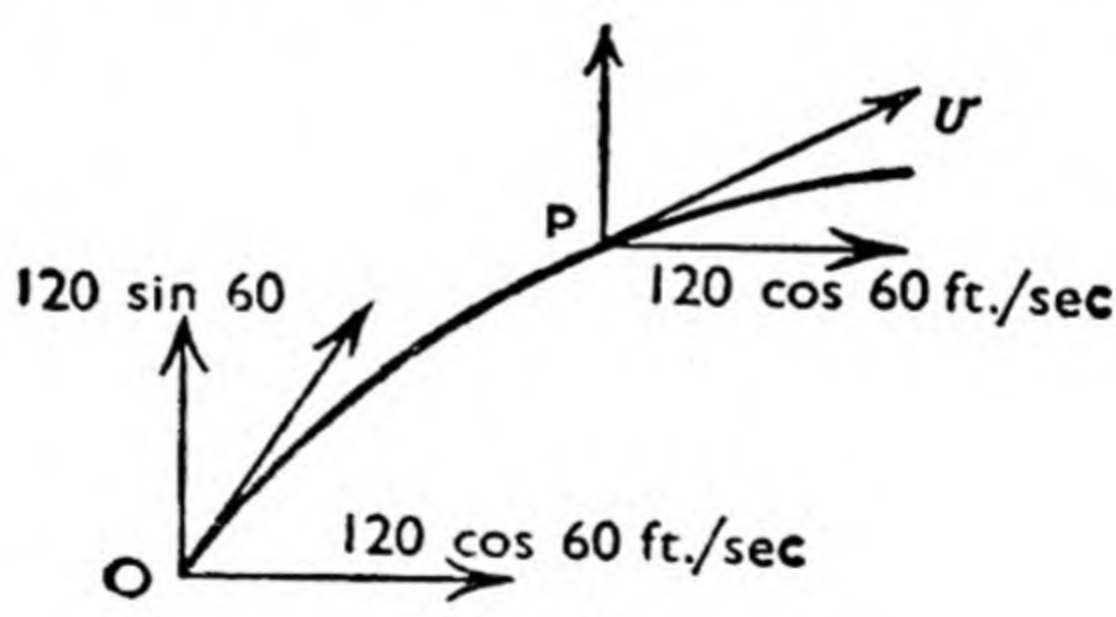


FIG. 348.

At O, the point of projection, the vertical component of velocity is 120 sin 60° ft./sec., and the horizontal component is 120 cos 60°. After 2 sec., when the body is at P, the vertical component of velocity is (120 sin 60° − 2 . g) ft./sec., *i.e.*,

$$(120 \times 0{\cdot}8660 - 64) \text{ ft./sec. } (g = 32) = 39{\cdot}92 \text{ ft./sec.}$$

At P the horizontal component of velocity is 120 cos 60° = 60 ft./sec. The magnitude, v ft./sec., of the resultant velocity is given by

$$v = \sqrt{39{\cdot}92^2 + 60^2} = 72 \text{ ft./sec.}$$

Let α be the angle made by the resultant velocity at P with the horizontal,

$$\tan\alpha = \frac{39\cdot92}{60} = 0\cdot6653,$$

$$\alpha = 33^\circ\ 38'.$$

Example (3).—*A golf ball is struck from the ground and pitches 50 yd. farther on after rising to a maximum height of 12 ft. Find in feet per second the velocity with which it leaves the ground.* (*O.C.*)

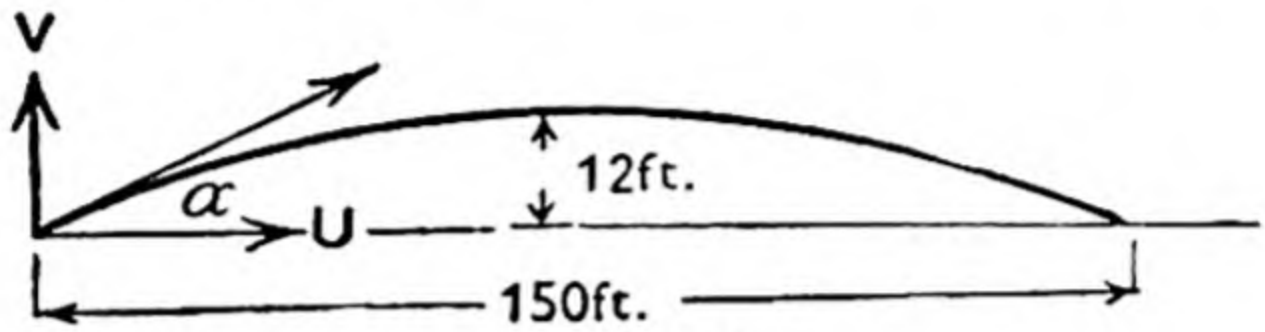


FIG. 349.

Let t sec. be the time taken by the ball to reach its maximum height from the instant of projection.

Considering the motion in a vertical direction, the ball falls vertically 12 ft. in t sec., its initial velocity vertically downwards being zero, and its acceleration is g ft./sec.^{2}

Therefore
$$12 = \tfrac{1}{2}gt^2,$$
$$12 = 16t^2, \quad (g = 32)$$
$$t = \sqrt{\frac{12}{16}} = \frac{\sqrt{3}}{2} \text{ sec.}$$

If V ft./sec. be the vertical component of the velocity of projection, then

$$V = g \cdot \frac{\sqrt{3}}{2} = 16\sqrt{3} \text{ ft./sec.}$$

If U ft./sec. is the constant horizontal component of velocity

$$U \times \frac{\sqrt{3}}{2} \times 2 = 150,$$

$$U = \frac{150}{\sqrt{3}} = 50 \cdot \sqrt{3} \text{ ft./sec.}$$

Hence the initial velocity of projection is

$$\sqrt{V^2 + U^2} = \sqrt{(256 \times 3 + 2500 \times 3)} = 90\cdot9 \text{ ft./sec.}$$

If the direction of projection makes α° with the horizontal,

$$\tan\alpha = \frac{16\sqrt{3}}{50\sqrt{3}} = 0\cdot32,$$

$$\alpha = 17^\circ\ 45'.$$

Example (4).—*A stone is thrown with velocity* 80 *ft./sec. from the top of a vertical cliff* 200 *ft. high and at an angle of* 60° *to the horizontal plane through the point of projection. Find where and at what angle the stone strikes the horizontal plane through the base of the cliff.*

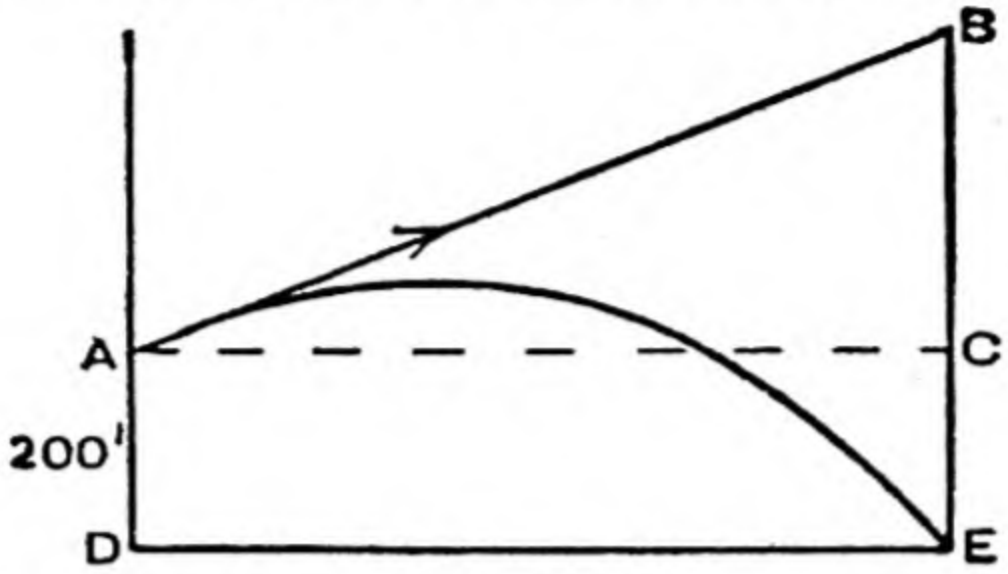


FIG. 350.

Let t sec. be the total time of flight.
Initial vertical component of velocity $= 80 \sin 60° = 40\sqrt{3}$ ft./sec.
Referring to the figure,

$$CB = 40\sqrt{3} \cdot t = 69{\cdot}28\ t.$$

The effect of gravity causes the vertical displacement BE, and

$$BE = \tfrac{1}{2}gt^2.$$

Hence, since $EC = EB - CB$,

$$200 = \tfrac{1}{2}gt^2 - 69{\cdot}28t,$$
$$16t^2 - 69{\cdot}28t - 200 = 0,$$
$$2t^2 - 8{\cdot}66t - 25 = 0,$$
$$t = 6{\cdot}31.$$

(The negative value of t is omitted as impossible.)

The horizontal distance travelled in 6·31 sec. is equal to the horizontal component of velocity $\times 6{\cdot}31 = 80 \cos 60° \times 6{\cdot}31 = 252$ ft.

When the stone strikes the horizontal plane its vertical component of velocity $= 69{\cdot}28 - 32 \times 6{\cdot}31 = -132{\cdot}6$ ft./sec.

The horizontal component of velocity $= 40$ ft./sec.

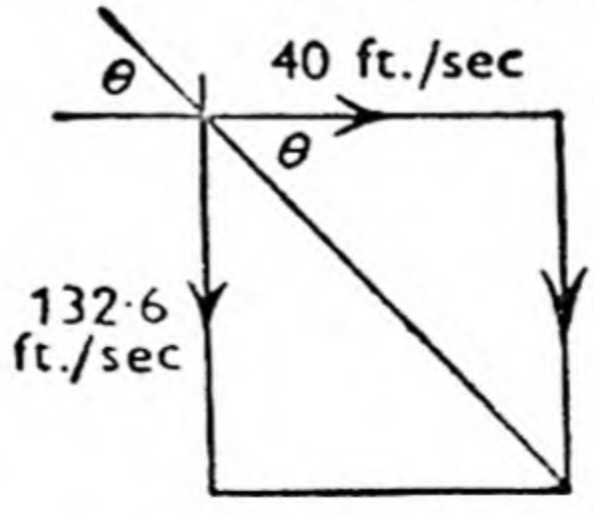


FIG. 351.

If θ is the angle made by the stone with the horizontal,

$$\tan\theta = \frac{132{\cdot}6}{40} = 3{\cdot}315,$$

and

$$\theta = 73^\circ\ 13'.$$

185. Formula for Range on a Horizontal Plane.

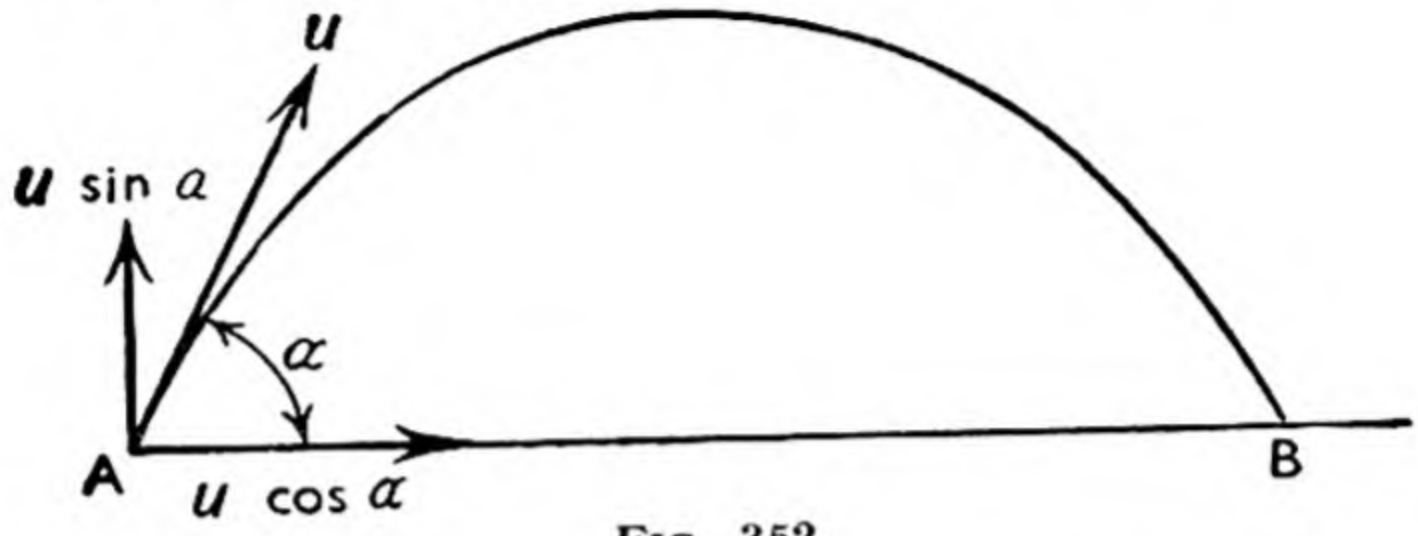


FIG. 352.

Suppose that a body is projected with velocity u ft./sec. in a direction making an angle α to the horizontal plane through the point of projection.

The vertical component of velocity $= u \sin\alpha$.

" " " " after t sec. $= u\sin\alpha - gt$.

At the point where the projectile is at its maximum height,

$$u\sin\alpha - gt = 0,$$

$$t = \frac{u\sin\alpha}{g}.$$

The total time of flight of the body is therefore $\frac{2u\sin\alpha}{g}$.

The horizontal distance travelled in this time is found by multiplying the horizontal component of velocity by the total time of flight.

Horizontal range $AB = u\cos\alpha\ \frac{2u\sin\alpha}{g} = \frac{2u^2\sin\alpha\cos\alpha}{g}$

$$= \frac{u^2\sin 2\alpha}{g}.$$

If we keep the speed of projection constant, the range has

its greatest value when the inclination to the horizontal gives to sin 2α its greatest value, *i.e.*, when $2\alpha = 90°$.

In this case, sin $2\alpha = 1$, and the maximum range for the given speed of projection is $\frac{u^2}{g}$, the angle of projection α being 45°.

EXAMPLES XLIX

1. A stone is projected with horizontal and vertical components of velocity 40 and 32 ft./sec. respectively. Find the maximum height reached by the stone, the time taken to reach this height and the distance travelled by the stone horizontally when it is again at the same horizontal level as the point of projection.

2. A stone is projected with a velocity of 100 ft./sec. at an angle of 30° to the horizontal. Find the range on a horizontal plane through the point of projection.

3. A shot is fired at an angle of 45° to the horizontal with a muzzle velocity of 1,000 ft./sec. How high is it after 10 sec. ? What are then its vertical and horizontal components of velocity ?

4. A ball is thrown with vertical and horizontal components of velocity 30 and 40 ft./sec. respectively. How long does it take before it strikes the vertical wall of a building 60 ft. away ? At what height above the point of projection does it strike the wall ?

5. A ball is projected so as just to pass over a roof top 64 ft. high and at a horizontal distance of 128 ft. Find the magnitude and direction of the velocity of projection.

6. Deduce a formula connecting the speed of a body moving vertically upwards and the height to which it will rise.

A ball is thrown at an angle of 45° to the horizontal so that at the top of its flight it enters a window 36 ft. above the thrower. Find the speed at which it is thrown and the distance of the wall containing the window from the thrower. (*G.S.*)

7. State the theorem of the composition of velocities.

An aeroplane travelling horizontally at a height of 1,600 ft. and at a speed of 90 m.p.h. drops a projectile when passing vertically over a certain spot. Find how far from the spot the projectile reaches the ground and also its velocity of impact. (*C.*)

8. A projectile has a muzzle velocity of 1,650 ft./sec. Find the range in yards on a horizontal plane if the angle of projection is 15°. Find the error at this range due to an error of 15′ in the angle of elevation of the gun. (*O.C.*)

9. A golf ball is projected at an angle of 45° with the vertical, and it travels a distance of 200 yd., measured horizontally, before striking the ground. Plot a graph to show how the horizontal and vertical components of velocity of the ball alter with the time, and from this graph find the greatest height to which the ball rises. (Neglect the action of the air on the ball.) (*G.S.*)

10. A small object slides down a smooth roof, inclined to the horizontal at an angle of 30°, through a distance of 8 ft. and strikes the ground 24 ft. below the point at which it leaves the roof. Find the horizontal distance

that it has described between leaving the roof and striking the ground. [Note : Acceleration of object down slope is $g \sin 30^\circ$ ft./sec.^{2}] (*O.C.*)

11. A golf ball is struck with a horizontal velocity three times its vertical velocity and pitches on a green on the same level as the tee and 225 yd. distant. Find its initial velocity, its time of flight and the height to which it rises (air resistance to be neglected).

12. A boy stands 55 ft. away from the vertical wall of a house. He throws a ball with a velocity of 60 ft./sec. from a height of 6 ft. from the ground. If the ball is projected at an angle of 60° with the horizontal, find the height at which it strikes the house. Show that the ball reaches its maximum height before striking the house. (*O.C.*)

13. A batsman skies a ball and is caught by a fieldsman standing 20 yd. from him, who has to wait 5 sec. for the ball to reach him. Find the initial velocity of the ball and the maximum height. (Assume that the bat and the fieldsman's hands are in the same horizontal plane.) (*O.C.*)

14. A bowler delivers a ball horizontally at 96 ft./sec., his hand being 6 ft. 3 in. above the ground. How far from him does the ball pitch and what is its time of flight ? If another ball takes twice as long to pitch in the same spot, how high must it rise in its flight ? (*O.C.*)

15. Draw a graph of the path of a cricket ball thrown with a velocity of 80 ft./sec. at an angle of 45° with the horizontal, assuming air resistance to be negligible. (Scale, 30 ft. to the inch.) (*O.C.*)

16. A boy stands 15 ft. away from a house and throws a ball so that it just reaches the height of a window-sill 25 ft. above him and falls into the room. With what velocity and at what angle of elevation must he throw the ball ? (*O.C.*)

17. A football is to be kicked so as just to pass over a bar 12 ft. high and 60 ft. away. If the ball is kicked with a velocity of 60 ft./sec., form two equations connecting α, the angle of projection, and t, the number of seconds between foot and bar. Eliminate t and prove that $15 \tan \alpha - 4 \sec^2 \alpha = 3$. Deduce the quadratic for $\tan \alpha$, and find the two possible angles of projection. (*O.C.*)

18. At a range of 100 yd. *in vacuo* a bullet fired horizontally strikes the target 3 in. below the point at which its initial direction produced would cut the target. Find the initial velocity of the bullet. (*O.C.*)

19. A golf ball is struck so as to start its motion with a vertical velocity one-third of its horizontal velocity and to pitch on a green on the same level as the tee or starting point and 144 yd. away. What height does it reach when its horizontal distance from the tee is 40 yd. ? (*O.C.*)

20. A cricket ball is thrown at an angle of 45° with the ground and pitches 100 yd. from the thrower, whose height above the ground (supposed horizontal) may be disregarded. With what velocity must the ball be thrown and how high will it rise in the air ? Would a different elevation give a better result ? (*O.C.*)

CHAPTER XX

RELATIVE VELOCITY

186. Relative Velocity.

Suppose that two bodies are moving in the same straight line in the same direction with equal speeds. Each body under these circumstances is said to be at rest relative to the other.

If the bodies are moving with equal *velocities* along parallel lines, each body is again said to be at rest relative to the other.

In each case the *displacement* of one body relative to the other is constant, *i.e.*, the distance from one body to the other remains constant in magnitude, and the bearing of one body from the other also remains constant.

If the displacement of one body relative to another is changing in magnitude, direction, or both, then each body is said to have a velocity relative to the other.

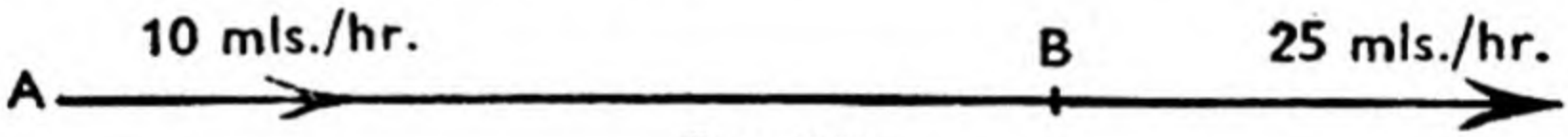


FIG. 353.

Let A and B be two bodies moving in the same straight line.

For convenience let AB be the easterly direction.

If both bodies are moving in the easterly direction, A at 10 m.p.h. and B at 25 m.p.h., the velocity of B relative to A is the displacement of B relative to A in unit time, *i.e.*, 15 m.p.h. in an easterly direction.

The velocity of A relative to B is 15 m.p.h. in a westerly direction.

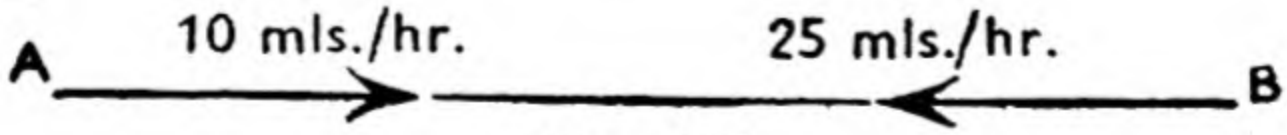


FIG. 354.

If the velocity of B is reversed in direction the velocity of

B relative to A is 35 m.p.h. in a westerly direction, and of A relative to B, 35 m.p.h. in an easterly direction.

187. Let us now suppose that the two bodies are not moving in the same straight line.

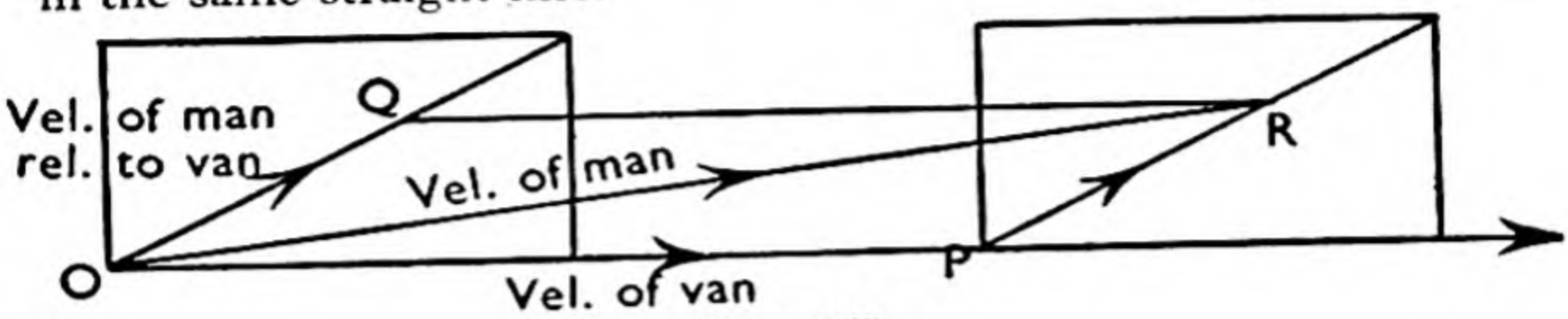


FIG. 355.

Consider a railway van which is travelling with uniform velocity represented by OP. The positions of the van at the beginning and end of 1 sec. are shown in the figure.

At the beginning of the second a man, situated at O, begins to walk diagonally across the carriage to the opposite corner. OQ represents the velocity of the man *relative to the carriage*.

The resultant velocity of the man is obtained by compounding the velocities of the carriage and of the man relative to the carriage, *i.e.*, by compounding OP and OQ, thus giving the velocity represented by OR.

The result may be summed up in the form of an equation,

Velocity of man = Velocity of carriage + Velocity of man relative to carriage.

In the same manner if OP represents the velocity of a stream and OQ the velocity with which a boat is rowed, or the velocity of the boat *relative to the stream*, then OR represents the resultant velocity of the boat. In this case,

Velocity of boat = Velocity of stream + Velocity of boat relative to stream.

If OP represents the velocity of the wind and OR the resultant velocity (or, as it is sometimes called, the "*true*" velocity *) of an aeroplane, what is represented by OQ ?

We may suppose that in the above cases OP represents the velocity of an observer, who remains in the corner of the van ;

* True velocity may be taken to mean velocity relative to the earth.

who floats with the stream; who floats in a balloon with the wind; then in each case OQ or PR represents the velocity of the observed body *relative to the observer*.

In each case, to find the velocity of the observed body we must compound with the velocity of the observer the velocity of the observed body relative to the observer, *i.e.*,

Velocity of observed body = Velocity of observer + Velocity of observed body relative to the observer.

The diagonal of this parallelogram of velocities always represents the velocity of the observed body or the "true" velocity of one of the bodies.

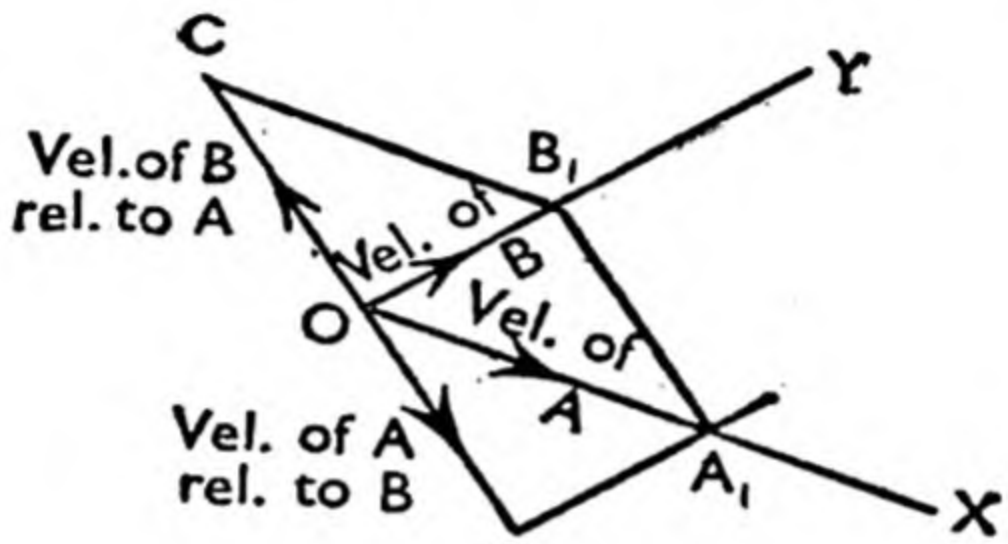


FIG. 356.

188. Let OX and OY represent two straight roads.

Two men A and B start from O together. In unit time A reaches A_1 and B, B_1.

OA_1 represents the velocity of A,
OB_1 " " " B.

Relative to A, B has received the displacement A_1B_1 in unit time.

Hence A_1B_1 represents the velocity of B relative to A. If the parallelogram OA_1B_1C is completed, OC represents the velocity of B relative to A.

Thus, exactly as in the previous cases, the diagonal OB_1 represents the resultant velocity of B, which is compounded of the velocity of A and the velocity of B relative to A.

Velocity of B = Velocity of A + Velocity of B relative to A.

In precisely the same manner the parallelogram $OB_1A_1C_1$ may be used to give the velocity of A relative to B. This velocity is simply the reverse of the velocity of B relative to A, and is represented by B_1A_1 or by OC_1.

The velocity of a body B may be regarded as the resultant of the velocity of any other body O and the velocity of B relative to O.

The velocity of a body B relative to any other body O may be regarded as the velocity of B, as seen by an observer travelling with O.

189. Example (1).—*To a car travelling due east at 30 m.p.h. an aeroplane appears to be travelling with a velocity of 60 m.p.h. in a northerly direction. What is the true velocity of the plane?*

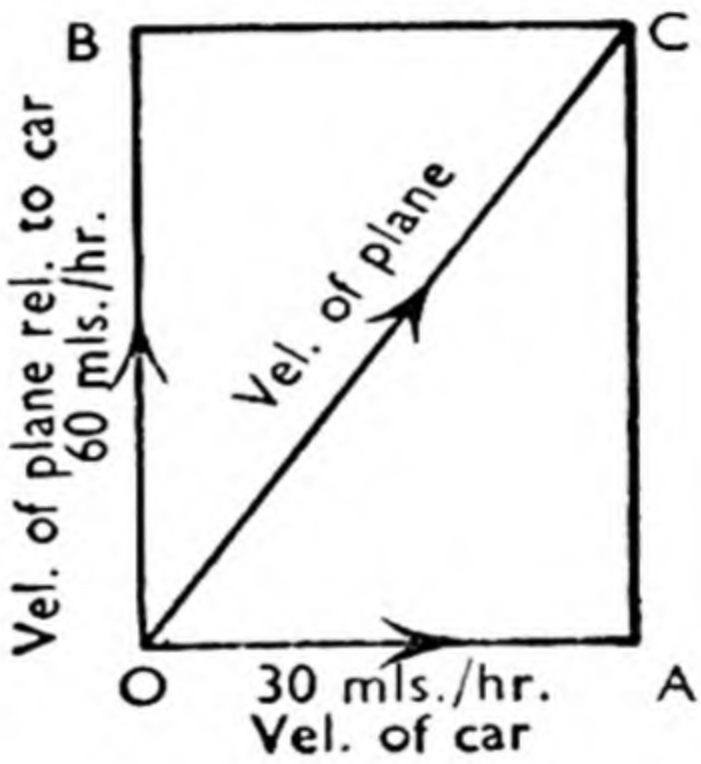


FIG. 357.

The observed body is the aeroplane.

Velocity of plane = Velocity of car + Velocity of plane relative to car.

Draw OA to represent the velocity of the car, OB to represent the velocity of the plane relative to the car. Complete the parallelogram OACB.

OC represents the velocity of the plane.

$$\tan COA = \frac{CA}{OA} = 2. \qquad \angle COA = 63° 26'.$$

$$OC = \sqrt{(30^2 + 60^2)} = 67{\cdot}1. \qquad \text{(Right-angled } \triangle OAC.)$$

Velocity of plane = 67·1 m.p.h. in a direction making an angle of 63° 26′ with the direction of the car.

Example (2).—*Two cyclists set off from the same place along two roads, one due south at 12 m.p.h. and the other south-east at 15 m.p.h. What is the velocity of the first cyclist relative to the second?*

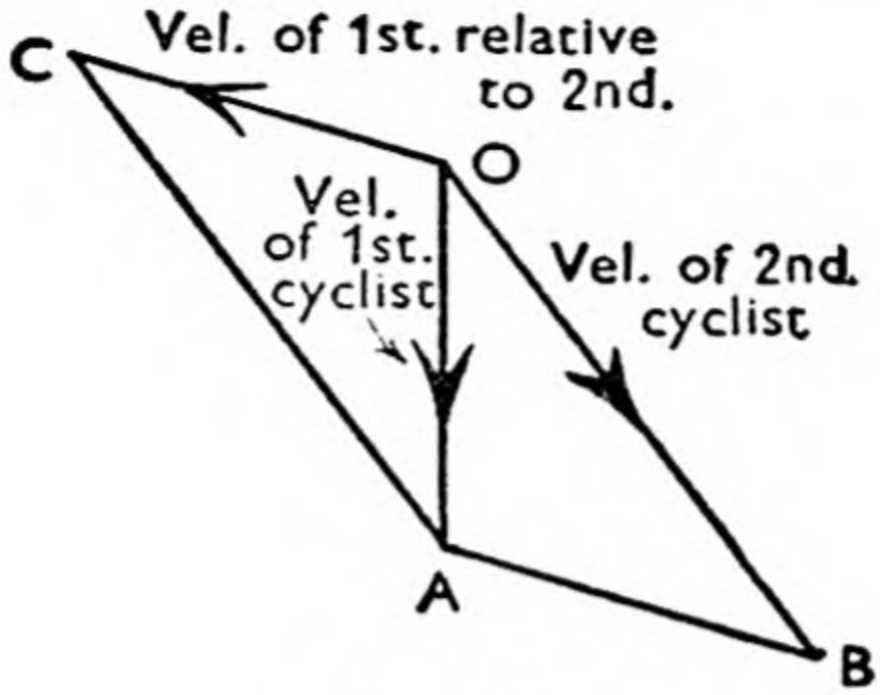


FIG. 358.

The velocity of the first cyclist A may be compounded of the velocity of the second cyclist, and the velocity of the first cyclist relative to the second.

Draw OA to represent the velocity of the first cyclist (and incidentally the diagonal of the parallelogram of velocities) 12 m.p.h. due south.

Draw OB to represent the velocity of the second cyclist, 15 m.p.h. south-east.

Complete the parallelogram OBAC, having OA as the diagonal.

OC represents the velocity which must be compounded with that of the second cyclist to give the velocity of the first, *i.e.*, OC represents the velocity of the first body relative to the second.

By calculation,

$$AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cos AOB$$
$$= 12^2 + 15^2 - 2 \times 12 \times 15 \times \cos 45°.$$
$$AB = 10{\cdot}7.$$

By the sine rule,

$$\frac{AB}{\sin AOB} = \frac{OB}{\sin OAB},$$

and

$$\sin OAB = \frac{OB \sin AOB}{AB} = \frac{15 \times 0{\cdot}7071}{10{\cdot}63} = 0{\cdot}9919.$$

$$\angle OAB = 97° 18'.$$

The velocity of the first cyclist relative to the second is 10·7 m.p.h. in a direction making 97° 18′ with the direction of the first cyclist.

190. Change in Velocity.

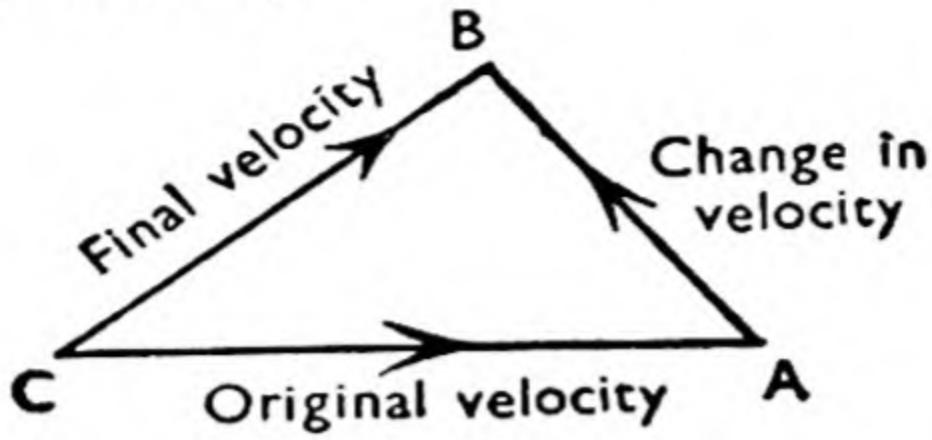


FIG. 359.

Suppose that CA represents the velocity of a cricket ball before it is struck by a bat and CB the velocity of the ball after striking the bat.

Then AB represents the change in velocity which the ball has undergone. (Incidentally the blow imparted by the bat to the ball is delivered in the direction AB.) The triangle of velocities which has been drawn above illustrates the statement,

Original velocity + Change in velocity = Final velocity.

191.

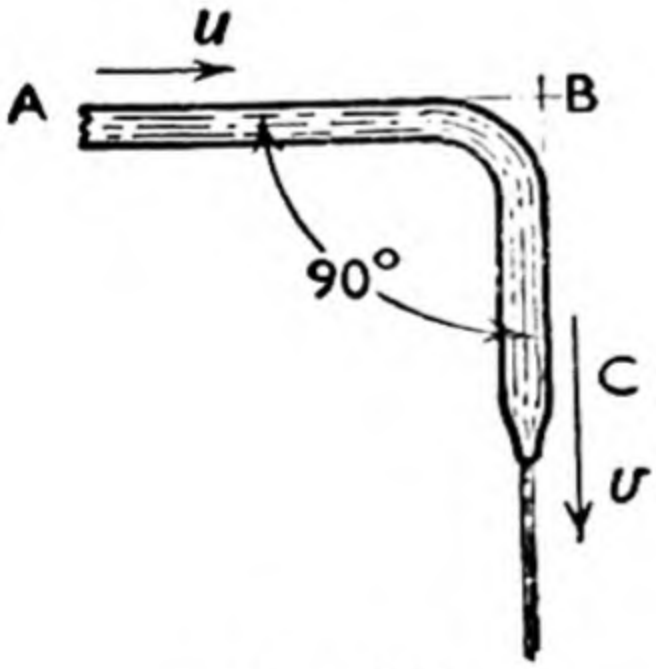


FIG. 360.

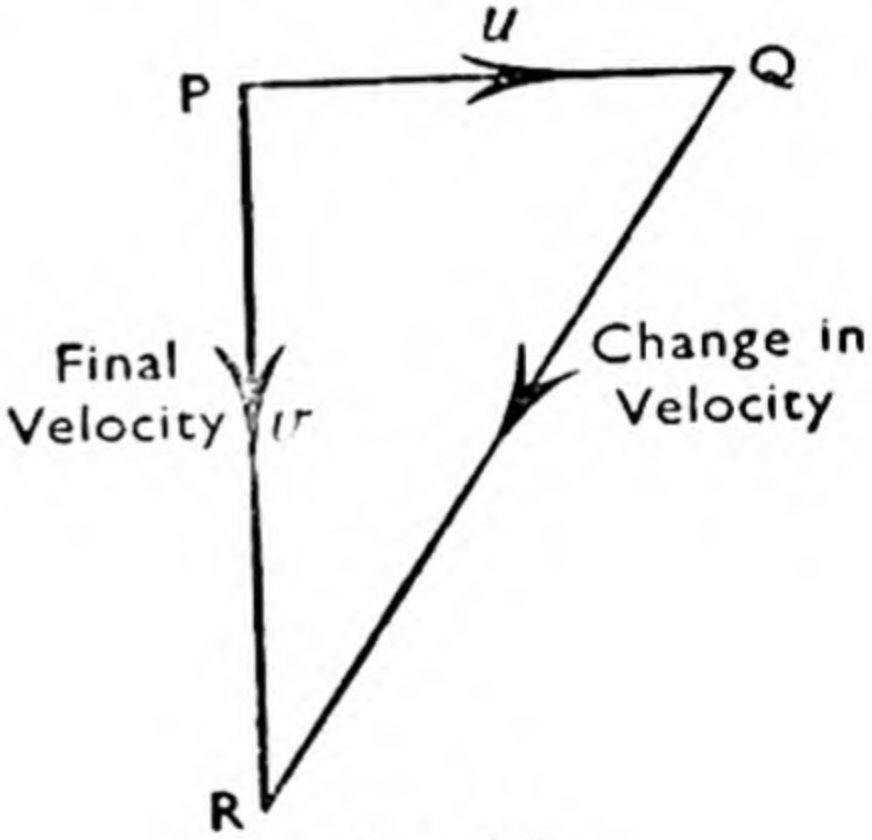


FIG. 361.

Suppose that water flows along a pipe which has a right-angle bend in it and finally issues from a nozzle at the end of the pipe

Let u be the velocity of the water in the direction AB and v the velocity with which the water issues from the nozzle in the direction BC.

To find the change in velocity undergone by the water we may proceed as follows :

Draw PQ to represent the initial velocity u and PR to represent the final velocity v, then QR represents the change in velocity and is $\sqrt{u^2+v^2}$ in a direction making $\tan^{-1}\frac{u}{v}$ with the direction PR.

EXAMPLES L

Graphical Solutions are Satisfactory

1. To an observer in a train travelling due east at 40 m.p.h. an aeroplane appears to be travelling due north at 75 m.p.h. Find the true course and speed of the aeroplane. (*O.C.*)

2. A motor boat runs at a uniform speed across a river $\frac{1}{2}$ mile wide in 5 min. if the tide is running down steadily at 2 m.p.h. Show in a diagram (drawn to scale) the direction in which the boat must be steered if its actual path is at right angles to the direction of the tide. Find the speed of the boat independent of the tide, expressing your answer in miles per hour. (*G.S.*)

3. A destroyer whose speed is 20 m.p.h. (independent of the tide) has to reach a point B, which is 5 miles distant, in a direction 30° north of east from its original position A. If the tide is running from north to south at 4 m.p.h., show in a diagram (drawn to scale) the direction in which the boat must be steered. Find also the time occupied by the journey. (*G.S.*)

4. Describe how to find the velocity of a body A relative to another body B which is moving in a direction inclined to that of A's motion.

The flow of a river is 3 m.p.h. and a boat A is drifting with the stream. Another boat B is headed at an angle of 45° downstream with a velocity of 10 m.p.h., but its actual velocity and course are affected by the stream. Determine the velocity of B relative to A. What would be the velocity of B relative to A if A were fixed ? (*G.S.*)

5. Explain the meaning of uniform velocity, variable velocity and relative velocity.

An aeroplane is headed 30° north of east against a wind from the east, and it appears to a stationary observer on the earth to be travelling at a velocity of 60 m.p.h. due north. Determine, graphically or otherwise, the velocity of the wind and the velocity of the aeroplane in still air. (*G.S.*)

6. State and illustrate the parallelogram of velocities. If a body has two velocities in different directions, explain how the resultant velocity may be found graphically.

To a person on a boat steaming due east at 10 m.p.h. the clouds appear to be moving in a direction 30° west of north at 15 m.p.h., find the true direction of the wind. (*L.M.*)

7. A train is travelling due north at 60 m.p.h., and to a passenger in it an aeroplane appears to be travelling due east. The true direction of the plane is 30° north of east. What is the speed of the plane and its apparent speed to an occupant of the train ?

If there is wind of 30 m.p.h. from the south, what would be the speed of the plane in still air ? (*L.M.*)

8. What is meant by relative velocity and resultant velocity ?

A boat is steaming at 8 m.p.h. due east, and the smoke leaving the funnel moves at 4 m.p.h. in a direction north-east, what is the velocity of the wind ? What will be the apparent speed and direction of the steamer to a passenger in another ship going due north at 10 m.p.h. ? (*L.M.*)

9. Two ships A and B are steaming at the same speed due north and due east respectively. Explain, with the help of a diagram, how you would find the apparent speed and direction of B to an observer on A.

To an observer on a ship A another ship C appears to be steaming at 20 m.p.h. in a direction 60° west of north. If A itself is steaming due north at 10 m.p.h., what is the direction and speed of C ?

10. Explain the meaning of the terms resultant velocity and relative velocity.

A train A travels due north at 30 m.p.h., and a train B due west at 40 m.p.h. What is the velocity of A relative to B ? If an aeroplane were to fly with this velocity, find, graphically or otherwise, the velocity of each train relative to the aeroplane. (*L.M.*)

11. A man in an airship flying due east at 30 m.p.h. observes a motor boat which is speeding across a lake in a northerly direction at 40 m.p.h. What is the apparent direction of the speed of the boat ?

If a wind of 10 m.p.h. blows from the north, what must be the direction and speed of an airship for the boat to have the same apparent motion ? (*L.M.*)

12. What do you understand by the terms resultant velocity and relative velocity ?

Small raindrops would fall vertically with a constant velocity of 26·4 ft./sec. if there were no wind. When a wind blows at 6 m.p.h. from the north, in which direction and with what velocity must a man walk in order that the raindrops appear to him to be falling vertically ? If the man were to reverse his direction of walking, what would be the velocity of the raindrops relative to him ? Draw diagrams illustrating each case. (*L.M.*)

13. The length of the seats in a railway carriage is 8 ft., and the width of the carriage is 6 ft., and the train is travelling at 20 m.p.h. If a fly goes in a straight line from one corner to that diagonally opposite to it in 3 sec., what is the velocity of the fly relative to the ground ? Also what is the velocity of the fly relative to the ground on the return journey if this is made in the same time ? Summarise your answers in a diagram. (*L.M.*)

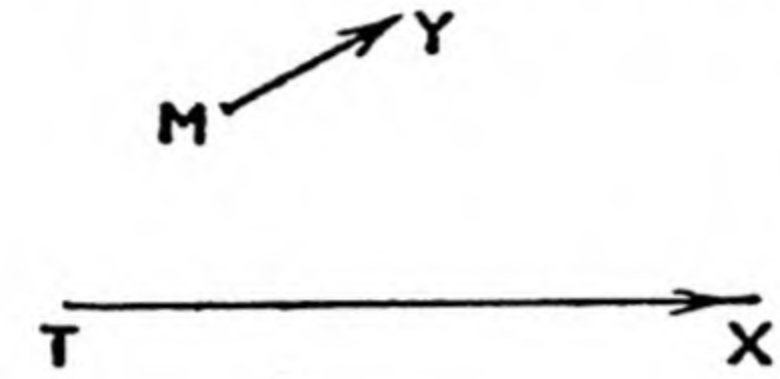
14. An aeroplane can travel at 80 m.p.h. in still air, and the wind is blowing at 30 m.p.h. Compare the time taken from A to B and back again if the direction of the wind all the time is (*a*) from A to B, (*b*) at right angles to AB, (*c*) at 45° to AB. (Resultant velocities in (*b*) and (*c*) may be found by a graphical method.) (*O.C.*)

15. A cricket ball moving along a line AB is struck so that an equal additional velocity is communicated to the ball in a direction making an angle of 60° with BA. Find graphically the magnitude and direction of the resultant velocity of the ball. *(O.C.)*

16. A billiard ball moving with a speed of 6 ft./sec. in a certain direction strikes on another ball and its subsequent velocity is 4 ft./sec. in a direction making 30° with its initial direction of motion. What change in velocity was communicated to the ball at the impact ?

192. The Relative Velocity of Two Bodies Moving in Different Parts of the Same Plane.

It has been stated that if two bodies are travelling with equal *velocities* along parallel paths, each body is at rest relative to the other. All bodies which are at rest relative to one of these bodies



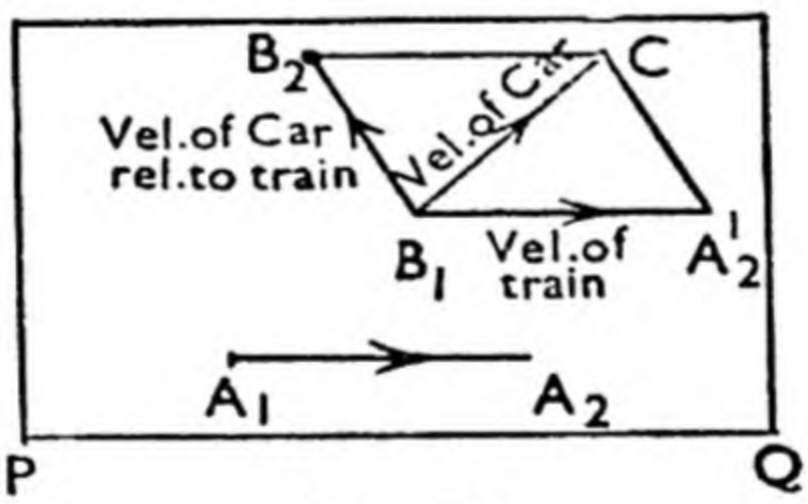


FIG. 362.

are also at rest relative to the other. The relative velocity of a third body to each of the two is also the same.

In the above figure T and M are simultaneous positions of a train and a motor car which are moving along TX and MY with uniform velocities.

An observer in the train marks on a sheet of paper A_1 and B_1 the respective positions of train and car at a certain instant. After unit time the observer again notes the position of the car

and represents it by the point B_2, taking PQ as the direction of the train's motion.

Now B_1B_2 represents the displacement of the car relative to the train in unit time.

Hence B_1B_2 represents the velocity of the car relative to the train exactly as observed by the passenger. What is the " true " velocity of the car ?

Evidently we have to take into account the motion of the observer himself.

To do this, let A_1A_2 represent the velocity of the observer.

In unit time a body which is at B_1 at the same instant as the car, and which remains at rest relative to the train, would move to A'_2, where $B_1A'_2 = A_1A_2$. Hence to find the resultant velocity of the car we must compound the velocity of the car relative to the train with the velocity of the train.

Again we have the same parallelogram of velocities :

Velocity of observed body = Velocity of observer + Velocity of observed body relative to observer.

Velocity of car = Velocity of train + Velocity of car relative to train.

In the figure, B_1C represents the velocity of the car.

It should be noted that one method of solving problems on relative velocity has been described. All problems on relative velocity in Examples LI. may be successfully solved by this one method. If the reader wishes for a practical hint, let him remember that the two so-called true or actual velocities always *diverge* from one point of the parallelogram, and that one of them is invariably the diagonal of the parallelogram.

193. Example (1).—*A steamer* B *is* 20 *sea miles due south of another vessel* A *at a certain time.* A *is steaming due east at* 10 *knots and* B *due north at* 12 *knots. What is the velocity of* B *relative to* A, *the shortest distance apart of the vessels and the time which they take to reach this position ?*

Let A_1 and B_1 represent the initial positions of the two vessels A and B.

Since B is the observed body, the velocity of B is regarded as the resultant of the velocity of A and the velocity of B relative to A.

Velocity of B = Velocity of A + Velocity of B relative to A.

Set off B_1B_2 along B_1A_1 to represent the velocity of B and the diagonal of the parallelogram of velocities.

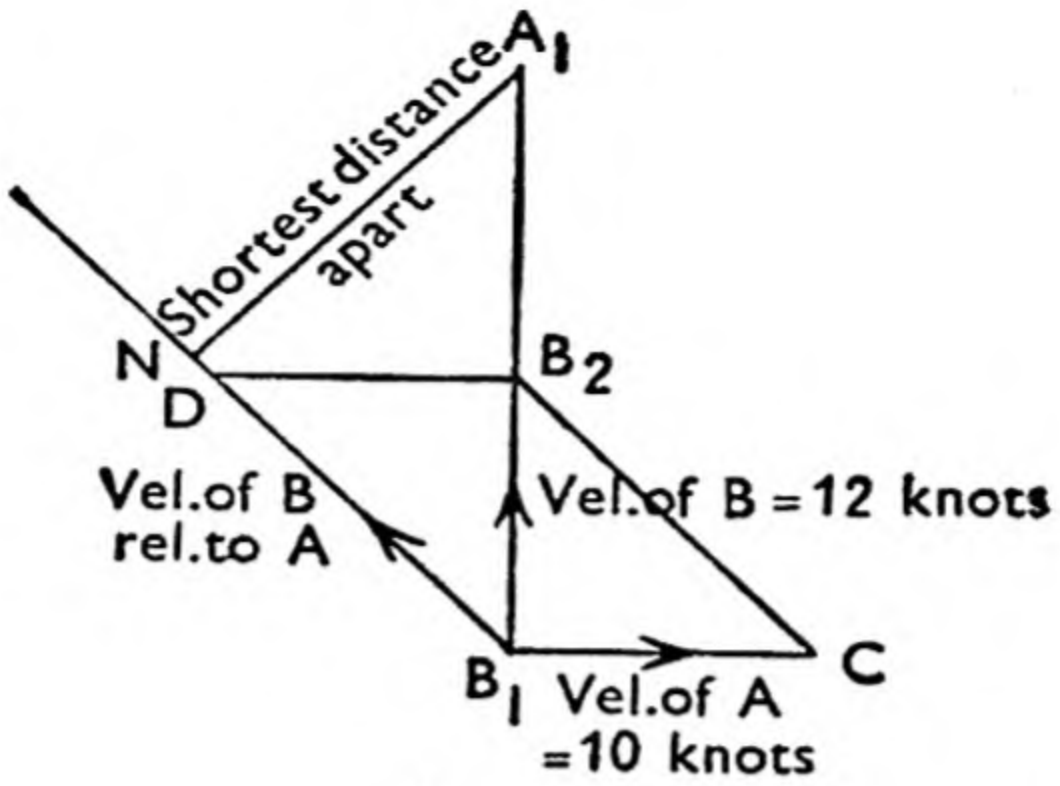


FIG. 363.

Draw B_1C to represent the velocity of A.
Complete the parallelogram B_1CB_2D.
Then B_1D represents the velocity of B relative to A.
The velocity of B relative to A $= \sqrt{(12^2 + 10^2)} = 15{\cdot}6$ knots in a direction making an angle whose tangent ratio is 0·8333 with the line B_1A_1, *i.e.*, 39° 48′ west of north.

If for convenience we regard A as being at rest, the path of B relative to A is along B_1D, and A_1 is the point at which A may be assumed to remain at rest.

The shortest distance apart of the vessels is therefore represented by the perpendicular A_1N drawn from A_1 to the path of B relative to A.

A_1N may be measured on a scale drawing; or by calculation, using the right-angled triangle B_1NA_1,

$$\sin 39° 48' = \frac{NA_1}{A_1B_1},$$

and therefore

$$A_1N = 20 \times 0{\cdot}6401 = 12{\cdot}8.$$

The shortest distance apart is 12·8 sea miles.

The time taken to reach this position is the time taken by B to travel the distance B_1N relative to A, *i.e.*, at the *speed of B relative to A*.

$$\text{Time} = \frac{B_1N}{15{\cdot}6} = \frac{20 \cos 39° 48'}{15{\cdot}6} = 59{\cdot}2 \text{ min.}$$

Note.—It should be remembered that a graphical solution on a suitably large scale has usually the advantage of being much more quickly carried out than a calculated solution and is in no way inferior to it.

Example (2).—*To a man travelling due north at 20 m.p.h. the wind appears to come from east. When he increases his speed to 40 m.p.h. the wind appears to come from the north-east. What is the "true" velocity of the wind?*

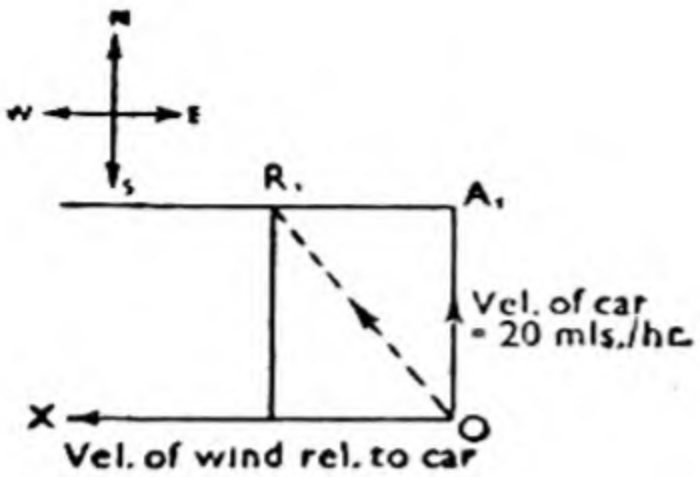


FIG. 364.

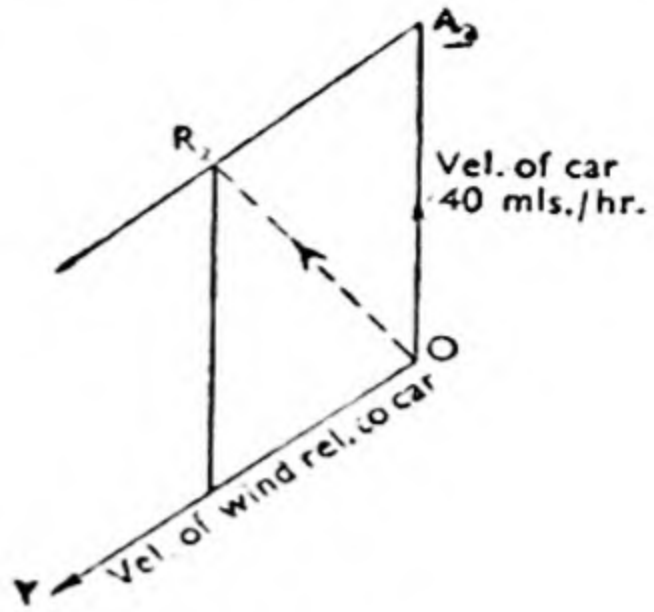


FIG. 365.

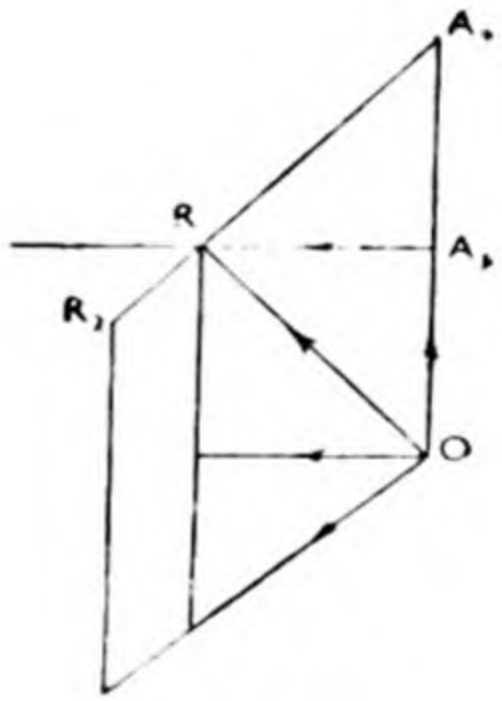


FIG. 366.

The observed body is the wind.

The velocity of the wind may be regarded as made up of two components, the velocity of the car and the velocity of the wind relative to the car.

Let OA_1 represent the velocity of the car 20 m.p.h. due north.

Draw OX in a direction due west.

Velocity of wind = Velocity of car + Velocity of wind relative to car.

Since the magnitude of the velocity of the wind relative to the car is not known, all we can state about this diagram is that the velocity of the wind is represented by OR_1, and R_1 lies along the line through A_1 parallel to OX.

In a similar manner, if OA_2 represents the velocity of 40 m.p.h. due north, and OY is drawn to represent the velocity of the wind relative to

the car, then OR_2 represents the resultant velocity of the wind, and R_2 lies on a line drawn through A_2 parallel to OY.

If these two diagrams are combined in one, then OR_1 and OR_2 cut at R, and since this is the one point common to the lines A_1R_1 and A_2R_2, OR represents the only possible velocity of the wind.

From the figure we see that $OR = RA_2$ and $ROA_2 = RA_2O = 45°$.

Hence the true velocity of the wind is,

$$\frac{40}{\sqrt{2}} = 20\sqrt{2} = 28{\cdot}3 \text{ m.p.h. in a direction } 45° \text{ west of north.}$$

194. To Find the Direction of Motion of a Body in order that it may meet a Second Moving Body :

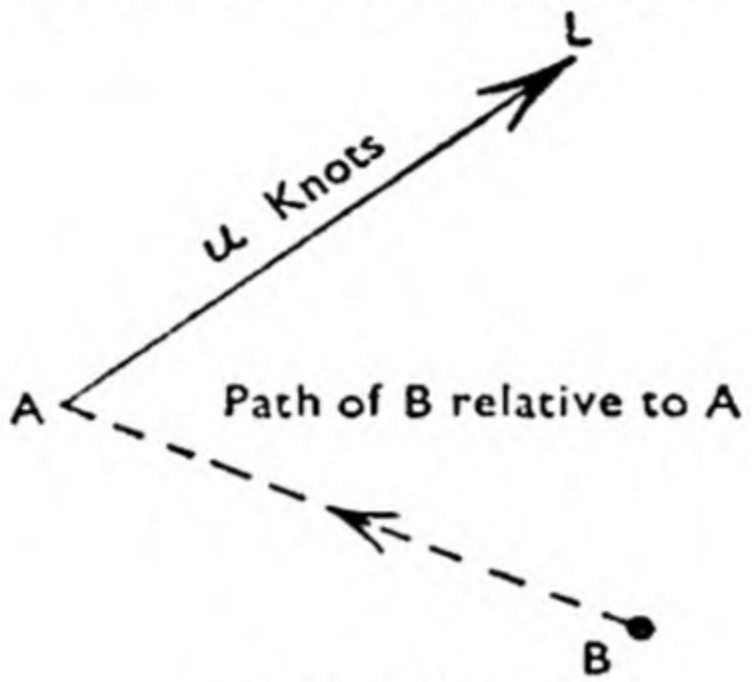


FIG. 367.

Let A and B be simultaneous positions of two ships, and let A proceed in the fixed direction AL at u knots.

The ship B is capable of a speed of v knots, and it is required to find the direction in which B must travel in order to meet A.

For B to meet A the velocity of B relative to A must be constantly directed towards the position occupied by A at any time during the motion of the ships.

Thus the path of B relative to A must be along BA, and the velocity of B relative to A must be directed along BA.

Now the velocity of B is the resultant of the velocity of A and the velocity of B relative to A :

Velocity of B = Velocity of A + Velocity of B relative to A.

Draw PQ // AL to represent the velocity of A,
PR // BA,
QS // BA,

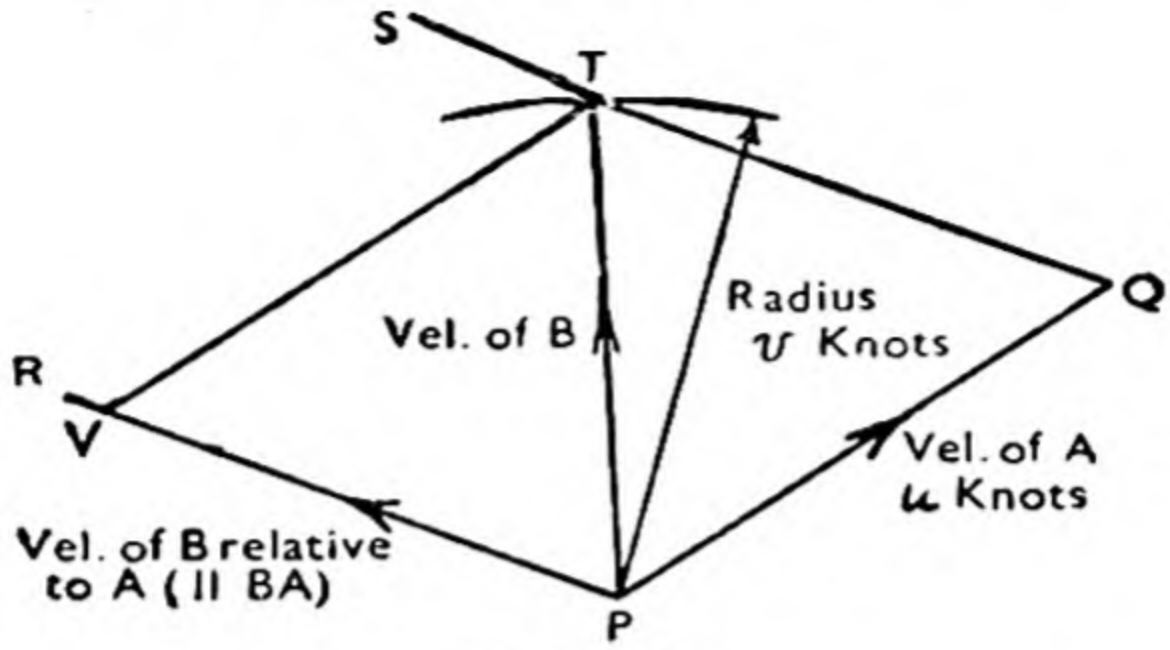


FIG. 368.

With centre P and radius to represent v knots the speed of B, draw an arc to cut QS at T. Complete parallelogram PQTV.

Then PT represents the velocity of B,

and PV represents the velocity of B relative to A.

The ship B must proceed in the direction PT with velocity v knots in order to meet A.

EXAMPLES LI

1. A steamer A is 10 miles due south of a steamer B. A is moving in a northerly direction at 12 m.p.h. and B is moving to the west with a speed of 9 m.p.h. Show on a diagram drawn to scale ($\frac{1}{2}$ in. $\equiv$ 1 m.p.h.) with what speed and in what direction A will appear to move to an observer on B. Find also from your diagram how near A will approach B. (G.S.)

2. Explain, with an example, what is meant by (*a*) relative velocity, (*b*) resultant velocity. Two boats are travelling, A north at 20 knots and B east at 30 knots. At noon B is 12 nautical miles north-west of A. When will the boats be nearest one another and how far apart will they be at that time? (1 knot = 1 nautical mile per hour.) (*G.S.*)

3. Distinguish between resultant velocity and relative velocity.

Two roads intersect, one going north and the other north-east. A car A is moving along the north road towards the north at a speed of 20 m.p.h. and a car B is moving north-east at the same speed. Determine the velocity of B relative to A.

At 1 P.M. B is at the junction of the two roads and A is south of it. At 2 P.M. B is due east of car A. Determine the position of car A at 1 P.M. (*O.C.*)

4. A steamer proceeding due west at 12 knots sights another steamer straight ahead of it at a distance of 10 sea miles proceeding due north at 15 knots. Find graphically or otherwise what will be their least distance apart in the subsequent motion. (1 knot = 1 sea mile per hour.) (*O.C.*)

5. A cruiser is proceeding due east at 20 knots and a destroyer, whose speed is 30 knots, is 10 sea miles due south of it. In what direction must the destroyer travel in order to meet the cruiser, and when will the meeting occur ? (*O.C.*)

6. At a certain instant the distance and bearing of a ship A from a second ship B are 3 miles and 60° east of north respectively. A is steaming at 10 m.p.h. on a course 30° east of north and B is steaming at 15 m.p.h. on a course 50° east of north. Find graphically or otherwise the distance and bearing of A from B when they are at their least distance from one another. (*O.C.*)

7. Two trains are travelling on lines which cross at right angles, one at 25 m.p.h. and the other at 20 m.p.h. Show graphically and calculate the magnitude of their relative velocities. Find the nearest distance apart of the trains if the slower train passes the crossing place 1 min. before the other.

8. To a motor car travelling due east at 30 m.p.h. another car appears to travel due south. When the first car increases its speed to 60 m.p.h., the second car appears to travel in the direction 30° west of south. Show that the true velocity of the second car is 60 m.p.h. in the direction 30° east of south.

9. To a motor car passenger travelling due south at 30 m.p.h. the wind appears to blow from the west. When the motor car speed is reduced to 15 m.p.h. the wind appears to come from the north-west. Find the true velocity and direction of the wind.

10. A cyclist rides round a circular track at 15 m.p.h. If there is a wind of 15 m.p.h. blowing from the east, draw a diagram showing the positions in which the wind appears (*a*) to be at rest, (*b*) blowing at 30 m.p.h., (*c*) blowing from the south-east, (*d*) blowing from the north-east. In cases (*c*) and (*d*) calculate the velocity of the wind relative to the cyclist.

11. An aeroplane traverses a course in the form of an equilateral triangle of side 60 miles when there is a wind of 60 m.p.h. blowing parallel to one side of the triangle in the direction in which the aeroplane traverses that side. Find the time taken by the aeroplane to complete the course, its air speed being 104 m.p.h.

12. A vessel A is travelling due north at 15 m.p.h. A vessel B is 10 miles due west of A and can travel at 20 m.p.h. Show that for B to pass just 5 miles from A, keeping a straight course, the velocity of B relative to A must make an angle of 30° with the easterly direction, and find the two possible times which may elapse before B is just 5 miles from A.

CHAPTER XXI

NEWTON'S SECOND LAW OF MOTION

195. Mass and Weight.

If we suspend a piece of wood by a vertical string and attach a dynamometer to it, the dynamometer when pulled quickly in a horizontal direction registers the horizontal force which, acting upon the wood, produces in it a certain change of motion in this direction. If we substitute a piece of lead of the same size and shape for the wood and repeat the operation, we find that the readings of the dynamometer required to produce something like the same change of motion in the lead in the same time as in the wood are very much greater than in the case of the wood. Consider a wooden flywheel whose bearings are horizontal and as smooth as possible. To set the wheel in motion suppose that a force is applied tangentially to the rim of the flywheel. In a similar manner the wheel may be stopped. Now take a steel flywheel identical in size and shape to the wooden one and repeat the operations, attempting to produce something like the same changes in motion in the two cases. We have no doubt after the experiment that it is more difficult to produce the same change of motion in the steel flywheel than in the wooden one.

Arrange two aluminium bodies to balance each other by passing a string connecting them over a smooth pulley. Set the bodies in motion vertically by pushing one of them downwards. Repeat the experiment with brass or iron bodies of the same size and shape as the aluminium ones. Once again there is no doubt that the brass or iron bodies are more difficult to set in motion than the aluminium bodies.

In all three cases described above we have taken the greatest care to eliminate any effects produced by the weights of the bodies. The results of the experiments show that there is a property of a body, *not its weight*, which is brought into play

when we change the motion of the body. It is this property which we call the **mass** of the body. The greater the effect of this property in the case of any body, the greater is the mass of the body.

That there is a close connection between the mass of a body and its weight is evident, for the above experiments have clearly indicated that the heavier a body the greater is its mass.

196. We have shown that the state of rest or of uniform motion in a straight line of a body is changed by the application to that body of a resultant force. The " weight force " of a body produces in that body a uniform acceleration of g ft./sec.^{2} when the body is allowed to fall freely. The weight force of a body at the surface of the moon is one-sixth of its weight force at the earth's surface, whereas at the surface of the planet Jupiter the weight force of the same body would be 2·7 times that at the earth's surface. We should therefore expect that a body would fall vertically with a smaller acceleration at the surface of the moon and a greater acceleration at the surface of Jupiter than at the surface of the earth. In fact, we should expect that the greater the resultant force which acts upon a body the greater will be its acceleration. Now bodies of equal weight when situated near each other fall freely with the same acceleration (neglecting air resistance) irrespective of size or shape, and since equal forces produce in them the same change of motion they have equal masses. Hence bodies which have equal weights have also equal masses.

197. The weight force provides a very convenient means of measuring mass, for if W_1 and W_2 are the weights of two bodies in the same units where masses are m_1 and m_2,

$$\frac{W_1}{W_2} = \frac{m_1}{m_2}.$$

It follows that the same balance used for comparing weights may also be used for comparing masses.

A piece of platinum preserved in the House of Commons is said to have a mass of 1 lb. and is the British standard of mass.

We have already referred to the fact that British engineers use the weight force of this standard mass of 1 lb. as their unit of force. It is rather unfortunate that they also frequently call this standard of force by the same name as that given to the mass, viz., 1 lb. We have, however, referred to the unit of force as 1 lb. wt., and shall continue to make this distinction; where the distinction is not made, the context usually indicates quite clearly which meaning of the word is intended.

The student should bear in mind that the change of motion of a body, to which a force is applied, is dependent upon the mass of the body and not upon its weight force. The mass of a body is constant, its weight force is not constant.

198. The Second Law of Motion.

We are now familiar with the fact that the constant force which we have called the weight force acting upon a body produces in the body a constant acceleration, and we naturally inquire what if the resultant force which acts upon a body is not the weight force but any force acting in any direction. Is the change of motion produced by such a force that of constant acceleration? We may expect to find that the answer to this question is YES, for the attraction of the earth upon a body is a force like any other force which may be measured by the tension or compression in a dynamometer spring, and whether the force acts vertically or in any other direction seems to be unimportant. We can show that our convictions are fully justified by the experiments described in the next paragraph.

199. The Fletcher Trolley.

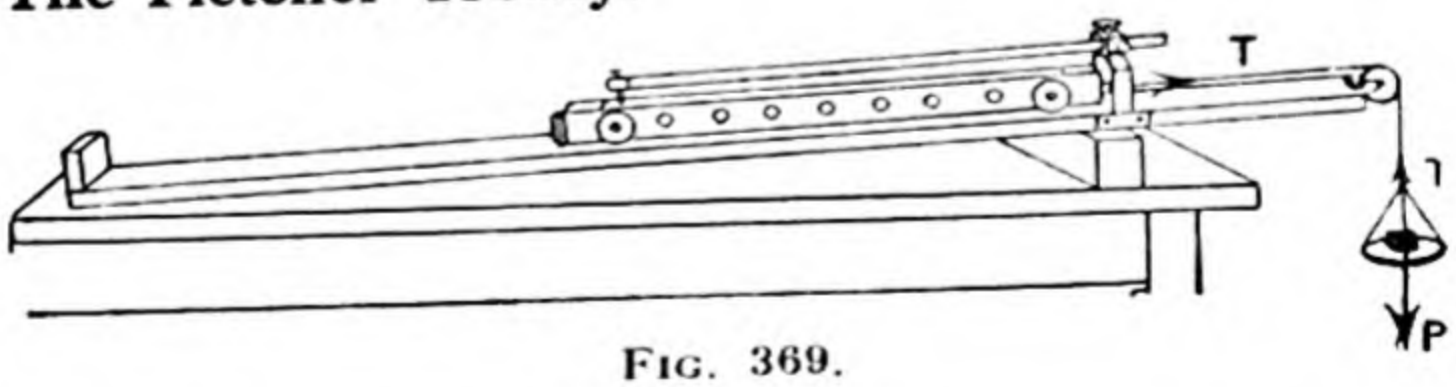


FIG. 369.

A Fletcher trolley is shown in the above figure. It consists of a trolley upon very light wheels which have very little

friction at their bearings. The trolley is attached by a string parallel to the inclined plane upon which the trolley rests, which passes over a smooth pulley at the top of the plane and then supports a pan of known weight.

Let W represent the weight of the trolley,
F, the frictional force brought into play when the trolley rolls along the plane,
P, the weight of the pan and added weights.

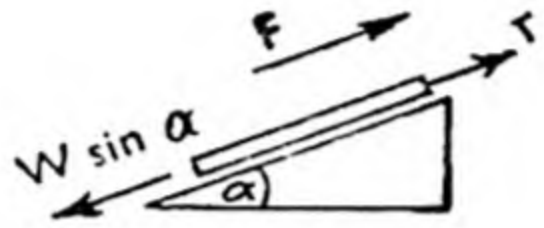


FIG. 370.

The forces acting on the trolley parallel to the inclined plane are :

W sin a, the component of the weight force of the trolley which acts down the plane,
T, the tension in the string which acts up the plane,
F, the frictional force which acts in the direction opposing the motion of the trolley.

The inclined plane is tilted so that when the trolley is released it runs down the plane with uniform velocity as far as may at first be judged by the eye. This may be checked by measuring the lengths of the waves made upon a sheet of paper pinned to the trolley by the brush attached to the vibrating bar. Since the time of vibration of the bar is constant, the lengths of the waves should be equal if the motion of the trolley is one of uniform velocity.

In this case the forces acting on the trolley parallel to the plane are in equilibrium, hence,

$$T + F - W \sin a = 0 \quad . \quad . \quad . \quad (1)$$

Moreover, the pan and its contents move vertically with uniform velocity, and therefore

$$T - P = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2),

$$P = T$$
$$= W \sin \alpha - F \text{ from (1).}$$

If now the scale pan and contents are detached from the trolley and it is allowed to run down the plane, the resultant force acting upon it is

$$W \sin \alpha - F.$$

But this force we have seen is equal to P, and therefore the resultant force acting on the trolley is the weight force of the pan and its contents.

The wave trace made by the brush attached to the vibrating bar for the accelerated motion of the trolley is now carefully taken. From it we have to show that the acceleration of the trolley is constant.

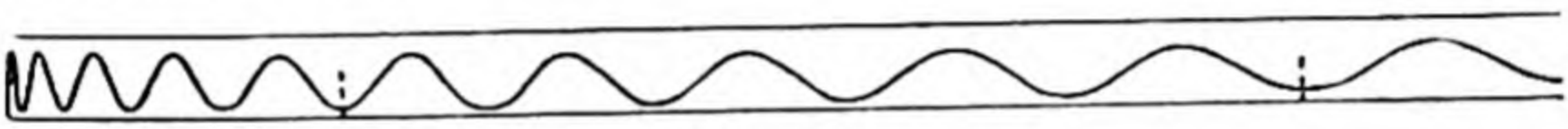


FIG. 371.

(1) Number of Periods of $\frac{1}{6}$ Sec.	(2) Distance in Millimetres.	(3) Lengths of Successive Waves. Increase in Distance per Unit Time.	(4) Increase in Velocity per Unit Time.
1	15		
		45	
2	60		27
		72	
3	132		31
		103	
4	235		26
		129	
5	364		32
		161	
6	525		28
		189	
7	714		

In the above table the first column of figures contains the number of vibrations of the steel rod, the second the

corresponding distance travelled by the trolley, and the third contains the lengths of the successive waves.

Since these waves show the *increase* in distance in successive vibrations, we have only to divide these distances by the times taken for the increases and we have the average velocity for these times. The vibrations take equal times, and we may therefore take the periodic time of the spring as our unit of time. In this case the entries in the third row are the average velocities in each successive unit of time. Since the velocity of the trolley is changing quite slowly we may assume the average velocity for a unit time to be the actual velocity at the middle instant of that time. By subtraction we therefore find the *change in velocity* per successive units of time, as shown in column 4.

This set of figures gives the increase in velocity per unit time, *i.e.*, the acceleration of the trolley ; which is seen to be reasonably constant. The average value of the acceleration is 29 units. *The experiment has shown that a constant accelerating force produces in a given body a constant acceleration in the direction of the force.*

The experiment is repeated for different inclinations of the plane, and the corresponding values of accelerating force and acceleration are shown in the table below :

$f = \frac{\text{Increase in Velocity}}{\text{Unit Time}}$ (Average Value).	Accelerating Force P.	$\frac{f}{P}$.
29	0·3	97
38	0·4	95
47	0·5	94

Since the values of $\frac{f}{P}$ are practically constant, we see that the *acceleration produced in a given body is proportional to the accelerating force.*

200. Let P represent the resultant force acting upon a body and f the acceleration which it produces. We have shown that $\frac{f}{P}$

is constant in value. By varying the mass and noting the resultant force P acting on the trolley together with the acceleration, we can show that

$$\frac{P}{mf} = \text{constant} \quad . \quad . \quad . \quad . \quad (i)$$

Now our unit of mass is already fixed; the pound, and the unit of acceleration is also fixed, 1 ft./sec.^{2} If we now choose our unit of force so that $P = 1$ when $m = 1$ and $f = 1$, the value of the constant is also unity.

This unit of force is therefore the force which produces in a mass of 1 lb. an acceleration of 1 ft./sec.^{2}, and is called the *Poundal.*

Using this unit we may write the above equation

$$\mathbf{P = mf} \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

The poundal is called an *Absolute* unit of force because it has a fixed value. The engineer's or gravitational unit of force, the weight of 1 lb., has not a fixed value, for although the difference is small, the value is different at different parts of the earth's surface.

1 poundal produces an acceleration of 1 ft./sec.^{2} in a mass of 1 lb., and since a force of 1 lb. wt. produces an acceleration g ft./sec.^{2} in a mass of 1 lb., it must be equal to g poundals, or

$$1 \text{ lb. wt.} = g\ (32{\cdot}2) \text{ poundals.}$$

The poundal is therefore equal to the weight of about half an ounce.

In a similar manner it may be seen that

$$1 \text{ ton wt.} = g\ (32{\cdot}2) \text{ tonals,}$$

where 1 tonal is the force which produces in a mass of 1 ton an acceleration of 1 ft./sec.^{2}

If the pound-weight unit of force is used instead of the poundal, equation (ii) becomes

$$P = \frac{mf}{g} \quad \text{or} \quad \frac{P \text{ (lb. wt.)}}{m \text{ (lb.)}} = \frac{f}{g}$$

In c.g.s. units the absolute unit of force is that which

produces in 1 gm. an acceleration of 1 cm./sec.^{2} and is called the *Dyne*.

Since 1 gm. wt. produces an acceleration of $g(=981)$ cm./sec.^{2} in a mass of 1 gm.

$$1 \text{ gm. wt.} = g(981) \text{ dynes.}$$

The dyne is therefore a very small force, equal to the weight of about a milligram.

The weight of a mass of m lb. is mg poundals ($g = 32{\cdot}2$, nearly) and that of a mass of m gm. is mg dynes ($g = 981$, nearly).

From equation (ii) *it will be seen that two bodies have equal masses if the same force produces in each mass the same acceleration.*

201. Momentum.

We have seen that the force required to produce a given acceleration in a body depends on the mass of the body. In the same way the force required to produce a given negative acceleration or retardation in a moving body depends on the mass of the body. It is well known that if a body is moving horizontally the force required to stop it in a given time depends on both the mass of the body and the speed with which it is moving. Suppose a body of mass m to be moving horizontally with speed v, what force is required to bring it to rest in time t? If the force is P, and f the retardation it produces, $P = mf$, and if it is reduced to rest in time t,

$$0 = v - ft, \quad f = \frac{v}{t},$$

therefore

$$P = \frac{mv}{t}$$

The force is therefore proportional to the product mv, *i.e.*, the product of the mass and velocity of the body.

This product is called the *Momentum* of the body and is an important quantity, particularly when one comes to deal with sudden changes of motion. It is what Newton called the "quantity of motion."

Since f, the acceleration, is the rate of change of v, the product mf represents the rate of change of momentum, and Newton's Second Law of Motion may be stated as follows :—

Change of momentum per unit time is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

Momentum is a vector quantity.

202. Example (1).—*A body of mass* 10 *lb. is acted on by a force of* 4 *lb. wt.* What acceleration is produced in the body ?

Taking $g = 32$, 4 lb. wt. $= 4 \times 32 = 128$ poundals.

Using $P = mf$,

$128 = 10f$, or $f = 12{\cdot}8$ ft./sec.^{2}

Example (2).—*A mass of* 15 *cwt. acquires a velocity of* 30 *m.p.h. when acted upon for* 1 *min. by a constant force. What is the value of this force in pounds-weight ?*

Many errors in dynamical problems occur through the use of wrong units, and it is safer always to reduce all measurements to foot-pound per second or c.g.s. units.

In this case we use foot-pound per second units.

$$30 \text{ m.p.h.} = \frac{30 \times 88}{60} = 44 \text{ ft./sec.}$$

$$1 \text{ min.} = 60 \text{ sec.}$$

$$15 \text{ cwt.} = 15 \times 112 = 1680 \text{ lb.}$$

To find the acceleration required to produce this velocity in 60 sec. we use

$$v = u \times ft.$$

Here $u = 0$, $v = 44$, $t = 60$,

therefore $44 = 60f$, or $f = \frac{11}{15}$ ft./sec.^{2}

To find the constant force necessary to produce this acceleration we use

$$P = mf.$$

Here $m = 1680$ lb., $f = \frac{11}{15}$ ft./sec.^{2},

therefore $P = \frac{1680 \times 11}{15}$ poundals $= \frac{1680 \times 11}{15 \times 32} = 38{\cdot}5$ lb. wt.

Example (3).—*The normal reaction on the driving wheels of a locomotive is* 100 *tons wt. If the coefficient of friction between the driving wheels and track is* 0·2, *what is the greatest driving force which can be exerted on the engine parallel to the track ? If the mass of the engine is* 200 *tons, what is the greatest acceleration which this force can produce in it if the resistance to motion is* 1,000 *lb. wt. ?*

If N is the normal reaction, and F the force of limiting friction,

$$\frac{F}{N} = \mu = 0{\cdot}2,$$

therefore $F = 100 \times 0{\cdot}2 = 20$ tons wt.

The greatest force that can be exerted by the rails on the engine parallel to the track is equal to this limiting friction and is therefore 20 tons wt.

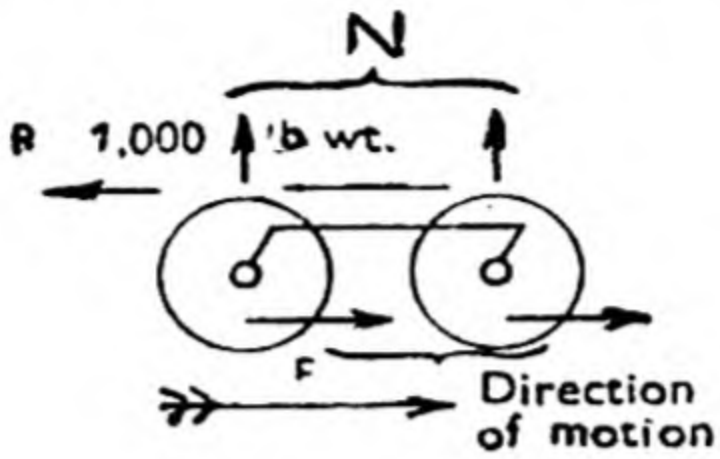


FIG. 372.

The resultant force on the engine parallel to the track is

$$(20 \times 2240 - 1000) = 43800 \text{ lb. wt.} = 43800 \times 32 \text{ poundals.}$$

Using $P = mf$,

$P = 43800 \times 32$, $m = 200 \times 2240$,

therefore $43800 \times 32 = 200 \times 2240 \times f$,

$$f = \frac{43800 \times 32}{200 \times 2240} = \frac{219}{70} = 3{\cdot}13 \text{ ft./sec.}^2$$

203. Pressure on the Floor of a Lift.

Suppose a body of mass m lb. is resting on a horizontal plane, such as the floor of a lift, which is moving upwards with acceleration f ft./sec.^{2} Let R be the reaction between the body and the plane. Since the body is moving upwards with acceleration f, it is evident that R is greater than the weight mg.

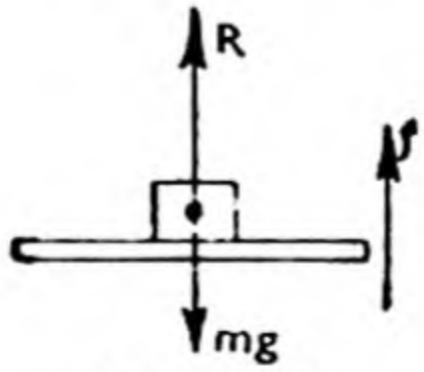


FIG. 373.

The resultant upward force acting on the body is $R - mg$, and

$$R - mg = mf,$$

therefore $$R = m(g + f).$$

If the plane is moving downwards with acceleration f, the

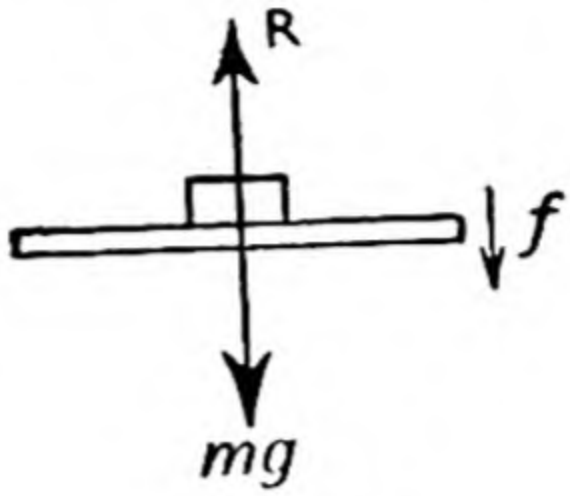


FIG. 374.

weight mg is now greater than R. The resultant downward force acting on the body is $mg - R$, and

$$mg - R = mf,$$

therefore

$$R = m(g - f).$$

If in the latter case $f = g$, then $R = 0$, *i.e.*, there is no pressure between the body and the plane.

Example.—*A man of weight* 10 *st. stands upon the floor of a lift which ascends with an acceleration of* 4 *ft./sec.*^{2} *What is the force exerted by the floor of the lift upon the man?*

Let R lb. wt. be the force exerted.

The resultant force acting on the mass vertically upwards is

$$(R - 140) \text{ lb. wt.} = (R - 140)32 \text{ poundals},$$

therefore

$$32(R - 140) = 140 \times 4,$$

$$R - 140 = \frac{560}{32} = \frac{70}{4},$$

$$R = 140 + 17{\cdot}5 = 157{\cdot}5 \text{ lb. wt.}$$

204. Motion on an Inclined Plane.

Suppose that a body of mass m is allowed to slide freely down a smooth plane inclined at an angle θ to the horizontal. The motion of the body along the plane is produced by the resolved part of its weight parallel to the plane, viz., $mg \sin \theta$, which,

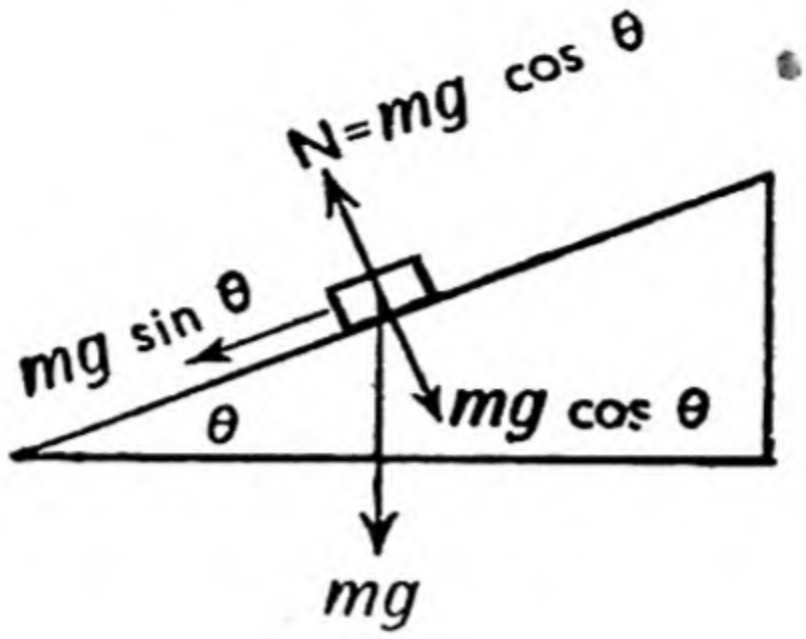


FIG. 375.

acting upon the mass m of the body, produces in it an acceleration f. By Newton's Second Law we have

$$mg \sin \theta = mf,$$

therefore

$$f = g \sin \theta.$$

It should be noticed that the reaction N of the plane on the body is perpendicular to the plane and does not affect the motion parallel to the plane.

If other forces act upon the mass, its motion along the inclined plane may be determined by finding the resultant of all resolved parts of forces acting on it parallel to the plane and then applying Newton's Second Law to the motion in this direction.

Example.—*A truck of weight 2 tons is free to move down an incline of 1 in 20.*

(a) *With what acceleration will it travel?* (b) *If the truck is to be dragged up the plane, what force parallel to the plane is required* (i) *to keep it moving at a uniform speed* (ii) *to produce in it an acceleration of $\frac{1}{2}$ ft./sec.^{2}?*

(*a*) The resolved part of the weight of the truck parallel to the track is $2 \times 2240g \times \frac{1}{20}$ poundals.

The mass of the truck is 2×2240 lb.

Hence, using $P = mf$,

$$2 \times 2240g \times \frac{1}{20} = 2 \times 2240 \times f,$$

$$f = \frac{g}{20} = 1 \cdot 6 \text{ ft./sec.}^2 \ (g = 32).$$

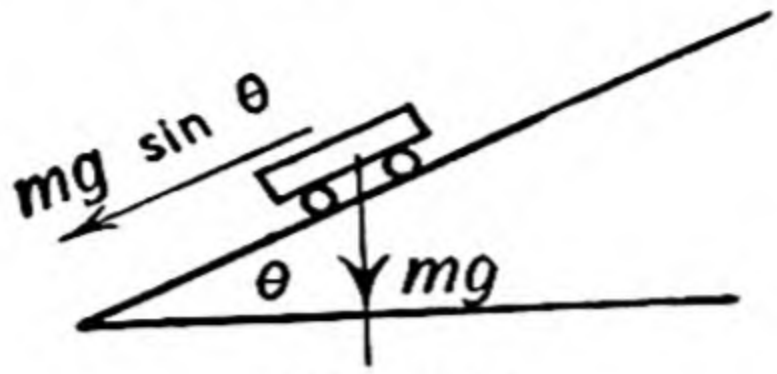


FIG. 376.

(*b*) (i) When the speed of the truck is constant, the resultant force parallel to the track is zero, and therefore

Force on the truck parallel to track

= Resolved part of the weight parallel to the track

$= 2 \times 2240g \times \frac{1}{20}$ poundals

= 224 lb. wt.

(ii) Consider only the resolved parts of forces parallel to the track.

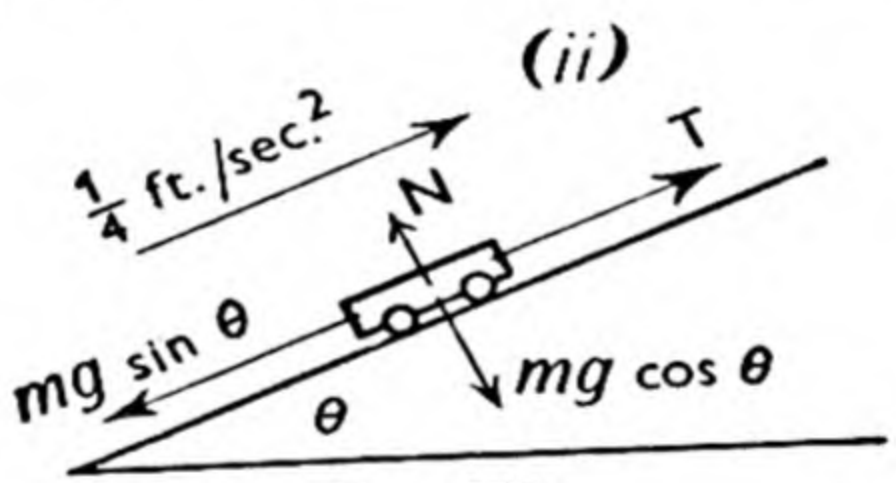


FIG. 377.

Let T poundals be the force on the truck parallel to the track. The resultant force parallel to the track up the plane is $\left(\mathrm{T} - 2 \times 2240 \times \frac{g}{20}\right)$ poundals.

Using $\mathrm{P} = mf$, we have, since $f = \frac{1}{4}$,

$$\left(\mathrm{T} - 2 \times 2240 \times \frac{g}{20}\right) = 2 \times 2240 \times \frac{1}{4},$$

therefore

$$\mathrm{T} = 2 \times 2240 \times \frac{g}{20} + 2 \times 2240 \times \tfrac{1}{4} \text{ poundals}$$

$$= 224 + \frac{2 \times 2240}{4 \times 32} \text{ lb. wt.}$$

$$\mathrm{T} = 224 + 35 = 259 \text{ lb. wt.}$$

EXAMPLES LII

NEWTON'S SECOND LAW

[$g = 32$ ft./sec.^{2} or 980 cm./sec.^{2}]

1. A force of 2 lb. wt. acts on a mass of 8 lb. What acceleration is produced in the mass ?

2. A force of 120 gm. wt. acts on a mass of 500 gm. What acceleration is produced in the mass ?

3. Express

(*a*) A force of 4 lb. wt. in poundals.
(*b*) A force of 10 gm. wt. in dynes.
(*c*) A force of 3,200 poundals in lb. wt.
(*d*) A force of $9{\cdot}8 \times 10^6$ dynes in kgm. wt.

4. What force in pounds weight acting on a mass of 100 lb. will produce in it an acceleration of 16 ft./sec.^{2} ?

5. What constant force in pounds weight acting on a mass of 8 lb. will produce in it a change of velocity of 24 ft./sec. in 2 sec. ?

6. The speed of a motor car of mass 16 cwt. increases to 30 m.p.h. from rest in 10 sec. If the force urging the car forward is constant, calculate its value in cwt.

7. The brakes of a car travelling at 20 m.p.h. are suddenly applied and it is brought to rest by a constant retarding force in a distance of 44 ft. Calculate the retardation of the car. If the mass of the car is 12 cwt., calculate in pound weight the retarding force.

8. A force of 10 lb. wt. acts on a mass of 20 lb. for 5 sec. Calculate the velocity acquired by the body and the distance which it travels while it is acquiring this velocity.

9. A bullet of mass 1 oz. is travelling at 1,200 ft./sec. when it strikes a target in which it embeds itself after penetrating 6 in. Calculate the average retardation of the bullet in the target and the average force exerted by the target on the bullet during the penetration.

10. An engine of mass 50 tons is travelling at a speed of 20 m.p.h. when steam is shut off. If the engine is retarded by resistances amounting to 7 lb. wt. per ton, calculate the retardation of the engine.

11. A mass of 20 lb. is held on a smooth plane inclined at 30° to the horizontal. If the mass is released, what will be its acceleration down the plane ? What force is producing this acceleration ?

12. Explain the terms poundal, pound weight. Describe an experiment to find the relation between them.

A mass of 10 cwt. acquires in 1 min. a velocity of 30 m.p.h. when acted upon by a constant force. Find the value of the force in pound weight.

13. Explain what is meant by force and define a unit in which it may be measured. A ball weighing 1·6 oz. and travelling at 150 ft./sec. is brought to rest in 6 in. What is the average force applied ? (*G.S.*)

14. Define force and momentum and show how they are related to one another.

If a car weighing 3,000 lb. and travelling at 25 m.p.h. is brought to rest in 20 yd., what is the average force acting ? (*G.S.*)

15. Define a unit of force in the foot-pound-second system.

A body of mass 10 lb. acted on by a constant force for 1 min. acquires thereby a velocity of 1,000 yd./min. Determine the value of the force. (*C.W.B.*)

16. Distinguish between pound, pound weight, poundal, and foot-pound. What force in pound weight will give a mass of 5 tons a velocity of 10 m.p.h. (*a*) in 1 min., (*b*) in 1 mile ? (*O.C.*)

17. Define the terms momentum, force, and work, and show how they are related.

A body of mass 24 lb. starts from rest under the action of a constant force and acquires a velocity of 10 m.p.h. in 4 sec. What is the value of the force measured in both the British units of force ? (*L.M.*)

18. What is meant by the terms force and momentum ? Show the relation between them. Define the units of force in the metric and British systems.

A motor car travelling at 30 m.p.h. is brought to rest in 30 yd. What is the average force in tons weight which is applied to the car if its mass is 1 ton ? (*L.M.*)

19. A lorry is travelling at 36 km. per hour. What force, expressed as a fraction of the weight of the lorry, is needed to stop it (*a*) in 10 sec., (*b*) in 10 m. ? (*O.C.*)

20. A carriage is slipped from a train travelling at 60 m.p.h., and the brakes are immediately applied to the carriage, giving a resistance equal to 0·06 of the weight of the carriage. Find how far from the spot at which it is desired to stop it is necessary to slip the carriage if it has to travel down a slope of 1 in 200. (*O.C.*)

21. Define poundal, pound weight and foot-pound.

A car of mass 25 cwt. is uniformly accelerated from 6 to 30 m.p.h. in 20 sec. Calculate the constant pull of the engine and the work done in the time when the car is travelling (*a*) along the level, (*b*) up an incline which rises 1 ft. vertically for every 10 ft. measured along the incline. State the units in each case. (Neglect all frictional forces.)

22. Define force and momentum. How are they related ?

A trolley weighing 5 lb. runs from rest for 5 ft. down an incline in 1·8 sec. What is the average force acting on it ? (*G.S.*)

23. Define poundal and pound weight.

A truck of weight 2 tons is to be hauled up an incline of 1 in 12. What constant force, applied parallel to the plane, is required to move it (*a*) with a constant velocity of 4 in./sec., (*b*) with a constant acceleration of 4 in./sec.^{2} ? (*L.M.*)

24. A balloon which weighs 1 ton is drifting horizontally, and 20 lb. of ballast are thrown out. With what acceleration will the balloon begin to ascend ? (*O.C.*)

25. A mass of 10 lb. is hung on a spring balance which is suspended from the roof of a lift. What will the readings of the balance be when the lift is

(*a*) Rising with uniform velocity 6 ft./sec.
(*b*) Rising with an acceleration of 3 ft./sec.^{2}
(*c*) Descending with an acceleration of 3 ft./sec.^{2} (*L.M.*)

26. A man in a lift holds a spring balance reading to tenths of a pound with a 7-lb. wt. hanging from it. The lift descends (*a*) with downward acceleration 4 ft./sec.^{2}, (*b*) with a uniform speed of 6 ft./sec., (*c*) with

upward acceleration of 2 ft./sec.^{2} What should be the readings of the spring balance ? (*L.M.*)

27. A train is moving along a level track with a constant acceleration of 10 m.p.h./min. How will a plumb-line suspended in the train hang ? (*L.M.*)

205. Action and Reaction.

According to Sir Isaac Newton, a body that pulls or presses another body is itself pulled or pressed by that other body, the force on the one body being equal in magnitude but opposite in direction to the force on the other. We have already called attention to this fact when bodies at rest were considered in an earlier chapter and have used the fact in a number of ways. We may not, however, have appreciated as Newton did that the principle is equally true when applied to moving bodies. His own method of making the fact clear can scarcely be improved upon, for in considering the action of one body upon another when the bodies are in motion, he imagines them to be connected by a light spring. The tension or compression in the spring exerts the same pull or push on each of the bodies. Thus the coupling between two coaches of a train in motion exerts equal but opposite forces upon each of the two coaches, and, to cite Newton's classic example, the apple which falls to the earth exerts the same attraction upon the earth as the earth upon the apple, but whereas the acceleration of the apple is readily observed, that of the earth, because of its greater mass, is too small to be measurable. If we take the weight of an apple as $\frac{1}{4}$ lb. and g as 32, we can calculate the acceleration of the earth, and if we suppose this to be constant while the apple falls 16 ft. the earth would move $\frac{2{\cdot}5}{10^{22}}$ in.

Newton's Third Law may be stated as follows :—

To every action there is an equal and opposite reaction.

We shall proceed to consider further examples showing the applications of the Second and Third Laws of Motion.

206. Accelerated Motion of Locomotive and Carriages.

We have already considered the part played by friction both

in assisting and hindering the motion of a locomotive. We shall now examine the motion more fully.

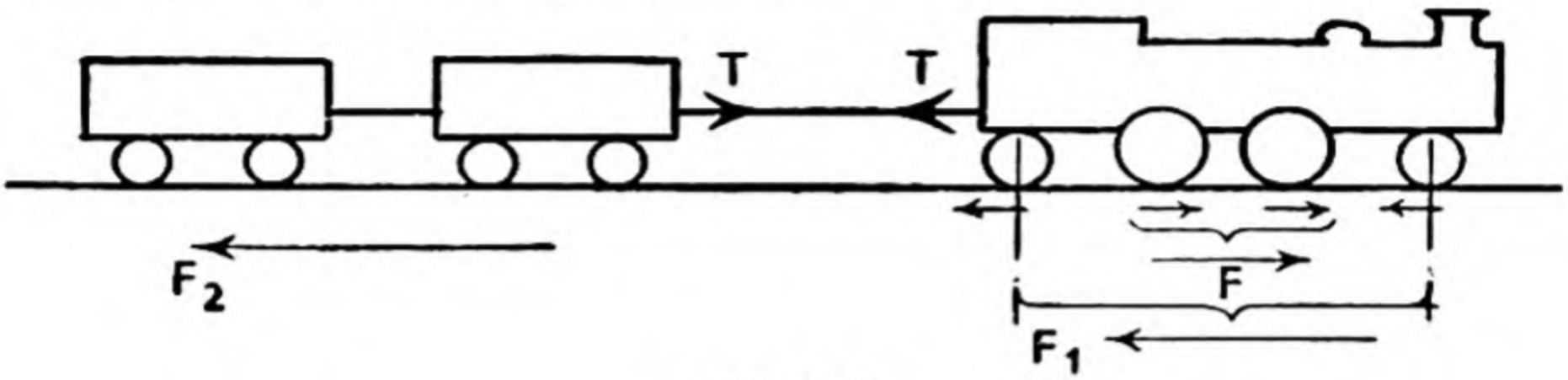


FIG. 378.

In the above figure F represents the force of friction acting on the driving wheels of the train, F_1 represents the force of friction opposing the motion of the locomotive, F_2 represents the force of friction opposing the motion of the carriages, T represents the tension in the coupling between the locomotive and the first carriage, M_1 represents the mass of the locomotive, M_2 represents the mass of the carriages, f represents the acceleration of the train.

Consider the motion of the locomotive.

The resultant force acting on it parallel to the track is

$$F - F_1 - T,$$

and therefore by Newton's Second Law

$$F - F_1 - T = M_1 f \quad . \quad . \quad . \quad . \quad (i)$$

Consider the motion of the carriages.

By Newton's Third Law the tension in the coupling acts in the opposite direction on the carriages to that in which it acts upon the locomotive.

Resultant force acting on the carriages, $T - F_2$, and therefore

$$T - F_2 = M_2 f \quad . \quad . \quad . \quad . \quad (ii)$$

Consider the motion of the train as a whole.

The tensions in the couplings do not affect the motion of the train, just as the tensions in the traces when a horse draws a cart do not affect the motion of the horse and cart as a whole.

Resultant force on the train, $F - F_1 - F_2$, and therefore

$$F - F_1 - F_2 = (M_1 + M_2) f \quad . \quad . \quad . \quad (iii)$$

Equation (iii) may be obtained by adding equations (i) and (ii).

Equation (iii) adds no information to (i) and (ii), but, on the other hand, when only the acceleration of the train is required, we see that the tension in the coupling between the locomotive and the carriages need not be considered.

207. Motion of Bodies Connected by Strings.

If two bodies are connected by an inextensible string the tension is the same throughout the length of the string, provided (i) that the string has no mass and (ii) that it only passes over perfectly smooth surfaces, such as a smooth pulley. In the examples that follow we shall assume that these conditions are satisfied and then the tension will always be the same throughout the string whether the bodies are at rest or in motion.

Example (1).—*A mass of 4 lb. rests on a smooth horizontal table. A light string is attached to it, and after passing over a smooth pulley at the edge of the table the other end of the string is attached to a mass of 2 lb., which hangs vertically. If the part of the string on the table is horizontal, what is the acceleration of each mass and the tension in the string?*

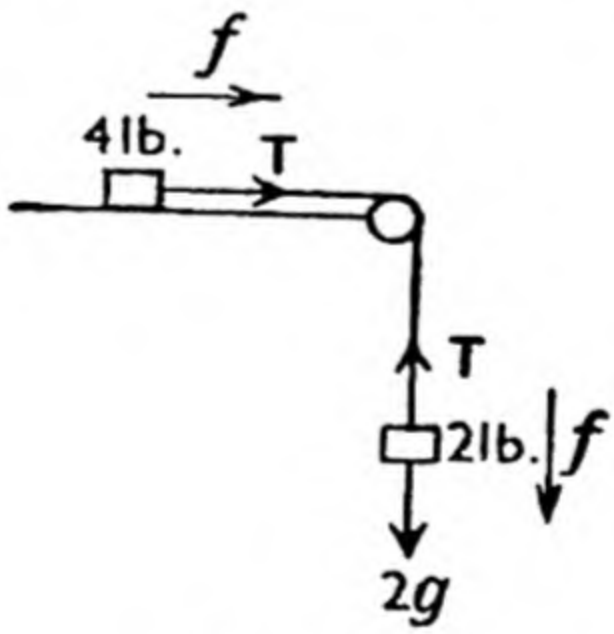


FIG. 379.

Since the string is inextensible, the acceleration of the two masses must be the same.

Let T poundals be the tension in the string and f the acceleration of each mass.

Consider the motion of the 4-lb. mass horizontally.

The only force acting on it is the tension T,

therefore $$T = 4f \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The 2-lb. mass is acted on by its weight $2g$ vertically downwards and the tension T vertically upwards.

The resultant downward force is therefore $2g - T$,

therefore $$2g - T = 2f \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Adding equations (i) and (ii), $2g = 6f$,

therefore $$f = \tfrac{1}{3}g = 10{\cdot}7 \text{ ft./sec.}^2$$

We can find T by substituting for f in equation (i)

$$T = \frac{4g}{3} = 42{\cdot}8 \text{ poundals.}$$

Example (2).—*Two particles of masses 4 and 2 lb. are connected by a tight inextensible string passing over a small smooth fixed pulley so that the parts of the string not in contact with the pulley are vertical. Find the resulting acceleration of the particles and the tension in the string.*

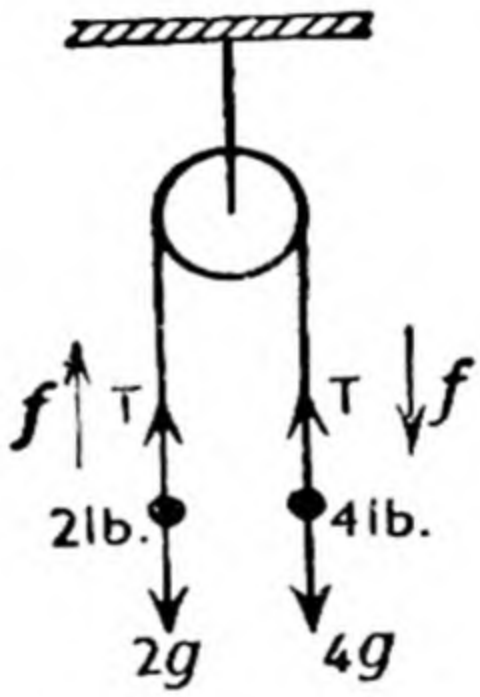


FIG. 380.

It is obvious that the 4-lb. mass will move downwards and that the acceleration of the two masses will be the same.

Let T poundals be the tension in the string and f the common acceleration.

The resultant force acting on the 4-lb. mass is $4g - T$ downwards,

therefore $$4g - T = 4f \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The resultant force acting on the 2-lb. mass is $T - 2g$ upwards,

therefore $$T - 2g = 2f \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Adding equations (i) and (ii), $2g = 6f$,

therefore $$f = \tfrac{1}{3}g = 10{\cdot}7 \text{ ft./sec.}^2$$

T can be found by substituting for f in either of the equations (i) or (ii).

Substituting in equation (i),

$$T = 4g - \frac{4}{3g} = \frac{8g}{3} = 85{\cdot}6 \text{ poundals.}$$

Note.—The force on the pulley vertically downwards is twice the tension in the string, *i.e.*, 171·2 poundals or $2\frac{2}{3}$ lb. wt.

Example (3).—*A mass of 10 lb. rests on a smooth horizontal table and is attached to masses of 4 and 5 lb. hanging vertically by two tight inextensible strings passing over two smooth pulleys at opposite edges of the table. Find the acceleration of the system and the tension in each string.*

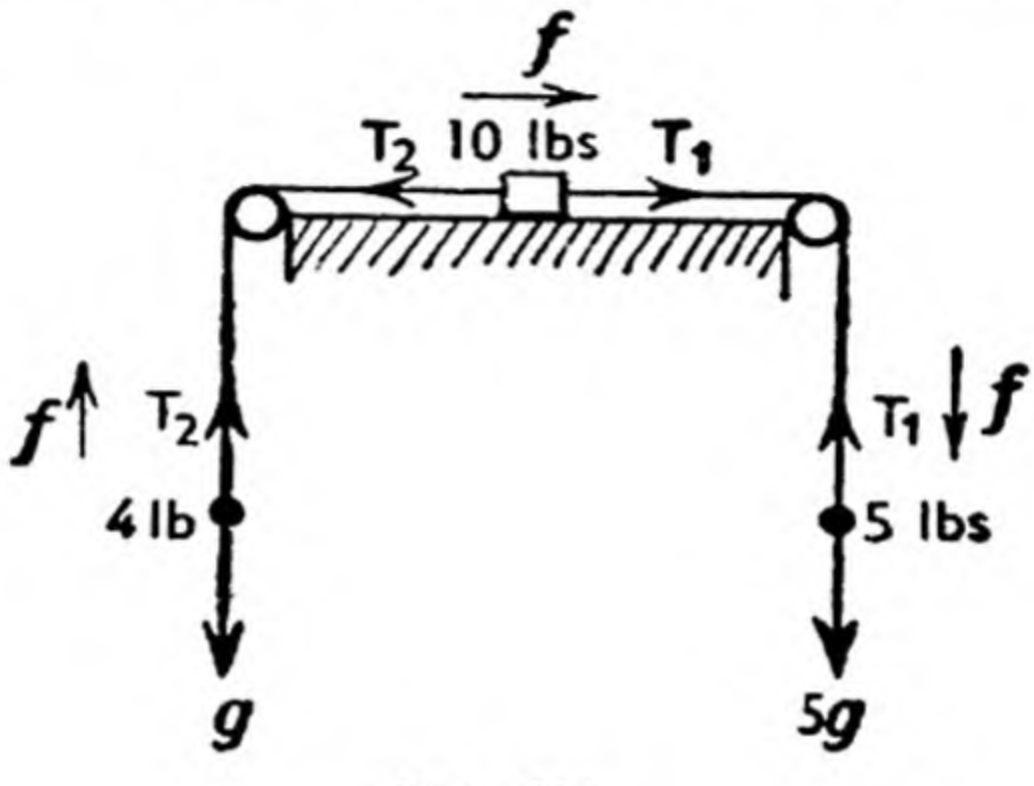


FIG. 381.

Let T_1 poundals be the tension in the string attached to the 5-lb. mass, T_2 that in the other and f the acceleration of the system.

It is clear that the 5-lb. mass will descend and the 4-lb. mass rise.

The resultant force on the 10-lb. mass is $T_1 - T_2$,

therefore $$T_1 - T_2 = 10f \quad . \quad . \quad . \quad . \quad \text{(i)}$$

For the 5-lb. mass, $$5g - T_1 = 5f \quad . \quad . \quad . \quad \text{(ii)}$$

For the 4-lb. mass, $$T_2 - 4g = 4f \quad . \quad . \quad . \quad \text{(iii)}$$

Adding equations (i), (ii) and (iii), $g = 19f$,

therefore $$f = \frac{g}{19} \text{ ft./sec.}^2$$

Substituting in (ii) and (iii),

$$T_1 = 4\tfrac{11}{19}g \text{ poundals, or } 4\tfrac{11}{19} \text{ lb. wt.}$$

$$T_2 = 4\tfrac{4}{19}g \text{ poundals, or } 4\tfrac{4}{19} \text{ lb. wt.}$$

Example (4).—*Two scale pans, each of mass 1 lb., are attached to the ends of a tight string which passes over a smooth pulley. A mass of 5 lb. is placed in one pan and a mass of 3 lb. in the other. Find the acceleration*

of the system, the force required to support the pulley and the pressure on each scale pan.

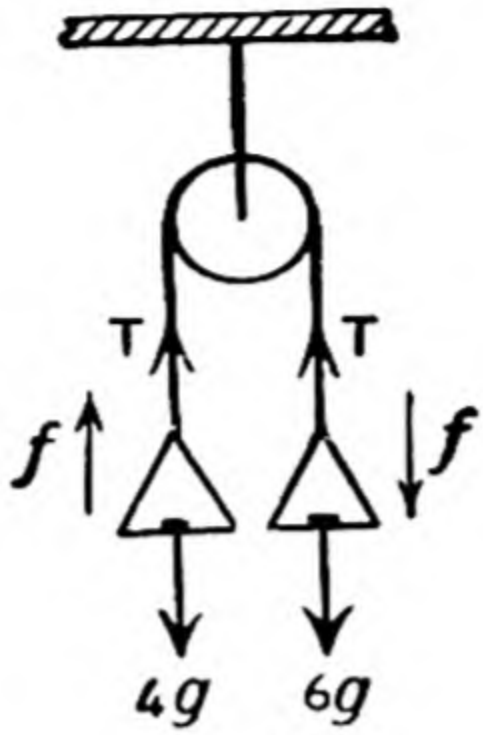


FIG. 382.

Let T poundals be the tension in the string and f the acceleration.

To find these we consider each scale pan with the mass in it as single bodies of masses, 6 and 4 lb.

For the former, $6g - T = 6f$ . . . (i)

For the latter, $T - 4g = 4f$ . . . (ii)

therefore $2g = 10f$, or $f = \frac{g}{5}$ ft./sec.^{2}

Substitute in (i), $T = 6g - \frac{6g}{5} = \frac{24g}{5}$ poundals $= 4\frac{4}{5}$ lb. wt.

The force required to support the pulley is twice this tension, *i.e.*, $9\frac{3}{5}$ lb. wt.

To find the pressure on the scale pans we have to consider the masses as separate from the pans.

If R_1 and R_2 are the pressures of the pans on the 5 and 3 lb. mass, for the 5-lb. mass $5g - R_1 = 5f$,

therefore $R_1 = 5g - 5f = 5g - g = 4g$ poundals $= 4$ lb. wt.

For the 3-lb. mass $R_2 - 3g = 3f$,

therefore $R_2 = 3g + 3f = 3g + \frac{3g}{5} = \frac{18g}{5}$ poundals $= 3\frac{3}{5}$ lb. wt.

EXAMPLES LIII

1. Distinguish between dyne, gram, and gram weight.

A bucket of coal of weight 50 kgm. is hauled up a shaft by a rope. What is the tension in the rope when the bucket is ascending with (*a*) an acceleration of 250 cm./sec.^{2}, (*b*) a uniform speed of 250 cm./sec. ? Give your answers in dynes and in kilograms weight. (*L.M.*)

2. Distinguish between mass and weight.

A body of mass 20 lb. is placed on a smooth horizontal table. Calculate the horizontal force required to move it through 20 ft. in 2·5 sec. from rest with uniform acceleration.

A light string passing over a smooth pulley at the edge of the table is attached to the 20-lb. mass. Determine what mass must be suspended from this string to produce the above motion. (*L.M.*)

3. A mass of 7 lb. is connected by a long cord, which passes over a " frictionless " pulley to a mass of 5 lb. on which is a detachable rider of mass 4 lb. The system is released, and after 4 sec. the 4-lb. rider is removed. What time will elapse from the moment the masses were released to the instant they once more pass the starting position ? It may be assumed that the cord and pulley are without mass. (*L.M.*)

4. A mass of 300 lb. is connected by a smooth cord, which hangs over a smooth peg, to a platform which weighs 100 lb. and supports a mass of 240 lb. The platform is released from rest, and after 30 sec. the mass falls from the platform. Where does this occur, and how much farther does the platform descend before coming to rest ? What time is taken for the platform to return to its original position ? (*L.M.*)

5. A body of mass 200 gm. at rest on a smooth horizontal table is attached by two threads, which pass over smooth pulleys, one at each end of the table, to masses of 150 and 130 gm. respectively, hanging freely. The parts of the thread on the table are aligned and parallel to the table. What is the velocity of the system 3 sec. after it is released from rest ? What is the tension in each thread during the motion ? (*L.M.*)

6. A body rests on a smooth plane, which is inclined at 30° to the horizontal, and is kept stationary by a light string which is parallel to the plane. The string passes over a smooth peg at the top and is attached to a scale pan which hangs freely and contains a mass of 280 gm. When the plane is lowered so that it becomes horizontal, and $246\frac{2}{3}$ gm. are removed from the scale pan, the body moves with an acceleration of 80 cm./sec.^{2} What is the mass (*a*) of the body, (*b*) of the scale pan ? (*L.M.*)

7. A trolley, which moves without friction on a smooth horizontal plane, is attached to a horizontal string, which passes over a light " frictionless " pulley. When a mass of 100 gm. is hung on the string the trolley moves with an acceleration of $61\frac{1}{4}$ cm./sec.^{2}, what acceleration would result if a mass of 500 gm. were added to the trolley ? (*L.M.*)

8. Two masses are attached to the ends of a light string passing over a smooth peg, one of the masses being m gm. Find the value of the second mass in order that the mass m may (*a*) ascend, (*b*) descend with an acceleration of $\frac{g}{a}$ cm./sec.^{2} Find also the tension in the string in each case. (*L.M.*)

9. A mass of m gm. is placed on a smooth horizontal table and is connected by a light string passing over a smooth pulley at the edge of the table to a mass of 50 gm. which hangs vertically. It is observed that the mass m, when released, acquires a velocity of 100 cm./sec. in passing over 100 cm. Determine the mass m.

10. Explain the equation : Force = Mass × Acceleration.

A mass of 7 lb. on a smooth horizontal table is connected by a string passing over a pulley at the edge of the table to a mass of 1 lb. hanging vertically. Determine the tension in the string when the system is moving and the distance the 1-lb. mass descends in 2 sec.

11. A mass of 970 gm. is placed on a smooth horizontal table and a mass of m gm. is attached to it by means of a light string passing over a smooth peg fixed at the edge of the table. When the bodies are allowed to move, it is observed that the mass on the table moves with uniform acceleration through 45 cm. in 3 sec. from rest. Calculate the value of m ($g = 980$ cm./sec.^{2}).

12. Masses of 1 lb. and 1 oz. are connected by a string passing over a smooth pulley. With what acceleration does the greater mass descend? After descending 1 ft. from rest this mass is brought to rest again on reaching the ground. What interval elapses before the string again becomes taut? (*O.C.*)

13. Masses of 125 and 120 gm. are connected by a light string which passes over a vertical pulley. The heavier mass is allowed to descend, and after 3 sec. the velocity observed is 54 cm./sec. What value do these figures as they stand give for g, the acceleration due to gravity, and what corrections are necessary? (*O.C.*)

14. A balloon weighing 768 lb. is descending with a constant acceleration of $1\frac{1}{2}$ ft./sec.^{2} when 54 lb. of ballast is suddenly thrown out. Find the resulting acceleration upwards. If the velocity of the balloon downwards was 5 ft./sec. when the ballast was thrown out, find how far the balloon falls before it begins to rise. (*O.C.*)

CHAPTER XXII

WORK AND ENERGY

208. Energy.

A body is said to possess *energy* if it is capable of doing work. This capacity may be due to a number of causes, *i.e.*, there are several forms of energy. When coal is burnt in the furnace of a steam engine, the energy stored in the coal is largely transferred to the heated steam in the boiler, and this steam, by exerting pressure on the piston, causes the latter to do mechanical work in rotating the wheels of the engine and moving it along.

When a steam engine or internal combustion engine is used to drive a dynamo, we have an example of heat energy being converted first into mechanical energy and then into electrical energy. The electric current from the dynamo can be used to drive an electric motor and the electrical energy is thus reconverted into mechanical energy.

In all cases where energy is transformed there is an apparent wastage, *i.e.*, the amount of the new form of energy is less than the amount of the original form. Experience and careful experiments show that most of this apparent wastage is due to heat energy being transferred to surrounding bodies or radiated into the air.

Thus, in a steam engine only part of the heat energy obtained from the burning coal is transferred to the water and steam in the boiler, while the steam is still hot after moving the piston, *i.e.*, the whole of the heat energy given to it has not been transformed into mechanical work. Friction between the moving parts of the machine used in the conversion also generates heat and so adds to the apparent loss of energy.

209. As the result of experience and many experiments it has been concluded that energy cannot be created or destroyed,

and this conclusion is embodied in what is known as the *Principle of Conservation of Energy*, which may be stated as follows :—

The total energy of a system of bodies is a quantity which cannot be increased or decreased by any action between the parts of the system, although it may be transformed into any of the forms which energy may assume.

In dynamics we are only concerned with purely mechanical energy, which we shall now consider more fully.

Water stored in a reservoir or tank above the level of the ground is capable of doing work in flowing to ground level by driving a water wheel or turbine. Similarly, a raised weight can do work in descending to the level from which it was raised.

In these cases the energy possessed by the body is due to its position, and such energy is called *Potential Energy*.

Another example of this kind of energy is provided by the mainspring of a clock when it is wound up. In the process of unwinding, work is done in driving the mechanism of the clock.

It will be noticed that in all these cases work has to be done on the body to get it into the position in which it possesses potential energy, and in the case of a raised weight we can show that the amount of work required to do this is equal to the energy of the body.

A moving body can do work when it is being brought to rest, and therefore possesses energy by virtue of its motion.

Energy of this kind is called *Kinetic Energy*.

210. Kinetic Energy.

If a mass m is moving with velocity v, we can calculate its kinetic energy by finding the work done in reducing it to rest.

The retardation f, necessary to reduce it to rest in a distance s, is given by

$$0 = v^2 - 2fs,$$

or

$$f = \frac{v^2}{2s}.$$

If P is the force required to produce this retardation, the work done is Ps,

but $$P = mf,$$

therefore $$Ps = m\frac{v^2}{2s}s,$$

therefore $$\text{Work done} = \tfrac{1}{2}mv^2.$$

The kinetic energy of a mass m moving with velocity v is therefore

$$\tfrac{1}{2}mv^2.$$

If m is in lb. and v in ft./sec., the expression $\frac{1}{2}mv^2$ gives the kinetic energy in foot-poundals.

If m is in gm. and v in cm./sec., $\frac{1}{2}mv^2$ gives the energy in ergs.

211. Kinetic Energy of a Falling Body.

If a mass of m lb. is raised vertically through h ft., the work done in raising it is mgh foot-poundals. This is equal to the potential energy of the mass in its raised position.

If the mass is now allowed to fall freely through h ft., the velocity v acquired is given by

$$v^2 = 2gh.$$

The kinetic energy of the mass is then

$$\tfrac{1}{2}mv^2 = mgh,$$

i.e., **the kinetic energy acquired in falling through a height h is equal to the work done in raising the body vertically through this height.**

The potential energy of the body in its raised position is converted into kinetic energy as the body falls, and in any position the gain in kinetic energy is equal to the loss in potential energy. It must be remembered that the potential energy of a body in a raised position, or the work required to raise it, depends entirely on the *vertical* height through which it is raised.

For example, if a body is drawn up the slope of a *smooth*

inclined plane the work required is exactly the same as if it were lifted vertically to the same height.

Similarly, the kinetic energy acquired by a body sliding down the plane is the same as that acquired by falling freely through the same vertical height.

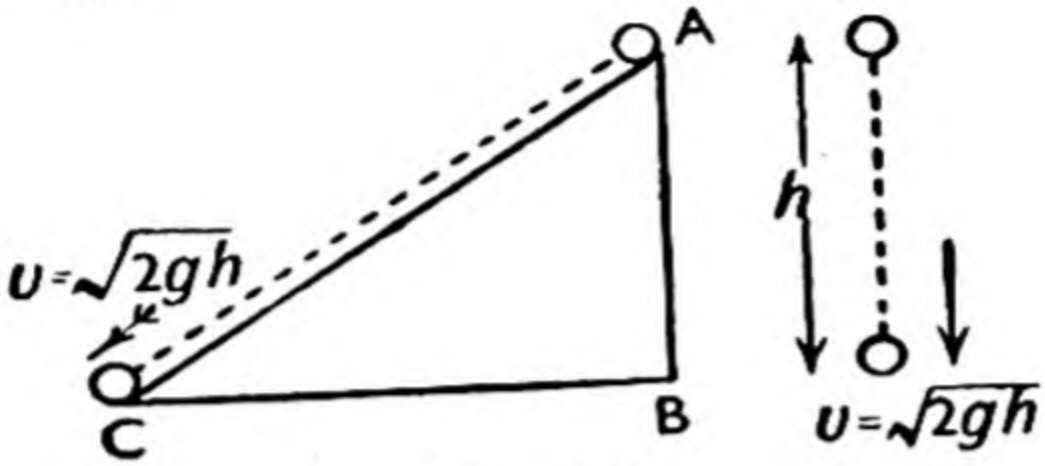


FIG. 383

In the figure, if AB is vertical and BC horizontal, the velocity acquired by a body which slides down AC is equal to that of a body which falls freely through the height AB ($=h$), *i.e.*, $\sqrt{2gh}$ for the gain in kinetic energy, $\frac{1}{2}mv^2$, is the same in each case and is equal to the loss in potential energy, mgh.

Thus $\frac{1}{2}mv^2 = mgh,$

$$v^2 = 2gh \text{ and } v = \sqrt{2gh}.$$

212. Gain in Kinetic Energy Produced by a Constant Force.

If a constant force of P lb. wt. acts on a mass of m lb. and f ft./sec.^{2} is the acceleration produced,

$$Pg = mf \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

If the body was initially at rest and v ft./sec. is the velocity it acquires while moving through s ft.,

$$v^2 = 2fs \text{ and } s = \frac{v^2}{2f} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The work done by the force P is Ps ft. lb. or Psg ft. poundals,

and $$Psg = mf \times \frac{v^2}{2f} = \tfrac{1}{2}mv^2.$$

Hence the work done by the force is equal to the gain in kinetic energy of the body.

If the body was initially travelling at u ft./sec., equation (ii) becomes

$$v^2 = u^2 + 2fs, \text{ or } s = \frac{v^2 - u^2}{2f},$$

therefore $$Pgs = mf \times \frac{(v^2 - u^2)}{2f} = \tfrac{1}{2}m(v^2 - u^2).$$

Hence the work done by P is equal to the gain in kinetic energy of the body.

If we know the initial and final velocities of a body and the distance travelled while a constant force acts on it, this relation enables us to calculate the force without finding the acceleration. If the force is not constant we can obtain in this way the constant force which would produce the same increase in kinetic energy, and the value of this force may be taken as the *average* value of the variable force which acts.

Example (1).—*A lorry of weight 2 tons is travelling at 30 m.p.h. on a level road. If it is brought to rest in 20 yd., what is the average force of resistance acting on the lorry?*

$$\text{Initial kinetic energy} = \frac{2 \times 2240 \times 44^2}{2 \times 32} = 70 \times 44^2 \text{ ft. lb.}$$

Since the whole of this energy is lost, the work done by the resistance R is equal to this number of foot-pounds,

therefore $$60R = 70 \times 44^2,$$

$$R = \frac{70 \times 44^2}{60} = 2258{\cdot}7 \text{ lb. wt.}$$

Example (2).—*A bullet of mass $\frac{1}{2}$ oz. travelling horizontally at 1,200 ft./sec. strikes a block of wood of thickness 1 ft. If it experiences an average resistance of 600 lb. wt., with what velocity does it emerge?*

Let v ft./sec. be the velocity with which the bullet emerges.

$$\text{Initial kinetic energy} = \tfrac{1}{2} \times \frac{1200^2}{32 \times 32} \text{ ft. lb.}$$

$$\text{Final kinetic energy} = \tfrac{1}{2} \times \frac{v^2}{32 \times 32} \text{ ft. lb.}$$

Work done by force of resistance $= 600 \times 1$ ft. lb.

Since work done by force = Change in kinetic energy,

$$600 = \frac{1200^2}{64 \times 32} - \frac{v^2}{64 \times 32},$$

therefore

$$\frac{v^2}{64 \times 32} = \frac{1200^2}{64 \times 32} - 600,$$

$$v^2 = 1200^2 - 600 \times 32 \times 64$$

$$= 1200(1200 - 16 \times 64)$$

$$= 9600(150 - 128) = 211200,$$

$$v = 459{\cdot}6 \text{ ft./sec.}$$

Note.—If the bullet were moving vertically we should have to allow for the change in potential energy as well as the change in kinetic energy.

EXAMPLES LIV

1. Calculate the kinetic energy of the mass in the following cases :—

(*a*) A mass of 5 lb. travelling at 10 ft./sec.
(*b*) A mass of $\frac{1}{2}$ oz. travelling at 1,000 ft./sec.
(*c*) A mass of 8 gm. travelling at 16 cm./sec.
(*d*) A mass of 2 tons travelling at 15 m.p.h.

2. (*a*) A mass of 4 lb. travelling at 20 ft./sec. is reduced to rest by the action of a constant force in a distance of 16 ft. Calculate the value of the force in poundals.

(*b*) A force of 10-lb. wt. acts on a mass of 5 lb. initially at rest. Calculate the distance through which the force acts if the final velocity of the mass is 24 ft./sec.

3. A bullet of mass $\frac{1}{2}$ oz. strikes a target with a velocity of 1,600 ft./sec. and penetrates 8 in. Calculate the average force of resistance exerted by the target on the bullet.

4. A mass of 8 lb. falls freely through a height of 25 ft. What kinetic energy has it acquired (*a*) in foot-poundals, (*b*) in foot-pounds weight ?

5. A body of mass 2 lb. is acted upon by a constant force which increases its speed from 15 to 25 ft./sec., while the body travels 20 ft. Calculate the magnitude of the force.

6. A truck of mass 2 tons is travelling at 12 ft./sec. on a horizontal track. If the truck is brought to rest in 10 yd. by a constant resistive force, find the value of this force in tons weight.

7. A bullet of mass $\frac{1}{2}$ oz. strikes a plank of wood of thickness 3 in. when travelling at 800 ft./sec. If its velocity is halved by the resistance of the plank, calculate the average value of this resistance in pounds weight.

8. A projectile weighs 24 lb. and its muzzle velocity just as it leaves the gun is 1,500 ft./sec. Calculate the average force exerted on it while in the barrel of the gun, assuming that it acts through a distance of 4 ft. 6 in.

9. What is meant by the term energy ? Distinguish between kinetic and potential energy. What is the kinetic energy of a bullet weighing $\frac{1}{4}$ oz. and travelling with a speed of 1,200 ft./sec. ? What fraction of its

energy has it lost after passing through a block of wood 15 in. thick which offers an average resistance of 400 lb. wt. ? (*L.M.*)

10. Show that when a body is falling freely under the action of gravity, the loss of potential energy when it has fallen a vertical distance h cm. from rest is equal to its gain of kinetic energy.

By making use of this fact, calculate how far a body of mass m gm. must fall vertically in order to acquire a velocity of 490 cm./sec. [No credit will be given for any other method.] (*L.M.*)

11. Find the loss of kinetic energy of a bullet weighing 175 gr. which enters a bank of earth with a velocity of 2,440 ft./sec. and leaves it with a velocity of 900 ft./sec. If the bank is 2 ft. 6 in. thick, find the average resisting force exerted on the bullet, being given that 7,000 gr. = 1 lb. (*L.M.*)

12. A rifle bullet loses 1/20th of its velocity in passing through a plank. Show that the bullet loses $\frac{39}{400}$ of its original kinetic energy.

13. What would be the velocity of the bullet in question 12 after it has passed through two such planks? How many planks are needed to reduce it to rest?

14. A vessel of mass 6,000 tons is moving at 20 ft./sec. when all power is shut off. If it travels 100 yd. before it comes to rest, calculate the average resistance to its motion.

15. A bullet weighing $\frac{1}{4}$ oz. has a velocity of 1,500 ft./sec. What is its kinetic energy? How far will it penetrate a fixed wooden block if the latter offers a constant resistance of 240 lb. wt. to the motion of the bullet? (*B.*)

16. What are kinetic energy, potential energy and work? In what units are these quantities measured? A train weighing 200 tons is moving at a speed of 100 m.p.h. when the steam is shut off and the brakes applied. The train stops after travelling for half a mile. What is the resistance in pounds weight due to the brakes, air resistance, etc.? ($g = 32$ ft./sec.^{2}) (*O.*)

17. How much kinetic energy is acquired by a mass of 2 lb. falling from rest through 50 ft.? How much more kinetic energy is acquired in falling through the next 50 ft.? (*O.C.*)

18. A car weighing 1 ton and travelling at 50 m.p.h. was brought to rest by applying brakes in 231 ft. What was the frictional force in pounds weight exerted on the tyres by the road, assuming that it is uniform? (*O.C.*)

19. What is meant by "The Conservation of Energy"?

A body of weight 2 lb. is lifted from the ground to a height of 10 ft. and then allowed to fall. Illustrate your answer to the first part of the question by considering this body (*a*) when it is 10 ft. above the ground, (*b*) when it has fallen 4 ft., (*c*) just before it reaches the ground, and (*d*) after it has struck the ground. (*J.*)

20. Explain what is meant by energy.

A mass of 10 lb. falls from rest under the acceleration of gravity. Find the gain in kinetic energy in each of the first 3 sec. (*L.M.*)

21. Explain the relation between force applied and work done. A motor car weighing 3,000 lb. and travelling on a level road at 30 m.p.h. is brought to rest by the brakes in 20 yd. What is the average force applied? (*L.M.*)

213. When several forces are acting on a body its motion is only changed if these forces have a resultant. The change in kinetic energy is that produced by the resultant force and is equal to the work done by that force. In the case of a locomotive moving with uniform speed the driving force is equal to the resistance to motion and the work done by the driving force is equal to the work done against the resistance.

If the driving force is greater than the resistance to motion, the engine accelerates, and the work done by the driving force is partly done against the resistance and the remainder of the work is used in increasing the kinetic energy of the engine.

Example (1).—*A motor car of mass* 1 *ton accelerates from rest to a speed of* 30 *m.p.h. in a distance of* 50 *yd. against a resistance to motion of* 20 *lb. wt. What is the average driving force? Using this average force, what is the greatest horse-power developed by the engine?*

Let P lb. wt. be the average driving force.

Resultant force on the car $= (P - 20)$ lb. wt.

Work done by resultant force $= (P - 20)150$ ft. lb.

Gain in kinetic energy of car $= \frac{2240 \times 44^2}{2 \times 32} = 35 \times 44^2$ ft. lb.,

therefore $(P - 20)150 = 35 \times 44^2$,

$P - 20 = 451 \cdot 7$,

$P = 471 \cdot 7$ lb. wt.

The greatest horse-power developed on the assumption that P is constant is when the car is moving at the greatest speed.

At 44 ft./sec. the work per second is $471 \cdot 7 \times 44$ ft. lb.,

therefore $\text{H.P.} = \frac{471 \cdot 7 \times 44}{550} = 37 \cdot 7$.

Example (2).—*A mass of* 2 *tons falls to the ground from a height of* 16 *ft. and penetrates* 1 *ft. before it is brought to rest. What is the average resistance of the ground?*

Let R tons wt. be the average resistance of the ground.

The resultant force acting vertically upwards on the mass during penetration is $(R - 2)$ tons wt.

The speed of the mass on striking the ground is

$\sqrt{2 \times 32 \times 16} = 32$ ft./sec.,

therefore Kinetic energy $= \frac{1}{2} \times \frac{2 \times 32^2}{32} = 32$ ft. tons.

Work done by resultant force = Loss of kinetic energy,

therefore $(R - 2)1 = 32$

$R = 34$ tons wt.

Example (3).—*A train travelling at 15 m.p.h. reaches the top of an incline of 1 in 100 when steam is shut off and the train runs down the incline against resistances amounting in all to 2 tons wt. If the mass of the train is 500 tons, find its velocity after it has travelled ¼ mile down the incline.*

Since the components of force perpendicular to the incline do no work they do not affect the motion parallel to the track.

The forces acting on the train parallel to the track are :

(1) The component of the weight of the train $= \frac{500}{100} = 5$ tons wt.

(2) The resistance to motion $= 2$ tons wt.

The resultant force accelerating the train is therefore 3 tons wt.

If v ft./sec. is the speed of the train after travelling ¼ mile or 1,320 ft., the gain in kinetic energy is

$$\tfrac{1}{2} \times \frac{500 \times v^2}{32} = \tfrac{1}{2} \frac{500 \times 22^2}{32} \text{ ft. tons.}$$

The work done by the resultant force is 3×1320 ft. tons,

therefore $$\frac{250v^2}{32} - \frac{250 \times 22^2}{32} = 3960.$$

therefore $$v^2 = \frac{3960 \times 32}{250} + 22^2 = 506{\cdot}9 + 484 = 990{\cdot}9.$$

$$v = 31{\cdot}5 \text{ ft./sec., or } 21{\cdot}5 \text{ m.p.h.}$$

214. Horse-power Transmitted during Accelerated Motion.

In order to produce an accelerated motion in a body it is necessary for a resultant force to act upon it, and therefore the motive force must be greater than the resistance to motion, *i.e.*, greater than the force required to maintain the body in motion at any uniform speed. If P is the motive force the transmitted horse-power $= \frac{Pv}{550}$, P being in pounds weight and v in feet per second, and it therefore follows that at any given speed the horse-power transmitted during accelerated motion is greater than that for uniform motion. Since the resultant force required to maintain an acceleration f ft./sec.^{2} in a body of mass M lb. is Mf poundals, the additional power required above that required for uniform speed v ft./sec. is

$$\frac{Mfv}{550g} \text{ horse-power.}$$

A locomotive working at a given horse-power may be capable of producing an accelerated motion in a train when the speed is 20 m.p.h., whereas at a speed of 30 m.p.h. it may be only capable of maintaining a uniform velocity, the transmitted horse-power being the same in both cases. The maximum speed at which a train can travel when the horse-power transmitted is the *uniform* speed of the train for which the whole of the power is expended in just overcoming the resistance to motion, and in this case the driving force is equal to the total resistance to motion.

Example.—*A train of weight* 400 *tons ascends an incline of* 1 *in* 100 *at a uniform speed of* 30 *m.p.h., the resistance to motion parallel to the track being* 15 *lb. wt./ton. What is the horse-power transmitted, and with what acceleration would the train travel at the particular speed of* 20 *m.p.h. on a horizontal track against the same resistance to motion if the horse-power transmitted is the same as before?*

Let P lb. wt. be the pull of the engine.

Since the motion is uniform, the resultant force on the train is zero.

$$P = 400 \times 15 + \text{Component of weight of train parallel to track}$$

$$= 6000 + 400 \times 2240 \times \frac{1}{100} \text{ lb. wt.}$$

$$= 14960 \text{ lb. wt.}$$

Horse-power transmitted at the uniform speed 44 ft./sec.

$$= \frac{14960 \times 44}{550} = 1196{\cdot}8.$$

Let Q lb. wt. be the force exerted by the engine on the level track. At the particular speed of 20 m.p.h. the horse-power transmitted must be the same as before,

$$\frac{Q \times 20 \times 88}{550 \times 60} = \frac{14960 \times 44}{550}.$$

$$Q = 22440 \text{ lb. wt.}$$

The resultant force on the train parallel to the track is

$$(22440 - 6000) = 16440 \text{ lb. wt.}$$

If f ft./sec.^{2} is the acceleration of the train,

$$16440 \times 32 = 400 \times 2240f,$$

therefore

$$f = \frac{1644 \times 32}{40 \times 2240} = \frac{1644}{40 \times 70} = 0{\cdot}587 \text{ ft./sec.}^2$$

EXAMPLES LV

1. A motor car of mass 15 cwt. accelerates from rest to a speed of 30 m.p.h. in a distance of 22 yd. against resistances to motion amounting to

40 lb. wt. Calculate the average force exerted on the car at the driving wheels.

2. A body of mass 50 lb. falls through a height of 9 ft. and penetrates 6 in. into the ground. Calculate the average force of resistance of the ground on the mass during the penetration.

3. A blacksmith strikes an iron bar 25 times per minute and brings the sledge-hammer to rest on the bar each time. If the striking velocity is 32 ft./sec. and the sledge-hammer weighs 14 lb., find his rate of working in horse-power.

4. Define the foot-pound.

When a truck of mass 25 cwt. reaches the foot of an incline which rises 1 ft. vertically for each 100 ft. along the slope, it has 2,800 ft. lb. of kinetic energy. If the frictional resistance to motion is 50 lb. wt., calculate how far up the slope the truck will travel before coming to rest. (*L.*)

5. A train of mass 200 tons is moving against a resistance of 10 lb. wt. per ton with a velocity of 15 m.p.h. The steam is then shut off; after travelling 80 ft. it comes to an incline of 1 in 20. How far along the incline will it travel before coming to rest? (The resistance parallel to the track may be assumed to be the same when travelling up the incline as on the level.)

6. An engine of weight 100 tons pulls carriages of total weight 200 tons up an incline of 1 in 80. The track resistance is equivalent to 100 lb. wt. for the engine and 200 lb. wt. for the carriages. If the train is travelling at a constant speed of 40 m.p.h., find the tension in the coupling between engine and carriages, and the horse-power at which the engine is working. (*B.*)

7. Define (*a*) work, (*b*) power. In what unit is work measured?

A motor car of mass 25 cwt. climbs a gradient of 1 in 20 at 30 m.p.h. Neglecting friction, find the horse-power. (1 H.P. = 550 ft. lb./sec.) (*C.*)

8. Define work and power, and state how each is measured.

When a train is travelling at a uniform speed of 60 m.p.h., the engine exerts a pull of $\frac{3}{4}$ ton wt. What is the horse-power developed by the engine? If the weight of the train is 140 tons, find the distance it would travel when the steam is shut off, assuming the retarding forces remain unchanged. (*L.M.*)

9. Define work and horse-power.

A car weighing 1,200 lb. accelerates on the level uniformly from rest to 20 m.p.h. in 4 sec. If frictional resistance is equivalent to 36 lb. wt., what is the average horse-power employed? (*G.S.*)

10. Explain the terms work, horse-power.

A motor car travelling on a level road at 20 m.p.h. maintains the same speed up an incline of 1 in 12. What extra horse-power is being used if the car weighs 3,000 lb.? (*G.S.*)

11. What horse-power is needed to draw a train of mass 200 tons up an incline of 1 in 224 at a speed of 15 m.p.h. when the frictional resistance to motion is 10 lb. wt. per ton? (*O.C.*)

12. Define work and horse-power.

A motor car of mass 25 cwt. is running at 25 m.p.h., and the resistance to its motion due to friction is 150 lb. wt. Find the work done by the engine in moving the car 440 yd. (*a*) along the level, (*b*) up an incline which rises 1 ft. for every 30 ft. measured along the plane. Calculate also the horse-power required in each case. (*G.S.*)

13. The resistance to a motor car weighing 1 ton is $(50 + 0{\cdot}04v^2)$ lb. wt. where v is the speed in miles per hour. Find the horse-power necessary to drive the car up a hill of 1 in 20 at a steady speed of 20 m.p.h. (*O.C.*)

14. Explain what you understand by potential energy and power. In what units are they measured in the c.g.s. system ?

A train, including the engine, weighs 100 tons. The resistances to motion on a level track are equivalent to a retarding force of 15 lb. wt. per ton. Find (*a*) the work done against this resistance when the train travels 1 mile, (*b*) the greatest speed at which the train can travel if the engine develops 120 H.P. (1 H.P. = 550 ft. lb./sec.) (*O.C.*)

15. Define kinetic energy.

A trolley starting from rest runs 150 yd. down an incline of 1 in 20 in 42 sec. If the trolley weighs 800 lb. and moves with uniform acceleration, calculate the changes in potential energy and in kinetic energy. Account for the difference between them. (*G.S.*)

16. Define kinetic energy and potential energy. State how each is measured and the unit in which it is expressed.

A toboggan weighing 250 lb. starts from rest at the top of a slope, accelerating uniformly, and acquiring a speed of 25 m.p.h. after it has run 400 ft. If the slope is such that it descends 1 ft. vertically for every 10 ft. of the length of the slope, calculate the energy expended in overcoming friction and air resistance. (*G.S.*)

17. State the principle of conservation of energy, and show that for a body falling freely under gravity the sum of its kinetic and potential energies is constant.

A body slides from rest down a plane inclined at 30° to the horizontal. After sliding a distance of 20 ft. down the plane it is found to be moving with a velocity of 16 ft./sec. How much energy has been lost in overcoming friction ? (*C.*)

18. Assuming that the resistance to motion of a train on the level is $0{\cdot}09v^2$ lb. wt./ton where v is the speed in miles per hour, find the horse-power required to draw a train of mass 300 tons up an incline of 1 in 150 at a uniform speed of 20 m.p.h. (*O.C.*)

215. Example (1).—*A water turbine is driven by a waterfall in which 12,000 gals. of water per minute fall through 18 ft. If 40 H.P. is developed, what fraction of the available energy is utilised ?* (*1 gal. of water weighs 10 lb.*)

The loss of potential energy per minute due to the water falling 18 ft. is $12000 \times 10 \times 18$ ft. lb.

The energy supplied per minute by the turbine is 40×33000 ft. lb.,

therefore $$\text{Fraction utilised} = \frac{40 \times 33000}{120000 \times 18} = 0{\cdot}61.$$

Example (2).—*A pump whose efficiency is 75 per cent. is ejecting 6 tons of water per minute with a speed of 60 ft./sec. If the water is ejected at the same level at which it is taken in when at rest, what is the useful horse-power developed and what horse-power is being supplied to the pump ?*

The kinetic energy of the water delivered per minute is

$$\frac{6 \times 2240 \times 30^2}{2 \times 32} \text{ ft. lb.}$$

Since the water is initially at rest this energy has been provided by the pump, and the horse-power developed is therefore

$$\frac{6 \times 2240 \times 30^2}{2 \times 32 \times 33000} = 5{\cdot}73 \text{ H.P.}$$

This is 75 per cent. or three-quarters of the horse-power supplied to the pump, which is therefore

$$\frac{4}{3} \times 5{\cdot}73 = 7{\cdot}64 \text{ H.P.}$$

Note.—If the water is taken in at a lower level than that at which it is delivered the horse-power of the pump would have to be greater. Work would have to be done in raising the water as well as in imparting kinetic energy to it.

Example (3).—*A mine shaft* 600 *ft. deep has a cross-section of* 100 *sq. ft. and contains water to a depth of* 240 *ft. If this water is received from the mine in* 5 *hr. and is delivered horizontally at the ground-level with a speed of* 40 *ft./sec., calculate* (a) *the potential energy given to the water*, (b) *the kinetic energy given to the water*, (c) *the average horse-power of the pump.* (1 *cu. ft. of water weighs* $62\frac{1}{2}$ *lb.*)

(*a*) The centre of gravity of the water is 120 ft. above the bottom of the shaft or 480 ft. below ground-level. The work done in raising the water is equal to the work required to raise a body of mass equal to that of the water from a depth of 480 ft.

$$\text{Mass of water raised} = 240 \times 100 \times 62{\cdot}5 \text{ lb.}$$

Work done, or gain in potential energy

$$= 2400 \times 625 \times 480 = 720{,}000{,}000 \text{ ft. lb.}$$

(*b*) Gain in kinetic energy $= \frac{1}{2} \times 2400 \times 625 \times 40^2$ ft.-poundals

$$= \frac{1200 \times 625 \times 1600}{32}$$

$$= 37{,}500{,}000 \text{ ft. lb.}$$

(*c*) The work done by the pump in 5 hr. is

$$72{,}000{,}000 + 37{,}500{,}000 = 757{,}500{,}000 \text{ ft. lb.,}$$

therefore $$\text{Work per minute} = \frac{757500000}{5 \times 60} \text{ ft. lb.,}$$

$$\text{Horse-power} = \frac{757500000}{300 \times 33000} = \frac{7575}{99} = 76{\cdot}5.$$

EXAMPLES LVI.

1. Calculate the work done in raising 10,000 gals. of water through a height of 60 ft. What additional work must be done if the water is to be delivered with a velocity of 48 ft./sec. ? (1 gal. of water weighs 10 lb.)

2. Calculate the horse-power of a pump required to raise 1,000 gals. of water per minute through a height of 20 ft. and to deliver the water

at a speed of 16 ft./sec., (*a*) assuming the pump to be 100 per cent. efficient, (*b*) assuming the efficiency of the pump to be 75 per cent.

3. An engine working at 2 H.P. pumps water through a height of 30 ft., the efficiency of the pumping machinery being 60 per cent. How much water is pumped in 1 hr.? (1 gal. of water weighs 10 lb.) (*O.C.*)

4. A mine shaft, 1,000 ft. deep and having a cross-section of 125 sq. ft., is filled with water to a depth of 400 ft. The water is pumped out by an engine of 200 H.P. and has a velocity of 50 ft./sec. on discharge at the surface. Find (*a*) the increase in potential energy of the water, (*b*) the kinetic energy imparted to the escaping water, (*c*) the time occupied in pumping. (1 cu. ft. of water weighs 62·5 lb.) (*L.M.*)

5. Define the terms force, work and kinetic energy, and state the units in which they are measured in the c.g.s. system.

An engine develops 10 H.P. and is used to pump water from a well. If the engine is 70 per cent. efficient, how many gallons of water can it raise 60 ft. in 1 min. and project the water horizontally at a speed of 30 m.p.h.? (1 gal. of water weighs 10 lb.) (*L.M.*)

6. What is meant by power?

A 5 H.P. engine is used to drive a pump which delivers 50 gals. of water a minute in a jet. Assuming that no energy is wasted, find the speed of the water in the jet. (1 H.P. = 550 ft. lb./sec.) (*L.M.*)

216. The Simple Pendulum.

A simple pendulum consists of a small heavy body, usually of lead or brass, which is suspended by a light thread from a fixed point. If a pendulum is made to oscillate it describes one *swing* in passing from one extreme position on one side of the vertical through the point of support to the next extreme position on the other side of the vertical. It describes one *oscillation* between two consecutive instants at which it is passing through the vertical position *in the same direction*. The extent of one half swing is called the *amplitude* of the oscillation.

If a pendulum is set in oscillation, the thread never making with the vertical an angle greater than about 10°, we may show that the time of oscillation remains constant and is independent of the amplitude of the motion. By means of a stop-watch take the time for groups of twenty-five oscillations, starting and stopping the watch as the pendulum passes through the vertical position. It will be observed that these intervals of time are equal, and therefore it may be concluded that the time for one oscillation of the pendulum is constant. Although the pendulum of a clock is not a simple pendulum, it possesses this same property which enables it to govern the rate at which wheels in the

mechanism of the clock rotate. It has been stated that this property of a pendulum was discovered by Galileo. It is said that in 1641, the year before his death, although blind, he dictated to his son a description of a pendulum clock including the escapement device. The clock was later constructed by Huyghens, an astronomer and a contemporary of Sir Isaac Newton.

The length of a simple pendulum is the distance from the point of suspension to the centre of gravity of the bob. It may readily be shown that the greater the length of the pendulum the longer is its time of oscillation. To determine the law connecting the time of oscillation and the length of a pendulum, the following experiment may be performed :—

EXPERIMENT.—The thread of a simple pendulum is supported by gripping it between two pieces of cork. It is then easy to alter the length

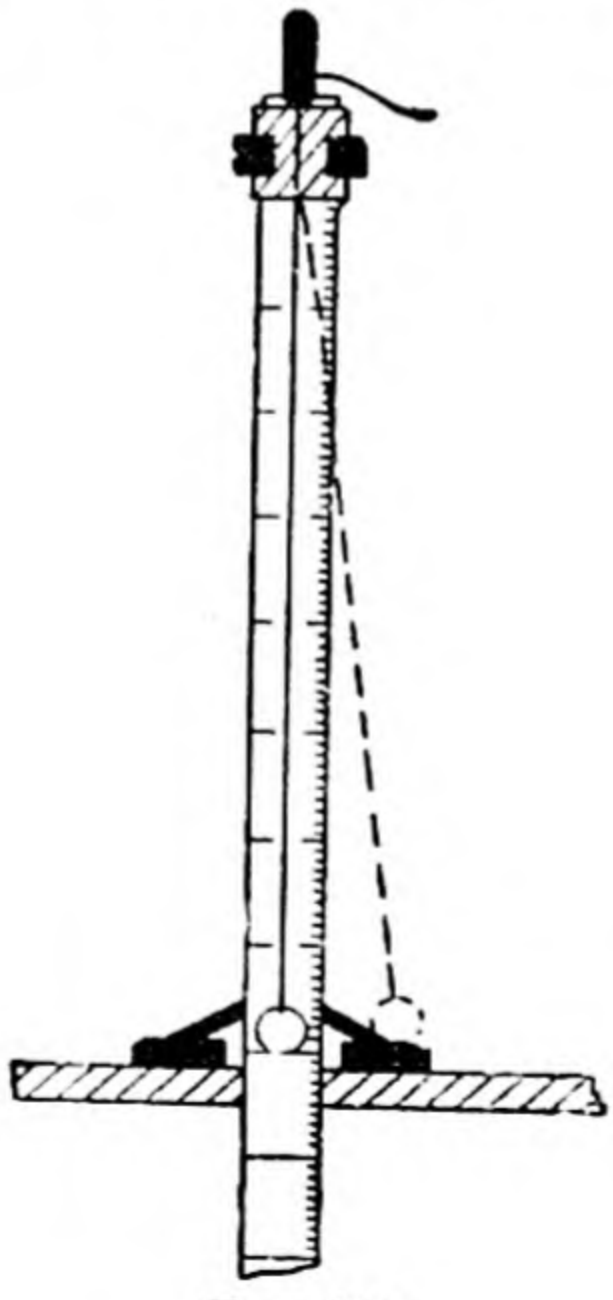


FIG. 384.

T.	l.	$\sqrt{l}$.	$\frac{T}{\sqrt{l}}$.
20·0	25	5	4·00
24·0	36	6	4·00
27·9	49	7	3·99
32·1	64	8	4·03
35·9	81	9	3·99
40·0	100	10	4·00
44·0	121	11	4·00
48·1	144	12	4·01

FIG. 385.

of the pendulum by pulling on the thread. Behind the pendulum a scale divided in centimetres is arranged in a vertical position. If a sphere of lead is used for the bob its diameter should first be determined, for it is convenient to measure the distance from the lowest point of the bob to the point of suspension, and the length of the pendulum may then be found. Using a pendulum of length 25 cm., find the time for 20 oscillations by means of a stop-watch. Repeat for another 20 oscillations. The two times should agree accurately. If they do not agree the experiment should be repeated. In a similar manner, find the times for 20 oscillations of pendulums of lengths 36, 49, 64, 81, 100, 121, 144 cm. Tabulate the results as shown in Fig. 385.

T sec. represents the time for 20 oscillations and l cm. the length of the pendulum. In column (4) the ratio $\frac{T}{\sqrt{l}}$ is calculated, and it is seen to be of constant value. Taking the average value of this constant,

$$T = 4{\cdot}00\sqrt{l},$$

hence the period of oscillation of the pendulum t is given by

$$t = \frac{T}{20} = 0{\cdot}200\sqrt{l}.$$

The period of oscillation of a simple pendulum varies directly as the square root of its length.

The values of l have been made perfect squares of simple numbers for convenience in calculation, but if the experiment is repeated using any values of the length, the ratio, $\frac{t}{\sqrt{l}}$ will be seen to have the same constant value.

If the values of t are plotted against a base, $\sqrt{l}$, the points lie on a straight line, and the gradient of this graph gives the best value of the ratio $\frac{t}{\sqrt{l}}$.

Now it may be shown that for a simple pendulum

$$t = 2\pi\sqrt{\frac{l}{g}},$$

and we may therefore compare the value of the ratio $\frac{t}{\sqrt{l}}$, determined experimentally with the value $\frac{2\pi}{\sqrt{g}}$ obtained from the formula. If l is in centimetres, g is 981, and if l is in feet, g is 32·15.

EXAMPLES LVII

1. If the bob of a simple pendulum is drawn aside and let go when the string is taut so that its vertical fall is 3 in., what is its velocity in the lower position? Neglect air resistance. (*C.W.B.*)

2. The bob of a simple pendulum is of mass 4 lb. If it passes through its lowest position with kinetic energy equal to 72 ft.-poundals, what is the greatest height it attains above this position?

3. Describe an experiment you would perform to verify the law connecting the time of oscillation of a simple pendulum and its length.

4. The time of oscillation of a simple pendulum of length 3·26 ft. is 2 sec. Without using the full formula for the periodic time of a simple pendulum, calculate the time of oscillation of a pendulum of length 2 ft.

5. Given that the length of a second's pendulum is 100 cm., determine without using the *full* formula (*a*) the time of swing of a pendulum 64 cm. long, (*b*) the length of a pendulum to beat periods of 1·6 sec. (*S.C.*)

6. Describe fully how you would investigate the relation between the length of a simple pendulum and its periodic time. State the result you would expect, and show clearly how you would deduce it from your observations. Calculate the acceleration due to gravity at a place where a simple pendulum of length 100 cm. makes 200 complete oscillations in 401 sec. (*J.*)

7. Find an expression for the maximum kinetic energy of the bob mass M of a simple pendulum of length l cm. due to a horizontal displacement of d cm., and show that for small displacements the maximum velocity is approximately proportional to the displacement. (*O.C.*)

217. The Energy of a Pendulum.

The bob of a simple pendulum is pulled aside, the string remaining taut, and is then released, the pendulum swinging to and fro.

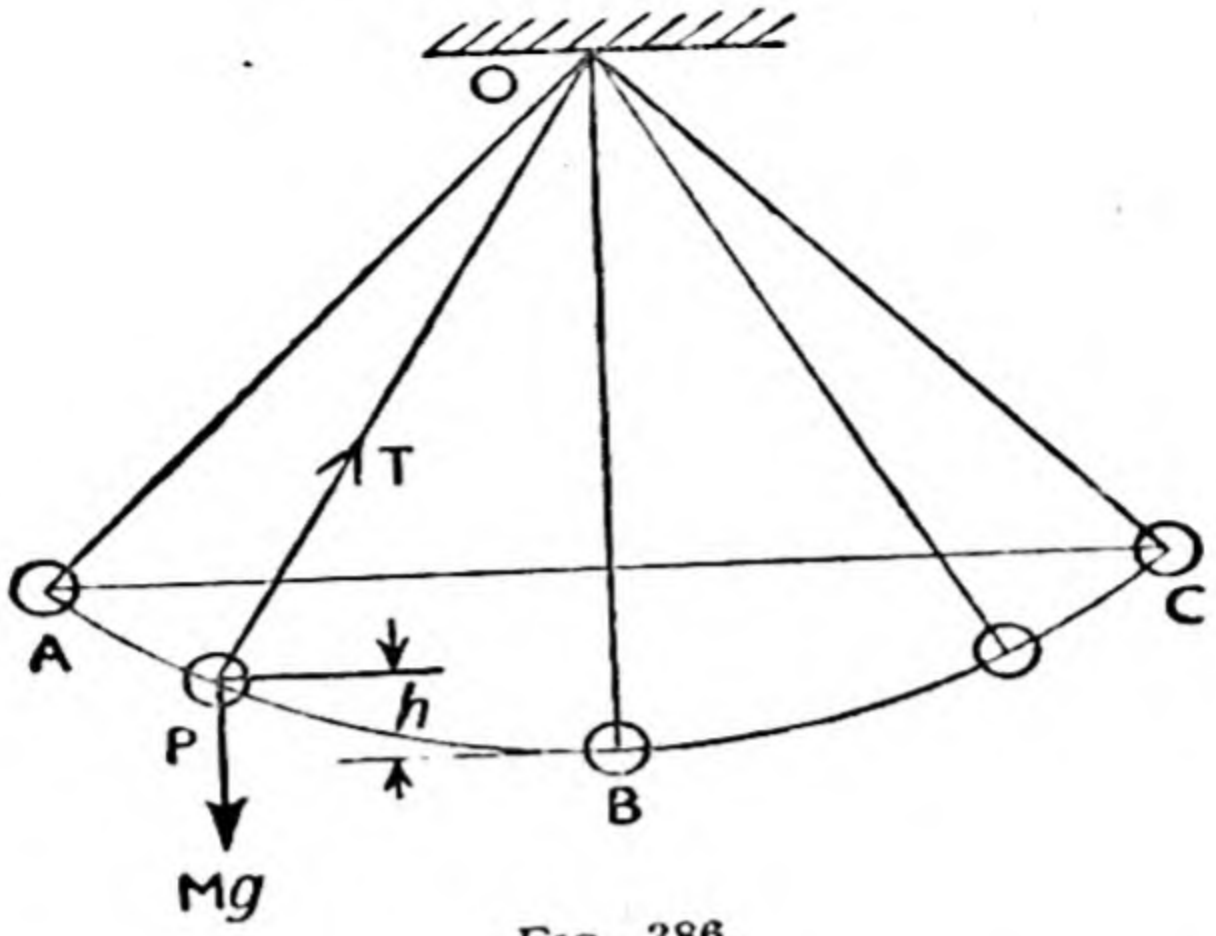


FIG. 386.

Suppose that there is negligible air resistance and that there is no resistance to motion at the point of support O.

The forces acting on the bob at any position such as OP are :

(i) The weight of the bob ;
(ii) The tension in the string OP.

During the motion of the pendulum the bob is always moving in a direction perpendicular to that in which the tension acts, and therefore the tension in the string does no work.

When the bob moves downwards through a vertical distance h the weight force Mg acts through a distance h in its own direction and therefore does work measured by Mgh. Mgh also measures the loss in potential energy of the bob when it moves vertically downwards a distance h. Since the work done by gravity has been converted into an equal amount of kinetic energy, it follows that the loss of potential energy is equal to the gain of kinetic energy. Again, when the bob is rising, the gain in potential energy is equal to the corresponding loss in kinetic energy.

Suppose that the bob starts its motion from a position of rest a height H above the lowest position it takes up during the subsequent motion, then in the first position A the kinetic

energy is zero, and if the potential energy is taken as zero at the lowest position B, at A the potential energy of the bob is MgH. Since at B the potential energy is zero and the whole of the energy of the pendulum is kinetic energy, $MgH = \frac{1}{2}Mv^2$ where v is the velocity of the pendulum at B. This transformation of energy from potential to kinetic is exactly reversed in the motion, which follows from B to C, and therefore the bob should again rise to a height H above B.

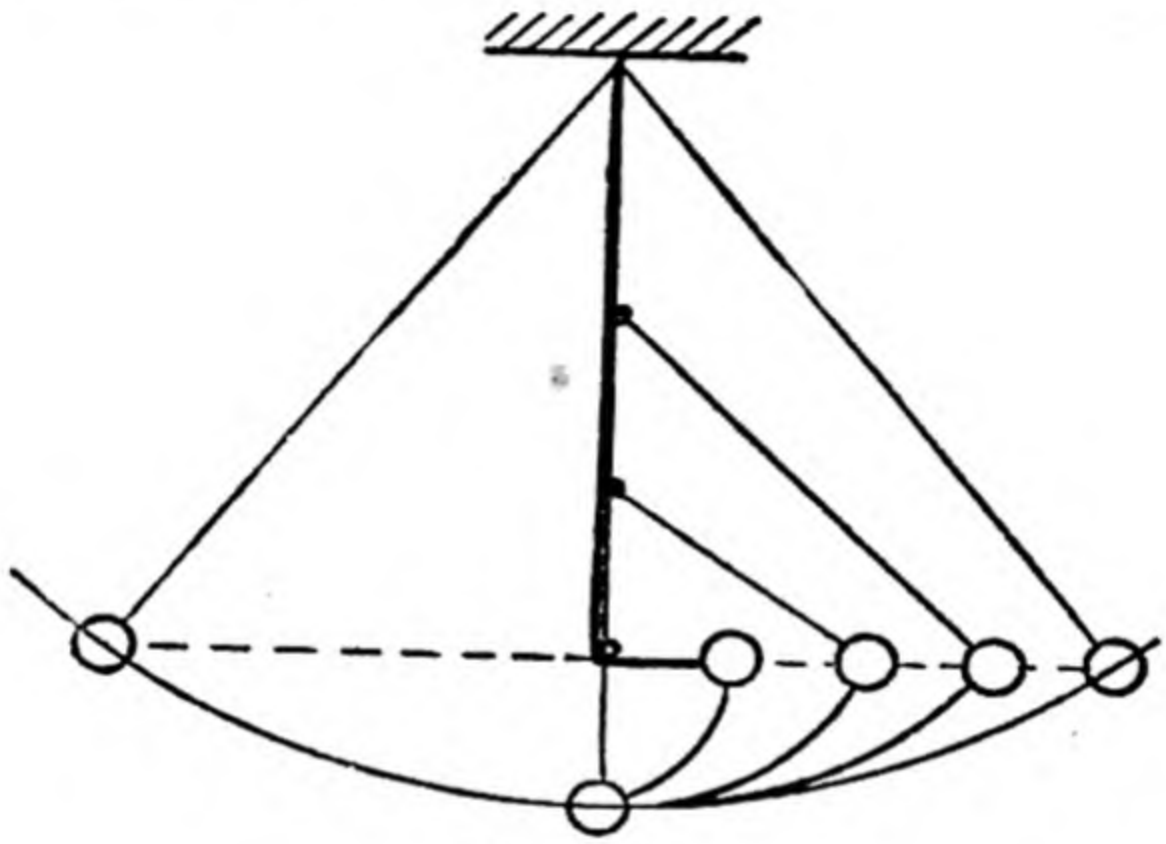


FIG. 387.

A little thought will show that if a thin, smooth peg interrupts the motion of the string at any position, as shown in the figure, the bob will continue to reach the same height as before. Taking the peg to have negligible thickness so that the string does not shorten when it wraps itself round the peg, consider the subsequent motion of the bob, placing the peg in various different positions.

Attach a piece of lead to a light string and swing the lead in a circle, allowing the string finally to wrap itself round your finger. From theoretical considerations only, how should the kinetic energy of the lead when it reaches your finger compare with its total energy when the string first began to wrap itself round your finger ?

218. Units of Heat Energy.

We have seen that heat is a form of energy. Just as there are units of mechanical energy, such as the foot-pound and the centimetre-gram, in terms of which an amount of mechanical energy may be measured, so there are units of heat energy in terms of which a quantity of heat may be expressed. Since equal masses of different substances require different amounts of heat to raise their temperatures by equal amounts, it is necessary to choose a standard substance. Water is chosen as a standard substance, and the following are definitions of units of heat energy in common use .—

One British Thermal Unit (B.Th.U.) is the amount of heat required to raise the temperature of 1 *lb. of water* 1° *F.*

One Centigrade Heat Unit (C.H.U.) is the amount of heat required to raise the temperature of 1 *lb. of water* 1° *C.*

One Calorie is the amount of heat required to raise the temperature of 1 *gm. of water* 1° *C.*

One Therm of heat energy (used by gas companies) = 100,000 *B.Th.U.*

Since the mechanical energy of machines is so frequently supplied to them in the form of heat energy, it is interesting to note that if 1 lb. of average coal is burnt and all the available heat energy is obtained from it, this heat energy is equivalent to 12,000,000 ft. lb. of work. In the same way 1 cu. ft. of average coal gas measured at atmospheric pressure contains energy equivalent to 500,000 ft. lb. of work.

The energy from coal is for the most part wasted when it is used to supply heat to a steam engine, for the most efficient steam engine does not return in the form of mechanical energy as much as 10 per cent. of the heat energy available in the coal.

219. The Mechanical Equivalent of Heat.

When two bodies, such as two pieces of wood, are rubbed vigorously against each other, mechanical energy is transformed into heat energy ; kinetic energy of descending water at the foot

of a waterfall is converted into heat energy (as may be observed from the rising clouds of steam).

If a certain amount of mechanical energy is converted entirely into heat energy, is the quantity of heat energy produced the same in whatever way the conversion takes place? This was the first question which *James Prescott Joule* (1818-89) set himself to answer. He began his experiments in 1843, and throughout his lifetime continued his experiments, which were distinguished by their thoroughness and reliability.

By measuring the quantity of heat energy produced when a known amount of mechanical energy was converted into heat energy, he was able to show that *to produce a unit quantity of heat energy required the expenditure of a constant amount of mechanical energy, and this amount of mechanical energy was called the mechanical equivalent of heat.*

220. Joule's Experiment.—The apparatus used by Joule is shown in Fig. 388. Water is contained in the copper vessel A inside which are fixed vanes having wards cut in them. A central spindle carries vanes shaped to fit the wards in the fixed vanes. When this spindle is rotated the water in the vessel is set in motion by the moving vanes and is then brought to rest by striking against the fixed vanes. The *kinetic energy* of the water is thus converted into heat energy, and the vessel and its contents undergo a rise in temperature. The spindle is connected to a drum round which lengths of string are wound side by side. The strings are then passed over the rims of wooden wheels which are made to rotate by two leaden weights supported by strings wound round their axles. The loss in potential energy of the weights, which are allowed to fall through measured distances a number of times, is equal to the kinetic energy imparted to the water and therefore to the heat energy produced, and then taken up by the vessel and its contents. The heat required to raise the temperature of the copper vessel and spindle through 1° F. is equal to that required to raise the temperature of a certain mass of water through 1° F. This mass of water is called *the water equivalent of the copper vessel and spindle.* If this mass is added to the mass of water in the vessel and the result is M lb., the number of units of heat given to the vessel and its contents when the rise in temperature is from $t°$ to T° F. is $M(T - t)$ B.Th.U. Since the falling weights acquire a speed of a few inches per second at the end of their descent, allowance must be made for the kinetic energy which they have acquired and which is not converted into heat energy. Suppose that E ft. lb. of mechanical energy have been transformed into heat energy, then

$M(T - t)$ B.Th.U. of heat energy are equivalent to E ft. lb. of mechanical energy,

1 B.Th.U. is equivalent to $\frac{E}{M(T - t)}$ ft. lb. of mechanical energy.

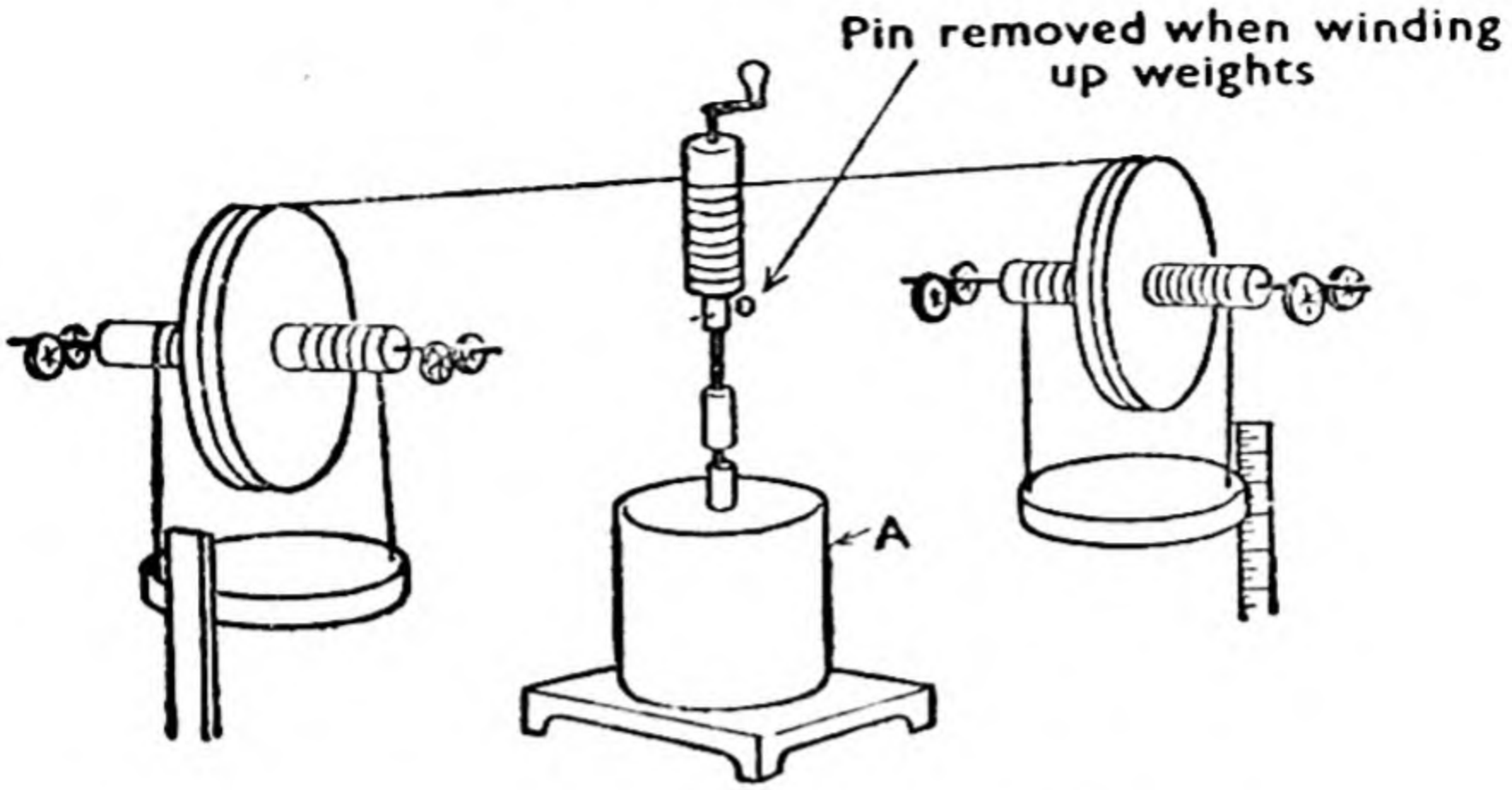


FIG. 388.

From experiments of this nature carried out in the year 1878 Joule found that 1 B.Th.U. = 775 ft. lb.

Later experiments have given the more accurate result,

1 B.Th.U. = 778 ft. lb.

and **1 C.H.U. = 1400 ft. lb.**

This is equivalent to the following :

1 calorie = $4{\cdot}19 \times 10^7$ ergs = 4·27 metre-grams.

Note.—The electrical unit of power, 1 watt = 10^7 ergs/sec. = 1 joule/sec. and therefore 1 calorie = 4·19 joules/sec.

221. Heat Efficiency of an Engine.

Suppose that 100 centigrade heat units of energy are transformed into work without loss, then the amount of work obtained is

$$100 \times 1400 = 140{,}000 \text{ ft. lb.}$$

The transformation from heat energy to mechanical energy is not possible without a certain amount of waste. The ratio of the mechanical energy made available to the heat energy expended is called the heat efficiency of the engine.

Example.—*40 B.Th.U. are converted into mechanical energy, the heat efficiency being* 15 *per cent. Calculate the mechanical energy produced.*

$$\text{Heat efficiency} = \frac{\text{Mechanical energy produced}}{\text{Heat energy input}}.$$

$$40 \text{ B.Th.U.} = 40 \times 778 \text{ ft. lb.},$$

$$\text{therefore} \quad \text{Mechanical energy} = \frac{15}{100} \times 40 \times 778 = 4668 \text{ ft. lb.}$$

222. Combined Efficiency.

The mechanical efficiency of an engine has already been defined, and it is readily seen that the combined efficiency of an engine is equal to the product of heat efficiency and the mechanical efficiency, for,

$$\text{Combined efficiency}$$

$$= \frac{\text{Work output}}{\text{Heat input}}$$

$$= \frac{\text{Mechanical work made available}}{\text{Heat input}} \times \frac{\text{Work output}}{\text{Mechanical work made available}}$$

$$= \text{Heat efficiency} \times \text{mechanical efficiency.}$$

Suppose the heat efficiency of an engine is 10 per cent. and the mechanical efficiency 80 per cent., the fraction of the original heat energy given out as useful work is

$$\frac{10}{100} \times \frac{80}{100} = \frac{8}{100},$$

the combined efficiency of the engine is therefore 8 per cent.

If the heat energy available is 1,000 C.H.U. per min., the power developed in the cylinder is,

$$\frac{10}{100} \times \frac{1000 \times 1400}{33000} = \frac{140}{33} = 4{\cdot}24 \text{ H.P.}$$

and the power output is

$$\frac{80}{100} \times \frac{140}{33} = 3{\cdot}4 \text{ H.P.}$$

3·4 H.P. is 8 per cent. of the power supplied in the form of heat.

EXAMPLES LVIII

1. A waterfall is 3,000 ft. high. How much hotter is the water at the bottom than at the top of the fall if all the kinetic energy of the falling water is converted into heat? (To raise the temperature of 1 lb. of water 1° F. requires 778 ft. lb. of energy.)

2. A body of mass m and velocity v is brought to rest by a constant force F in a distance s. Show that the deceleration of the body is $\frac{v^2}{2s}$. Hence prove that the kinetic energy of the body is $\frac{1}{2}mv^2$.

A waterfall is 1,000 m. high. How much hotter is the water at the bottom than at the top of the fall if all the kinetic energy of the falling water is converted into heat? (Mechanical equivalent of heat $= 4{\cdot}2 \times 10^7$ ergs/calorie.) (*O.C.*)

3. Convert to their equivalents in B.Th.U. (*a*) 100,000 ft. lb., (*b*) 13 ft. tons.

4. 5,000 C.H.U. are supplied to an engine, and 1,400,000 ft. lb. of work are made available in the cylinder. Calculate the heat efficiency of the engine.

5. The power developed in a steam engine cylinder is 3 H.P. If its heat efficiency is 10 per cent., what heat in C.H.U. must it receive per second?

6. 20 C.H.U. enter the cylinder of an engine per stroke. The heat efficiency is 15 per cent. and the mechanical efficiency 80 per cent. What is the brake horse-power developed if 165 working strokes are made per minute?

7. Calculate the overall efficiency of an engine if the heat efficiency is 12 per cent. and the mechanical efficiency is 75 per cent. If 6,000 C.H.U. are supplied per minute, calculate the brake horse-power.

8. A car travels 20 miles per gal. of petrol used. If the heat value of the petrol is 20,000 B.Th.U. per lb. and it weighs 7·8 lb. per gal., what is the horse-power transmitted when the speed is 60 m.p.h. and the overall efficiency is 11 per cent.?

223. The Energy of a Flywheel.

A flywheel consists of a heavy wheel mounted on bearings which are as smooth as possible. If the wheel is

made to rotate at a speed of n revs./sec. the linear speed of a particle at a distance r from the axis of the wheel is $2\pi rn$. For a flywheel of given mass rotating at a certain speed the energy stored in it is greater the further the mass is situated from the axis of the wheel. If a flywheel is examined it will be found that this arrangement of the mass is carried out as far as is consistent with strength and compactness. Flywheels are coupled to machinery on which the demand for energy fluctuates; the presence of the flywheel prevents the machinery from "racing" when the load is small and provides an additional supply of energy when the demand is heavy. Thus the screw propeller of a steamship may in a heavy sea be lifted almost out of the water by the pitching of the ship and the load on the ship's engines greatly reduced. In these circumstances the presence of the flywheel prevents violent increase in the engine speed, for the engines continue to work at only slightly increased speed, the work done being stored in the flywheel and not wasted.

The energy of a flywheel is the same as if all its mass were concentrated at the rim of a wheel whose radius is called the "mean radius" or the "radius of gyration" of the wheel. If this radius has once been determined, the energy stored in the flywheel at any angular speed can at once be written down.

$$\text{Kinetic energy of flywheel} = \frac{mv^2}{2g},$$

where m is the mass of the wheel and v the velocity of a point on the circle whose radius is equal to the radius of gyration.

Example (1).—*420 ft. tons wt. of energy are supplied to a flywheel of mass 10 tons, which is initially at rest, and finally rotates 90 times per minute. Find the mean radius of the wheel.*

If v ft./sec. is the velocity of a point on the circle of mean radius,

$$\frac{10v^2}{2g} = 420, \text{ i.e., } v = 51{\cdot}9 \text{ ft./sec.},$$

$$v = 2\pi rn,$$

$$r = \frac{51{\cdot}9}{2 \times 3{\cdot}14 \times 1\frac{1}{2}} = 5{\cdot}51 \text{ ft.}$$

Example (2).—*A wheel of mass 1,000 lb. and mean radius 2 ft. has a speed of 120 revs. per min. If 500 ft. lb. of energy are taken from it, what is its new speed in revolutions per minute?*

Initial energy stored in the wheel $= \frac{1000(2\pi \cdot 2 \cdot 2)^2}{2g}$ ft. lb.

Let n revs. per sec. be the final speed of the wheel,

$$\frac{1000(2\pi 2n)^2}{2g} = 987 - 500 = 487,$$

$$n^2 = \frac{487 \times 64}{1000 \times 16\pi^2},$$

$n = 1{\cdot}40$ revs. per sec. or 84 revs. per min.

Example (3).—*A flywheel is set in motion by a mass of 4 lb. attached to a rope which is wound round the axle of the wheel. If the mass descends 9 ft. from rest in 12 sec. and its acceleration is constant, find the kinetic energy of the wheel and axle. (The bearings of the wheel may be assumed to be smooth.) What is the tension on the rope during the motion?*

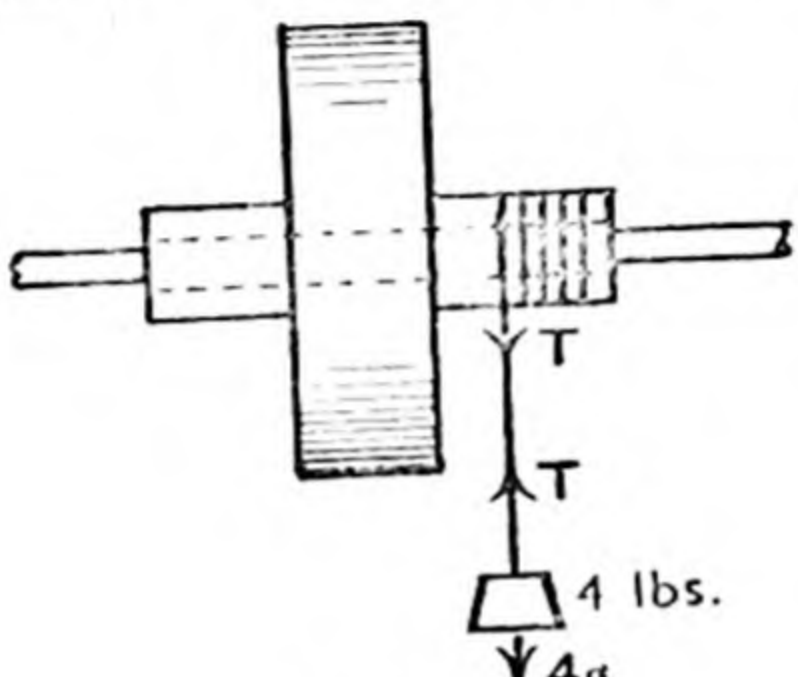


FIG. 389.

The only force which does work is the weight of the 4 lb. mass, and this work is equal to the loss in potential energy of the system.

The loss of potential energy is equal to the gain in kinetic energy of the wheel and axle and the mass of 4 lb.

Let v ft./sec. be the final velocity of the 4-lb. mass.

Consider the motion of the 4-lb. mass.

Since distance travelled = average velocity × time,

$$9 = \frac{v}{2} \cdot 12,$$

$$v = 1\tfrac{1}{2}.$$

Loss of potential energy of 4-lb. mass $= 4 \times 9 = 36$ ft. lb.

Gain in kinetic energy of 4-lb. mass $= \frac{4 \times (1\frac{1}{2})^2}{2 \times 32} = \frac{9}{64}$ ft. lb.

Hence gain in kinetic energy of flywheel and axle

$$= 36 - \frac{9}{64} = 35\tfrac{55}{64} = 35{\cdot}9 \text{ ft. lb.}$$

The resultant force on the mass of 4 lb. during this motion is $(4g - T)$ poundals.

Since the work done by this force = change in kinetic energy of the 4-lb. mass,

$$(4g - T)9 = \frac{9}{64} \times g,$$

$$4g - T = \frac{1}{2},$$

$$T = 4g - \frac{1}{2} \text{ poundals},$$

therefore $$T = 3\tfrac{63}{64} \text{ lb. wt.}$$

EXAMPLES LIX

1. The axle of a flywheel has a diameter of 4 in. and the flywheel is set in motion by a weight of 60 lb. supported by a cord wrapped round the axle. If the weight descends 8 ft. from rest and has then acquired a speed of 4 ft./sec., calculate the kinetic energy stored in the flywheel and the kinetic energy of the 60-lb. mass.

FIG. 390.

2. A flywheel of 8 in. diameter is set in motion by a weight of 1 lb. attached to a string wound round the axle, which is supposed frictionless and is 1 in. in diameter. When the weight has descended 2 ft., the speed of the rim of the wheel is 3 ft./sec. What is the kinetic energy of the flywheel at this stage? What will the velocity of the rim be when the weight has descended 4 ft.? (*O.C.*)

3. A flywheel of diameter 2 ft. has frictionless horizontal bearings and the axle is of diameter 6 in. The flywheel is set in motion by a 40 lb. wt. attached to a string which is wound round the axle. When the weight has descended 5 ft. the speed of the rim of the wheel is 10 ft./sec. What is the kinetic energy of the flywheel and the kinetic energy of the 40-lb. mass at this time? Assuming that the tension in the string is constant, by considering the forces acting on the 40-lb. mass and the change in the kinetic energy of the mass, calculate the value of this tension.

4. A flywheel has an axle of diameter $3\frac{1}{2}$ in. The wheel is brought to rest by a friction band on the axle in which there is an effective tension of 50 lb. wt. If the flywheel makes 200 revs. while being reduced to rest, what energy was stored in it?

CHAPTER XXIII

IMPULSIVE FORCES

224. We have now to consider the effect of a force which acts for a very short time—too small to measure. Such cases are quite common—the blow of a hammer on a nail, the impact of two billiard balls, the impact of a bullet on a target, or of a dart on a dart-board. Let us consider the last case a little more closely. Directly the point of the dart hits the board it begins to penetrate and its motion is resisted by the board. By Newton's third law the force exerted by the board on the dart is equal and opposite to that exerted by the dart on the board at every instant of penetration. If the board is fixed to the wall of a house it cannot move appreciably and its resistance rapidly reduces the dart to rest and destroys its momentum.

When a rifle is fired, gas under great pressure is suddenly produced and the resultant force of the gas pressure drives the bullet along the barrel. At each instant the force exerted on the bullet is equal and opposite to that exerted on the rifle. As the latter is heavier than the bullet and is also held against the shoulder of the person firing, it recoils a correspondingly small distance, but the momentum generated in the rifle is equal and opposite to that generated in the bullet.

225. Impulse of a Force.

The impulse of a force P acting for a time t is measured by the product Pt.

Now if m is the mass of the body on which P is acting, the acceleration f produced is given by $P = mf$.

If u is the initial and v the final velocity of the body after time t,

$$v = u + ft$$

or $$ft = v - u,$$

therefore $$mft = m(v - u)$$

or $$Pt = m(v - u).$$

The impulse of the force is therefore equal to the momentum it generates or destroys.

Impulse of force = Change of momentum produced.

If a body is moving with a velocity u and an impulse is given to it which reverses its direction of motion, then in the last equation above the change of momentum produced by the impulse of the force is $m(v + u)$ and is greater than if the body is merely reduced to rest.

Thus a ball of mass m which strikes a wall at right angles with velocity u and then rebounds with velocity v is given an impulse measured by $m(v + u)$, and the average impulsive force P exerted by the wall on the ball during the time of impact t is given by the equation

$$Pt = m(v + u).$$

226. Impulsive Forces.

We can now see how to obtain a measure of the effect of a large force acting for a very short time. Such a force is called an impulsive force and its effect is measured by the change of momentum it produces. The shorter the time for which the force acts the greater the force must be to produce a given change of momentum.

If a bullet of $\frac{1}{2}$-oz. mass is moving horizontally at 2,000 ft./sec., the impulse of the force on the target which brings it to rest is

$$\frac{2000}{32} = 62\tfrac{1}{2} \text{ lb.-ft.-sec. units of impulse.} \quad (\tfrac{1}{2} \text{ oz.} = \tfrac{1}{32} \text{ lb.})$$

It must be understood that an impulsive force cannot be conveniently expressed in ordinary units of force, poundals or pounds weight, but we can get some idea of the magnitude in these units by assuming that the bullet is reduced to rest in some definitely very small time, say $\frac{1}{1000}$ sec.

To destroy a speed of 2,000 ft./sec. in this time the retardation f is given by

$$0 = 2000 - \frac{f}{1000}$$

or $$f = 2{,}000{,}000 \text{ ft./sec.}^2.$$

To produce this retardation in a mass of $\frac{1}{2}$ oz. requires a force of $\frac{2000000}{32}$ or 62,500 poundals.

227. Units of Impulse and Momentum.

Since an impulse is measured by the change of momentum it produces it is expressed in units of momentum. No special name has been given to the unit of momentum as in the case of force, work, etc. As, however, the momentum of a body is the product of its mass and its velocity it is expressed in terms of the units in which the mass and velocity are measured.

We therefore speak of the pound-foot-second or the gram-centimetre-second unit of momentum, according to whether we are working in British or metric units.

As mentioned above, it is only when we assume that an impulsive force acts for some definite time that we can express it in ordinary force units as the average force required to produce the given change of momentum in that time.

Example (1).—*A hammer-head of mass 2 lb. strikes the head of a nail horizontally with a velocity of 30 ft./sec. If the blow lasts $\frac{1}{80}$ sec., what is the impulse of the blow and the average force exerted by the nail on the hammer?*

Impulse of blow = Change of momentum of hammer
$= 2 \times 30 = 60$ lb.-ft.-sec. units.

If P is the average force in poundals exerted by the nail on the hammer,

$$\frac{P}{80} = 60,$$

therefore $$P = 4800 \text{ poundals or } 150 \text{ lb. wt.}$$

Example (2).—*A truck of mass* 3 *tons travelling at* 4 *ft./sec. strikes stationary buffers and rebounds with velocity* 2 *ft./sec. What is the impulse given to the truck and the average force acting on it if the time of impact is* $\frac{1}{4}$ *sec.?*

Momentum of truck before impact $= 3 \times 2240 \times 4$ lb.-ft.-sec. units.

Momentum of truck after impact $= 3 \times 2240 \times 2$ lb.-ft.-sec. units.

Total change in momentum $\equiv 3 \times 2240(4+2)$

$= 18 \times 2240$ lb.-ft.-sec. units.

(Note that the change is the *sum* of the momenta since the direction of motion of the truck is *reversed*.)

Let P lb. wt. be the average force acting on the truck during the impact;

therefore $$\frac{32P}{4} = 18 \times 2240, \qquad (g = 32)$$

$$P = \frac{18 \times 2240}{8}$$

$$= 5040 \text{ lb. wt.}$$

EXAMPLES LX

MOMENTUM. ($g = 32$ ft./sec.^{2}, or 980 cm./sec.^{2})

1. Calculate the momentum of a mass of 20 lb. moving with velocity 16 ft./sec. (*a*) in poundal seconds, (*b*) in pound weight seconds. What is the velocity of a mass of 64 lb. if it has the same momentum as that of the 20-lb. mass?

2. A mass of 12 lb. initially at rest is acted upon for 6 sec. by a force of 4 lb. wt. What is the change of momentum produced in the mass? What is the final velocity of the mass? If the same force acts for the same time on (*a*) a 24-lb. mass, (*b*) a 6-lb. mass, what would be the changes in velocity produced in these masses?

3. What force acting for 10 sec. on a mass of 50 lb. will produce in it a change of velocity of 15 m.p.h. (Answer in poundals.)

4. A tennis ball weighing 2 oz. is struck from rest by a tennis racket, and its initial speed after the blow is 80 ft./sec. If the time of impact was 1/10th sec., what was the average force on the racket during the blow?

5. A rubber ball of mass $\frac{1}{2}$ lb. strikes a vertical wall with a speed of 40 ft./sec. and leaves the wall with a speed of 32 ft./sec. What was the total change in the velocity of the ball due to the impact? If the time of impact was 1/20th sec., what was the average force exerted by the wall on the ball in pounds weight?

6. A mass of 8 lb. is travelling at a velocity of 48 ft./sec., and its velocity is reduced to 20 ft./sec. by the action of a constant force in 2 sec. What was the magnitude of the force? How much longer time would the force have to act in order to reduce the mass to rest?

7. A 3-ton truck travelling at 4 m.p.h. strikes a stationary pair of buffers and rebounds with a velocity of 3 m.p.h. If the time of impact was

½ sec., calculate the average force exerted on the stationary buffers during the impact.

8. A tennis ball weighing 2 oz., moving horizontally at 60 ft./sec., was hit straight back with a speed of 100 ft./sec. If contact lasted 1/20th sec., what average force was acting on the racket? What change of velocity would the impact cause in a mass of 14 oz.? (*O.C.*)

9. A force of 28 lb. wt. acts horizontally for 15 sec. on a truck of 1 ton on horizontal rails. If there is no friction, what will be the kinetic energy (in foot-pounds) of the truck at the end of the time? (*O.C.*)

10. Distinguish between kinetic energy and momentum.

The head of a hammer weighing 1 lb. and moving at 8 ft./sec. drives a nail ¼ in. into a wall. Find the average force of resistance and the time occupied by the action. (*L.M.*)

228. Example.—*A mass of* 100 *lb. falls* 25 *ft. to the ground and is reduced to rest in* 1/20*th sec. What is the average force of resistance of the ground on the mass?*

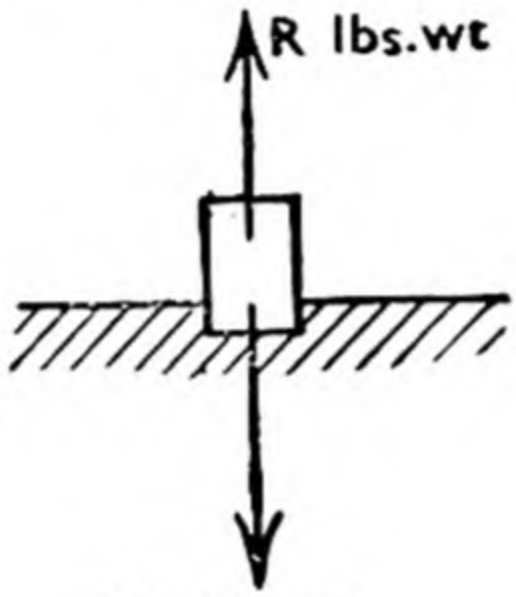


FIG. 391.

Let R lb. wt. be the average force of resistance of the ground acting vertically upwards.

The resultant force on the mass which destroys its movement is

$$(R - 100) \text{ lb. wt.}$$

$$\text{Impulse of resultant force on mass} = \frac{(R - 100)32}{20} \text{ lb.-ft.-sec.} \qquad (g = 32)$$

If v is the velocity of mass acquired in falling 25 ft.,

$$v^2 = 64 \times 25, \text{ or } v = 40 \text{ ft./sec.}$$

$$\text{Change of momentum of mass} = 100 \times 40 \text{ lb.-ft.-sec.},$$

therefore

$$\frac{(R - 100)32}{20} = 4000,$$

$$R - 100 = \frac{20 \times 4000}{32} = 2500,$$

$$R = 2600 \text{ lb. wt.}$$

Note.—Problems of this kind can be solved by considering the retardation required to bring the mass to rest. This method is preferable where the distance the mass penetrates into the ground is given instead of the time of penetration.

Velocity v acquired in falling 25 ft. is given by

$$v^2 = 64 \times 25, \text{ or } v = 40 \text{ ft./sec.}$$

Retardation f to reduce to rest in 1/20th sec. is given by

$$0 = 40 - \frac{f}{20},$$

therefore $$f = 800 \text{ ft./sec.}^2$$

Resultant force required to produce this retardation in a mass of 100 lb. is 80,000 poundals, or 2,500 lb. wt.

But Resultant force = Resistance of ground − Weight,

therefore Resistance of ground = 2500 + 100 = 2600 lb. wt.

EXAMPLES LXI

1. A mass of 40 lb. falls vertically and strikes the ground with velocity 30 ft./sec. If the mass is reduced to rest in 1/40th sec., what is the average force of resistance of the ground on the mass? Show that the mass penetrates 4½ in. into the ground.

2. A steel hammer of mass 10 lb. strikes an anvil vertically downwards at 12 ft./sec. and rebounds with velocity 4 ft./sec. If the time of the impact is 1/50th sec., find the average force exerted by the hammer on the anvil.

3. A mass weighing 4 lb. falls freely from a height of 3 ft. on to an anvil and rises 4 in. after the blow. If the time of impact is 1/50th sec., find the average force between the mass and the anvil. (*O.C.*)

4. A ball weighing 4 lb. is thrown vertically downwards at 10 ft./sec. on to a horizontal surface 9 ft. below and rebounds to the same height. Calculate the changes produced by the impact in (*a*) velocity, (*b*) momentum, (*c*) energy. If contact lasts for 1/20th sec., what is the average force acting on the ball during the impact? (*J.*)

5. A blacksmith strikes a sheet of iron with a sledge-hammer of mass 8 lb. If the striking velocity is 20 ft./sec. and the hammer-head has a striking face of area 3 sq. in., calculate the force exerted on the sheet of iron per square inch given that the time of impact of the hammer on the iron is 1/40th sec.

6. A rubber ball of mass ¼ lb. is thrown vertically downwards with a velocity of 40 ft./sec. from a height of 4 ft. After rebounding it reaches a height of 9 ft. If the time of impact with the ground was 1/10th sec., what was the average force exerted on the ball?

229. The Average Force of Reaction on a Machine Gun.

If the mass, muzzle velocity and the number of bullets fired per second from a machine gun are known, we can calculate the change in momentum per second of the bullets and therefore

the average force required to produce this change. The average force of reaction on the gun is equal and opposite to this.

Example.—*A machine gun fires* 500 *bullets per min., each of mass* $\frac{1}{2}$ *oz., with a muzzle velocity of* 2,400 *ft./sec. What is the average force on the gun and what is its average horse-power?*

$$\text{Momentum generated per second} = \frac{500}{32} \times \frac{2400}{60} = \frac{500 \times 5}{4} \text{ lb.-ft.-sec. units.}$$

The average force exerted is

$$\frac{500 \times 5}{4} \text{ poundals} = \frac{625}{32} \text{ lb. wt.,}$$

therefore the average force on the gun is $\frac{625}{32} = 19{\cdot}5$ lb. wt.

The kinetic energy generated in the bullets per second is

$$\tfrac{1}{2} \times \frac{500}{32 \times 60} \times 2400^2 \text{ ft.-poundals,}$$

$$\text{therefore} \qquad \text{Horse-power} = \frac{500 \times 2400^2}{64 \times 60 \times 32 \times 550} = 42{\cdot}5.$$

230. Impact of Water on a Fixed Surface.

To find the average pressure due to a jet of water impinging against a fixed surface, or a continued fall of rain on the ground, we have only to calculate the amount of momentum destroyed per second. The average force is measured by this amount of momentum.

Example.—*A jet delivers* 10 *gals. of water per second through a nozzle of cross-section* 4 *sq. in. If the jet is directed on to a vertical wall which it strikes at right angles, find the steady force exerted on the wall if the water does not rebound.*

(1 gal. of water weighs 10 lb. and 1 cu. ft. weighs 62·5 lb.)

Mass of water striking the wall per second $= 10 \cdot 10 = 100$ lb.

$$\text{Volume} = \frac{100}{62{\cdot}5} \text{ cu. ft.}$$

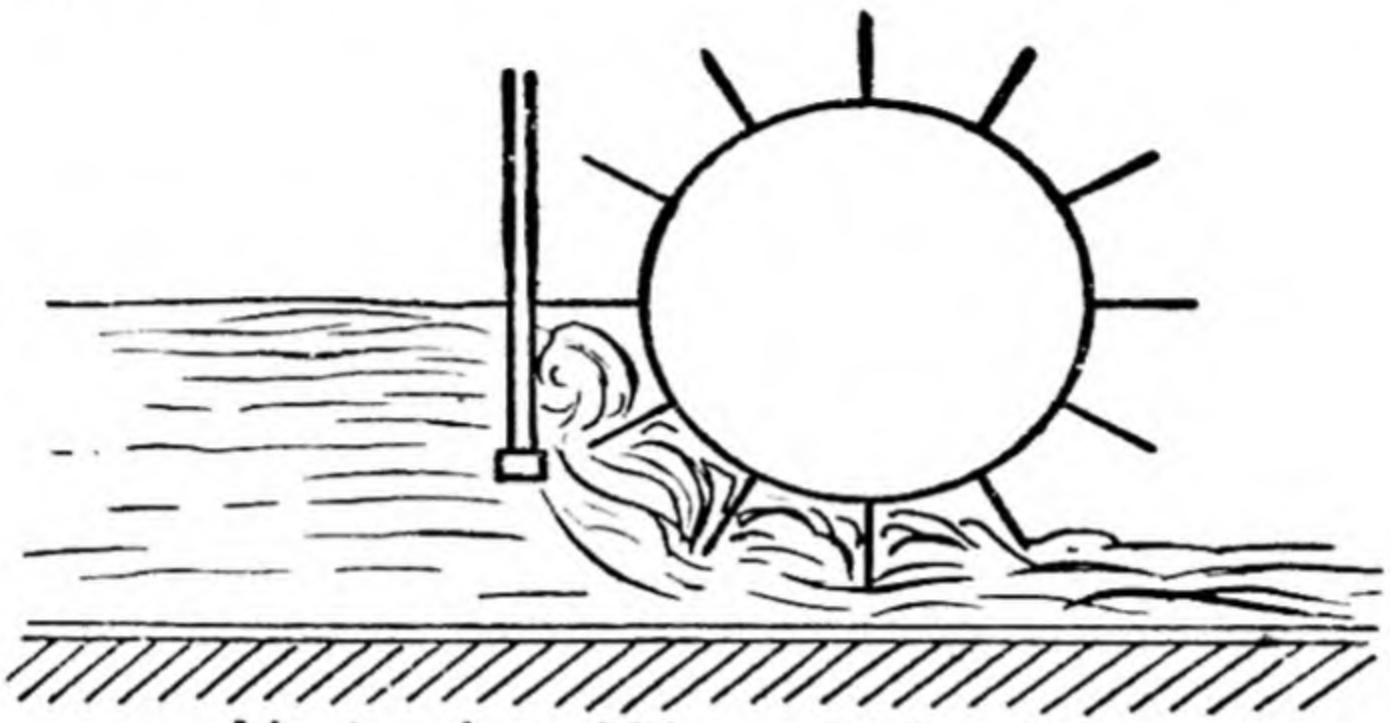
Since the cross-section of the nozzle is 4 sq. in. or $\frac{4}{144}$ sq. ft., the length of the column which issues from the nozzle per second is

$$\frac{100}{62{\cdot}5} \div \frac{4}{144} = 57{\cdot}6 \text{ ft.}$$

The speed of the jet of water is therefore 57·6 ft./sec.
Momentum destroyed per second $= 57{\cdot}6 \times 100 = 5760$ units.
The average force on the wall is therefore 5760 poundals $= 180$ lb. wt

231. Impact of a Jet of Water on Moving Vanes.

If a jet of water moving with velocity v impinges normally on a series of plane plates attached to the circumference of a wheel and moving with velocity u, the loss of momentum of the water provides a steady pressure which keeps the wheel rotating. This is the principle of the undershot water-wheel shown in the diagram.



Undershot Water Wheel

FIG. 392.

If m is the mass of water striking the vanes per second the loss of momentum per second is $m(v-u)$, and the average pressure on the vanes is $m(v-u)$ poundals.

Since the whole of the momentum is not destroyed the efficiency of this type of wheel is comparatively small, and in order to increase the efficiency the plane plates are replaced by cupped vanes as in the Pelton Wheel.

232. The Pelton Wheel.

If a supply of water is available at a great height the water may be lead through pipes without much loss of energy to a power station, where it passes through a special nozzle which restricts the water as smoothly as possible. The jet which issues has a very small cross-sectional area, but owing to the great height of the supply it is forced out of the nozzle with con-

siderable speed. The jet of water is then allowed to impinge on specially constructed buckets which are attached to the rim of the Pelton Wheel.

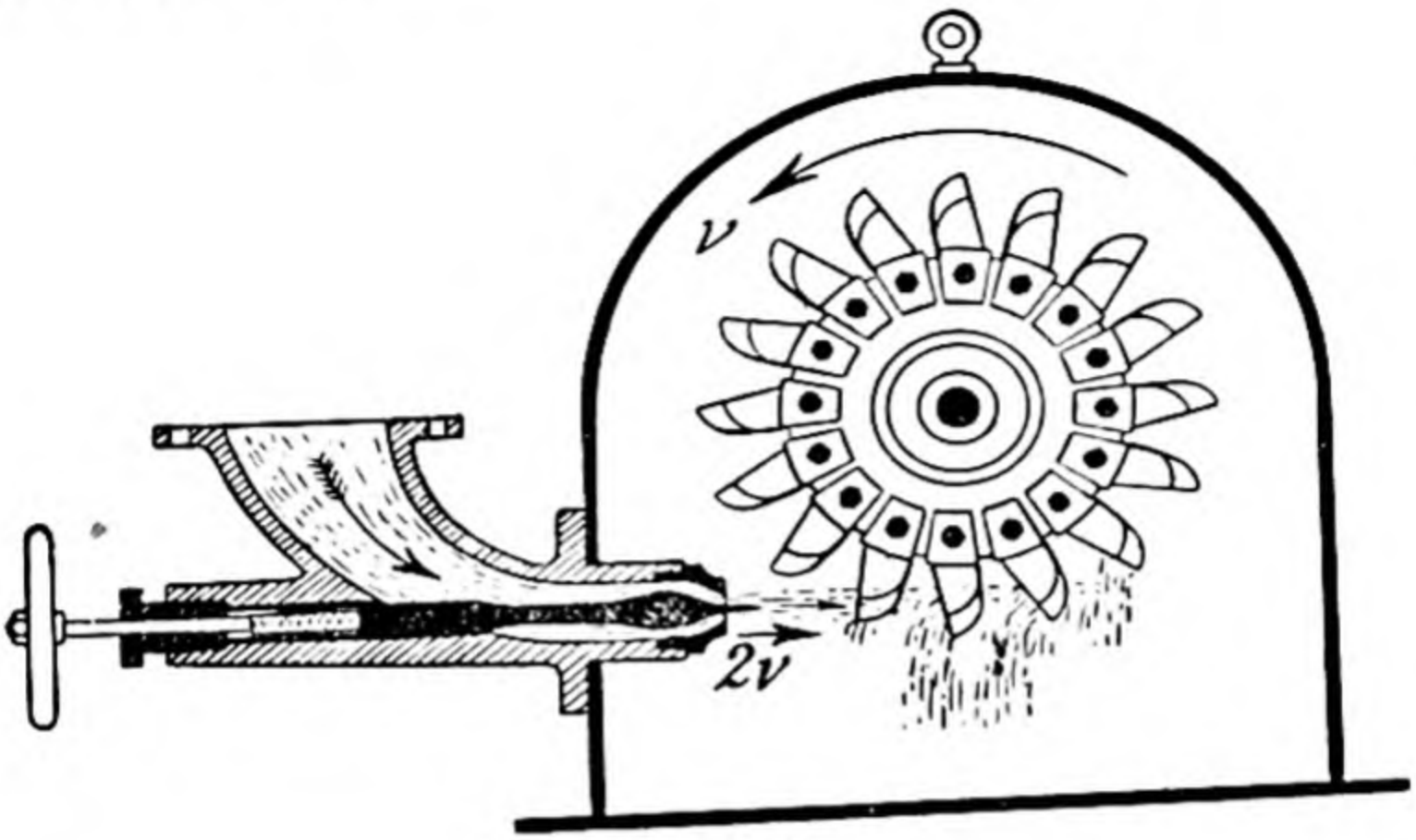


FIG. 393.

The shape of the vanes or buckets is shown in the figure. The jet impinges on the sharp edge which divides the bucket into two parts. The water runs smoothly round the curved surface and its tangential velocity relative to the surface of the bucket is practically unaltered.

If v is the velocity of the jet and u that of the bucket, the velocity of the water relative to the bucket is $v-u$, and the water leaves the lip of the bucket with this same velocity $v-u$ backwash, and therefore with velocity $u-(v-u)$ or $2u-v$ in the forward direction relative to the ground.

If $v=2u$, *i.e.*, if the velocity of the jet is twice that of the buckets, the water is reduced to rest and simply drops to the ground.

If m is the mass of water striking the wheel per second the momentum destroyed per second is mv, and the average pressure on the wheel is mv poundals.

Since the buckets are moving at $\frac{v}{2}$ ft./sec. the work done on

the wheel per second is

$$\tfrac{1}{2}mv^2 \text{ ft. poundals}$$

and the horse-power

$$\frac{mv^2}{2g \times 550}.$$

At Lac Fully, in Switzerland, the water is supplied from a height of 5,400 ft. above the jet and the water emerges with a speed of 590 ft./sec. through a nozzle of cross-section 1·48 sq. in.

The volume of water striking the buckets per second is

$$590 \times \frac{1{\cdot}48}{144} \text{ cu. ft.,}$$

the mass of this water is

$$\frac{590 \times 1{\cdot}48 \times 62{\cdot}5}{144} \text{ lb.,}$$

the momentum destroyed per second is

$$\frac{590^2 \times 1{\cdot}48 \times 62{\cdot}5}{144},$$

and this gives the average pressure in poundals. This is equal to 3,320 lb. wt.

The work done by the pressure per second is $3{,}320 \times 295$ ft. lb.

and the horse-power $= \dfrac{3320 \times 295}{550} = 1{,}780$ H.P. approximately.

EXAMPLES LXII

1. A machine gun fires 600 bullets a minute, each of mass $\frac{1}{2}$ oz., with a muzzle velocity of 2,400 ft./sec. Find the momentum produced in these bullets per second and hence the average force of reaction on the gun in pounds weight.

2. A jet of water of cross-section 3 sq. in. and velocity 40 ft./sec. impinges normally on a plane inelastic wall, so that the velocity of the water is destroyed on reaching the wall. Calculate in pounds weight the thrust on the wall. (*I.S.*)

3. Find the average pressure per square foot on the ground due to a rainfall of $\frac{1}{2}$ in. in 2 hr. The velocity of the rain on striking the ground is equal to 240 ft./sec., and 1 cu. ft. of rain-water weighs $62\frac{1}{2}$ lb.

4. The area of a sail is 400 sq. ft. and wind impinges on it at an angle of 30° to the sail. Assuming that a cubic foot of air weighs $1\frac{1}{4}$ oz. and that the speed of the wind is 20 m.p.h., calculate the force exerted on the sail. Assume (*a*) that the sail is not subjected to any force parallel to its surface, (*b*) that the speed of the sail may be neglected, (*c*) that the momentum of the wind perpendicular to the sail is entirely destroyed.

5. The velocity of flow in a water main of 8 in. diameter is 7 ft./sec., and at one place the main is bent through an angle of 60°. Find (*a*) the magnitude of the rate of charge of momentum, (*b*) the magnitude of the force acting on the pipe due to the change in direction of the pipe. (Take $\pi = \frac{22}{7}$; 1 cu. ft. of water weighs 62·5 lb.) (*O.C.*)

6. Force is sometimes defined as "rate of change of momentum." Give another way of defining it and show that the two definitions come to the same thing.

Taking wind pressure as due to change of momentum of the air, and a cubic foot of air as weighing $1\frac{1}{4}$ oz., find the pressure in pounds weight per square foot on a fixed surface at right angles to a gale of 60 m.p.h. (*O.C.*)

7. How is force measured? Find the number of dynes in a poundal. A jet of water (10 cwt. of water per minute) strikes a wall at right angles with a velocity of 60 ft./sec. Find in pounds weight the thrust of the jet on the wall. (*L.A.*)

233. Motion of a Shot and Gun.

When a gun is fired the explosive charge forms a large volume of gas at a very high pressure. This pressure acts equally on the shot and gun in the direction of the barrel and drives the shot out.

If the gun is free to move in the direction of the barrel the forward momentum generated in the shot at the instant it leaves the barrel is equal to the backward momentum generated in the gun. This is the case when the gun is placed on a smooth horizontal plane with the barrel horizontal. If the barrel of the gun is elevated the momentum of the gun is *not* equal and opposite to that of the shot, as any vertical momentum imparted to the gun is at once destroyed by the impulsive pressure of the plane on which it stands.

It should be noted that any spring or similar device for preventing recoil does not prevent the principle of momentum being applied.

At present we shall only consider cases when the gun is free to recoil in the direction of the barrel.

Example.—*A shell of mass 320 lb. is fired from a gun of mass 15 tons with a muzzle velocity of 1,500 ft./sec. What is the velocity of recoil of the gun? If the recoil is limited to 3 ft. by means of a spring, what is the average force exerted by the spring?*

Let V be the velocity of recoil of the gun.

Momentum of gun = Momentum of shot.

Therefore $$V = \frac{320 \times 1500}{2240} = \frac{100}{7} = 14\tfrac{2}{7} \text{ ft./sec.}$$

If f is the retardation required to destroy this velocity in 3 ft.,

$$0 = \left(\frac{100}{7}\right)^2 - 6f,$$

therefore $$f = \frac{1}{6}\left(\frac{100}{7}\right)^2.$$

If R is the average force exerted by the spring,

$$R = \frac{15 \times 2240 \times 100^2}{6 \times 49} \text{ poundals}$$

$$= \frac{15 \times 100^2}{6 \times 49 \times 32} \text{ tons wt.}$$

$$= 15{\cdot}9 \text{ tons wt.}$$

234. Impact of Two Bodies.

If two bodies A and B collide, then the action of A on B is, while they are in contact, equal and opposite to that of B on A. The impulse of A on B is therefore equal and opposite to that of B on A, and it follows that the changes in momentum of A and B are equal and opposite. The sum of the momentum of the two bodies, measured in the same direction, is therefore unaltered by their impact. This is an example of the principle of Conservation of Linear Momentum.

If a mass m, moving with velocity v, strikes a mass M, which is free to move in the direction of m's motion, and the two move on as a single body, there is no loss of momentum.

Hence if V is the velocity of the two together after impact,

$$(M + m)V = mv,$$

therefore $$V = \frac{m}{M + m}v.$$

It should be noted that, although there is no change in momentum due to the impact, there *is* a loss of kinetic energy.

The kinetic energy before impact is $\frac{1}{2}mv^2$.

The kinetic energy after impact is

$$\tfrac{1}{2}(M + m)V^2 = \tfrac{1}{2}\frac{m^2}{M + m}v^2$$

and this is less than $\frac{1}{2}mv^2$, since $M + m$ is greater than m.

Since kinetic energy is lost in nearly all cases of impact the principle of energy must never be used in dealing with cases where impulsive forces occur.

Example.—*A truck of mass 5 tons travelling at 20 ft./sec. runs into a stationary truck of mass 3 tons. If the two trucks become coupled so that they move on together after the impact, what is their common velocity?*

The total momentum before impact in the direction of the moving truck is $5 \times 20 = 100$ tons-ft.-sec. units.

If v ft./sec. is the common velocity in this direction after impact,

$$(5+3)v = 100,$$

therefore

$$v = \frac{100}{8} = 12\tfrac{1}{2} \text{ ft./sec.}$$

235. Conservation of Linear Momentum.

From the Third Law of Motion we can deduce a very important fact with regard to the total momentum of a system of bodies, such as a number of particles which are acting on each other either by attraction or repulsion at a distance, or by actual contact. If two of the bodies A and B act on each other, the action of A on B is equal and opposite to that of B on A, the action and reaction therefore generate equal and opposite amounts of momentum in the two bodies. Hence the total amount of momentum reckoned in any fixed direction is unaltered. This constitutes a very important principle known as the

Conservation of Linear Momentum.

In any system of mutually attracting or impinging particles, the linear momentum in any fixed direction remains unaltered unless there is an external force acting in that direction.

236. Hick's Balance.

In Fig. 394 OP represents a pendulum of length r and supported at O. AOB is the vertical diameter of the circle centre O.

Suppose that the pendulum describes *small* angular oscillations. In the figure, if OP is the extreme position reached by

the pendulum and PN is the perpendicular drawn from P to AB, it will be assumed that in the experiment the accuracy will permit us to neglect BN in comparison with $2r$.

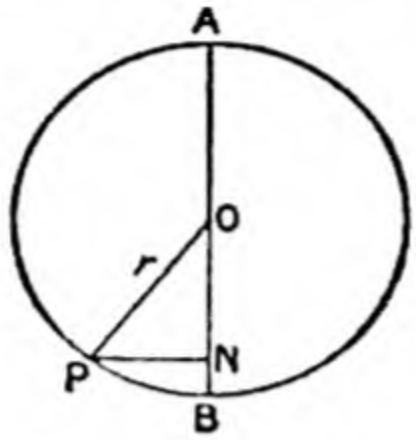


FIG. 394

Since PN is perpendicular to the diameter AB,

$$AN \,.\, NB = PN^2$$

$$BN(2r - BN) = PN^2.$$

Hence

$$BN = \frac{PN^2}{2r}.$$

If m is the mass of the pendulum bob and v is the velocity with which the bob passes through its lowest position, since in passing from B to P the loss of kinetic energy of the bob is equal to the gain in potential energy,

$$\tfrac{1}{2}mv^2 = mg \,.\, BN,$$

therefore

$$v^2 = 2g \,.\, BN$$

$$= 2g \,.\, \frac{PN^2}{2r}$$

$$= \frac{g}{r} \,.\, PN^2,$$

therefore

$$v = \sqrt{\frac{g}{r}} \,.\, PN.$$

For the same pendulum the speed with which the bob passes horizontally through the lowest position is proportional to the horizontal distance travelled from this position before the bob is

instantaneously at rest. The distance PN is therefore a measure of the velocity of the bob at its lowest point.

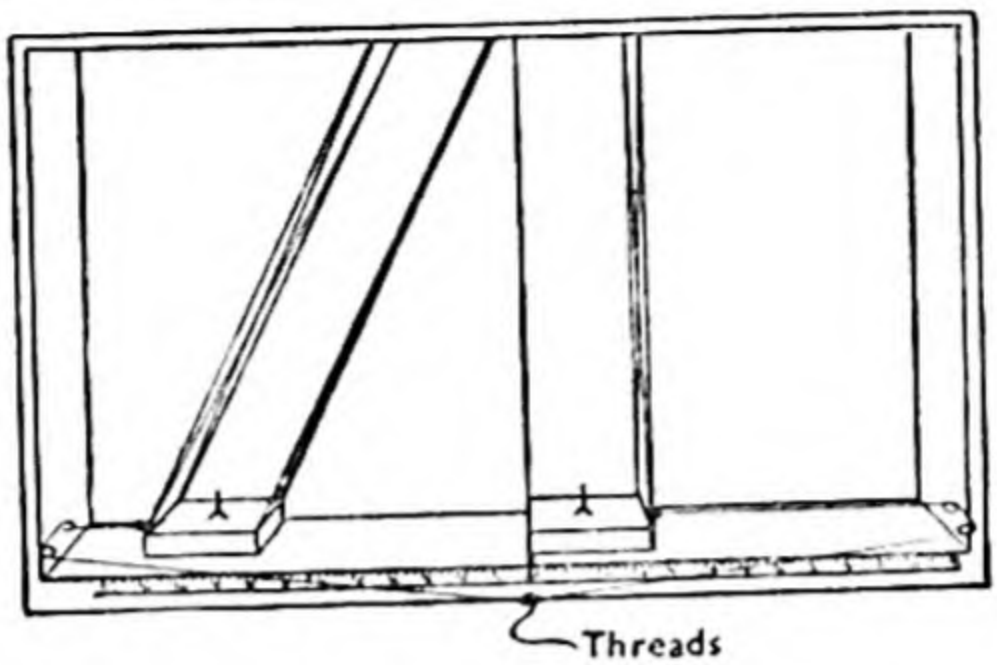


FIG. 395.

237. EXPERIMENT.—The ballistic pendulum, or Hick's Balance, shown in the figure above, consists of two platforms, each supported by eight strings. It will be seen that the pendulums are as long as possible so that the angular displacements are small. The platforms remain horizontal when in motion, and the horizontal displacement is shown by a pointer which moves over a horizontal scale. Weights may be attached to the platforms, as shown in the figure.

The faces of the platforms which come into contact are covered with cork and needle points, which project horizontally from them, penetrate the cork during a collision, and the platforms are thus held together and move as one body afterwards.

By tying threads to the midpoints of the extreme faces of the platforms they may be released together.

Place different weights on the platforms and arrange their displacements until there is no motion after the collision. The platforms always meet when they are both in a vertical position, for the time of swing of each pendulum is constant whatever the maximum displacement, provided that the angular displacement is small.

If M and m are the masses of the platforms with their loads and D and d the horizontal displacements; before the platforms are released it will be found that when there is no motion after the collision, $MD = md$, and since D and d are proportional to the speeds of the platforms on collision, viz., V and v, $MV = mv$. Draw one of the platforms aside and allow it to collide with the other which hangs vertically at rest. Note the displacement of the two platforms together. It will be found that if M is the mass of the platform and load which is drawn aside, D its displacement and d the common displacement,

$$MD = (M + m)d, \textit{ i.e.}, MV = (M + m)v.$$

The speeds V and v correspond to the displacements D and d.

Finally, let the platforms be drawn aside and released together. If

M and m have initial displacements D and d, and x is the common displacement after the collision, it should be found that

$$MD - md = (M + m)x.$$

Hence the algebraic sum of the momenta before collision in a chosen direction is equal to the momentum in the same direction after collision.

238. Impact of Inelastic Bodies.

We shall now consider more fully cases such as that mentioned in the last paragraph where two bodies collide. After collision one of two things may happen: (i) the bodies may remain together as a single body either at rest or in motion, or (ii) they may rebound and separate. In the former case the bodies are said to be *inelastic*, *i.e.*, they show no tendency to resume their former shape after being deformed, and in the second case they are said to be *elastic*, *i.e.*, they tend to retain their original shape.

We shall first take the case of inelastic bodies and suppose that they are moving in the same straight line at the moment they collide.

The principle used in all cases is that there is no change in momentum due to the collision, *i.e.*, the momentum after the impact is equal to the momentum before the impact. In this connection it must be remembered that momentum, like velocity, has direction as well as magnitude. If momentum in any particular direction is considered as positive, momentum in the opposite direction must be considered negative, and it is the *algebraic* sum of the momenta in any direction that is unaltered.

If two bodies of equal mass moving with equal velocities in opposite directions collide and remain together, the combined body will be at rest, as their momenta before impact were equal and opposite.

If a mass M moving with velocity V overtakes a mass m moving with velocity v in the same direction and the two remain as one body after impact, the common velocity is easily found as follows:

$$\text{Momentum before impact} = MV + mv.$$

$$\text{Total mass after impact} = M + m.$$

Therefore the common velocity $= \dfrac{MV + mv}{M + m}$.

If the masses are moving in opposite directions, the direction of motion after impact will be that in which the one with the greater momentum was moving. If this is M the momentum before impact in its direction of motion is $MV - mv$, and the common velocity after impact will be

$$\frac{MV - mv}{M + m}.$$

Example (1).—*A truck of 5 tons travelling at 20 ft./sec. collides with a second truck of mass 3 tons travelling in the opposite direction at 10 ft./sec. If the trucks remain together after the collision, what is their common velocity and what is the loss of kinetic energy due to the impact?*

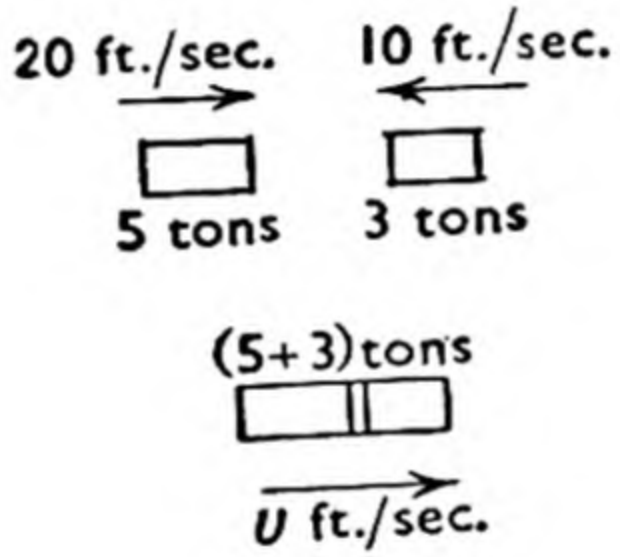


FIG. 396.

The 5-ton truck has the greater momentum, and we therefore take its direction of motion as the positive direction.

The algebraic sum of the momenta before impact is

$$5 \times 20 - 3 \times 10 = 100 - 30 = 70 \text{ tons-ft.-sec. units.}$$

If v ft./sec. is the common velocity after impact,

$$8v = 70,$$

$$v = 8\tfrac{3}{4} \text{ ft./sec.}$$

or

The kinetic energy before impact is

$$\frac{1}{2} \times \frac{5 \times 20^2}{32} + \frac{1}{2} \times \frac{3 \times 10^2}{32} = \frac{1000}{32} + \frac{150}{32} = \frac{1050}{32} = 35{\cdot}9 \text{ ft. tons.}$$

The kinetic energy after impact is

$$\frac{1}{2} \times \frac{8 \times 35^2}{32 \times 16} = \frac{35^2}{96} = 12{\cdot}76 \text{ ft. tons},$$

therefore loss in kinetic energy due to impact is

$$32{\cdot}8 - 12{\cdot}8 = 20 \text{ ft. tons.}$$

Example (2).—*A pile driver of mass 4 cwt. falls 9 ft. on to a pile of mass 1 ton. The pile is driven 6 in. into the ground. Find the common velocity of the pile and driver after the impact and the average resistance of the ground to the penetration of the pile.*

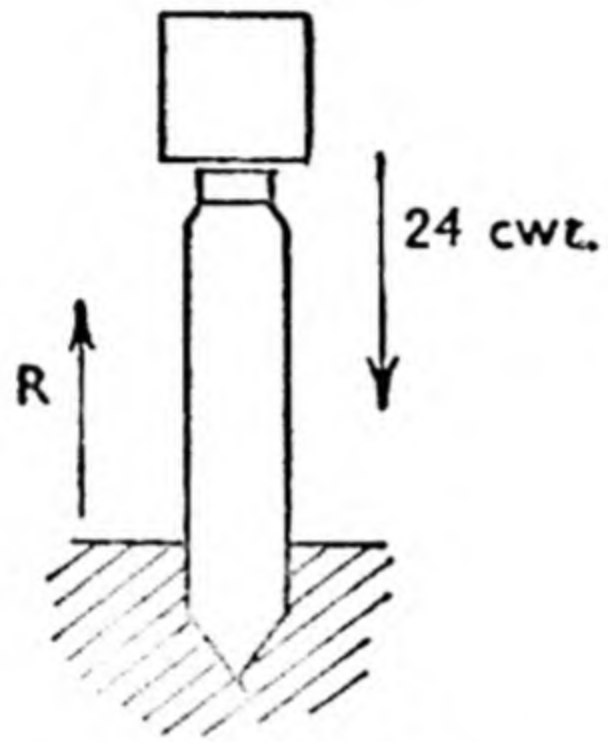


FIG. 397.

The velocity of the pile driver after falling 9 ft. is

$$\sqrt{2 \times 32 \times 9} = 24 \text{ ft./sec.} \qquad (v^2 = 2gh)$$

After impact the total mass in motion is 24 cwt., and if V is the common velocity of the pile and driver, since the momentum of pile and driver after impact = momentum of driver before impact,

$$24 \times 112V = 4 \times 24 \times 112,$$

therefore

$$V = 4 \text{ ft./sec.}$$

Now this velocity is destroyed in $\frac{1}{2}$ ft., and if f is the retardation

$$0 = 16 - 2f \cdot \tfrac{1}{2},$$

therefore,

$$f = 16 \text{ ft./sec.}^2,$$

and the resultant retarding force is

$$24 \times 112 \times 16 \text{ poundals} = 12 \times 112 = 1344 \text{ lb. wt.}$$

The resistance of the ground R must be equal to the retarding force plus the weight of the driver and pile, therefore

$$R = (12 \times 112 + 24 \times 112) \text{ lb. wt.}$$
$$= 36 \times 112 \text{ lb. wt.}$$
$$= 1{\cdot}8 \text{ tons wt.}$$

239. Shot and Target.

If a shot enters a target which is free to move and whose mass is large in comparison with that of the shot, the latter is brought to rest in a time so short that we may assume the target does not move appreciably before the shot is reduced to rest relative to the target.

The target and shot are therefore assumed to move on together as one body from the initial position of the target. The momentum of the shot and target after the impact is equal to the momentum of the shot before impact, and we can ignore the distance penetrated by the shot and the time taken to penetrate.

Example.—*A target of mass* 20 *lb. is suspended by a string. A bullet of mass* $\frac{1}{2}$ *oz. moving at a speed of* 1,200 *ft./sec. strikes the target horizontally. To what height will the target rise above its initial position? What is the loss of kinetic energy due to the impact* **as a fraction** *of the original kinetic energy of the bullet?*

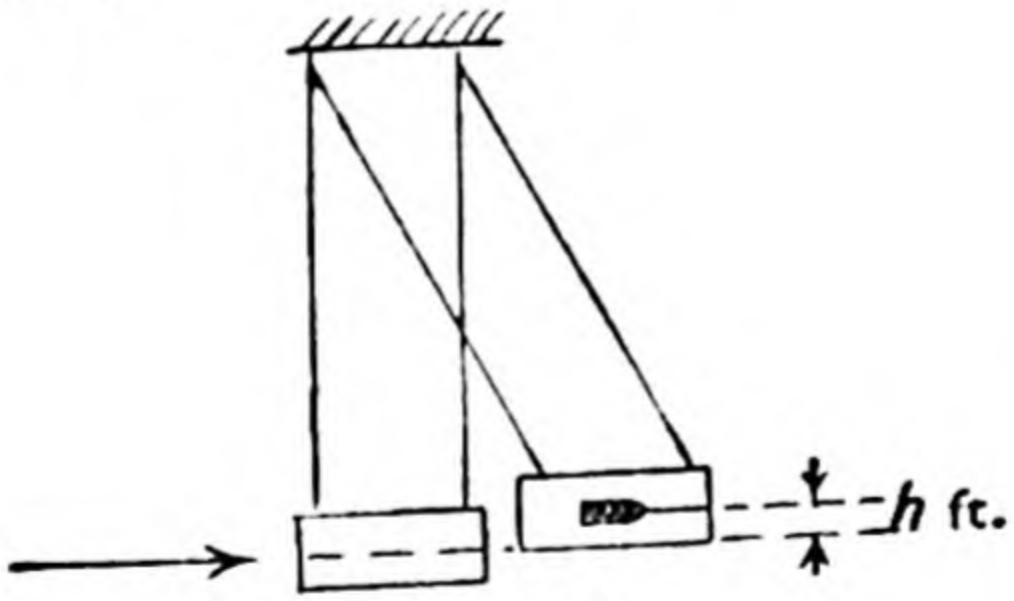


FIG. 398.

Let v ft./sec. be the common velocity of the bullet and target after the impact.

Momentum of bullet before impact $= \frac{1}{32} \times 1200$ lb.-ft.-sec. units.

Momentum of bullet after impact $= (20 + \frac{1}{32})v$ lb.-ft.-sec. units.

Therefore $$20\tfrac{1}{32}v = \frac{1200}{32},$$

therefore $$v = \frac{1200}{641} \text{ ft./sec.} = 1{\cdot}87 \text{ ft./sec.}$$

At the highest position reached by the target the gain in potential energy is equal to the kinetic energy of the target and bullet after the impact.

If h ft. is the height to which the target rises, the gain in potential energy is

$$32(20\tfrac{1}{32})h \text{ ft. poundals.}$$

The kinetic energy of target and bullet after impact is

$$\frac{1}{2} \times \frac{641}{32} \times \frac{1200^2}{641^2} = \frac{1200^2}{64 \times 641} \text{ ft. poundals,}$$

therefore $$641h = \frac{1200^2}{64 \times 641}.$$

Therefore $$h = \frac{1200^2}{64 \times 641^2} \text{ ft.}$$

$$= \frac{1200^2 \times 12}{64 \times 641^2} \text{ in.}$$

$$= 0{\cdot}66 \text{ in.}$$

The kinetic energy of the bullet before impact is

$$\tfrac{1}{2} \times \tfrac{1}{32} \times 1200^2 \text{ ft. poundals,}$$

therefore the energy loss $$= \frac{1200^2}{64}\left(1 - \frac{1}{641}\right) = \frac{1200^2 \times 640}{64 \times 641},$$

and the fractional loss $$= \frac{1200^2 \times 640}{64 \times 641} \times \frac{64}{1200^2} = \frac{640}{641}.$$

EXAMPLES LXIII

1. A bullet weighing $\frac{1}{2}$ oz. travelling at 1,600 ft./sec. strikes a target of mass 10 lb. which is free to move. If the bullet embeds itself in the target, find their common velocity.

2. A truck of mass 6 tons travelling at 4 ft./sec. collides with a truck of mass 8 tons which is at rest. The trucks are immediately coupled and proceed together after the impact. Calculate their common velocity and the loss of energy in foot-tons due to the impact.

3. A ball weighing 0·1 lb. travelling horizontally at 150 ft./sec. embeds itself in the centre of a suspended mass of clay. If the clay weighs 8 lb., with what velocity will they begin to move together? What is the change of kinetic energy? (*G.S.*)

4. (*a*) A body of mass 10 lb. is moving with a velocity of 10 ft./sec. Find its momentum and its kinetic energy.

For how long must a force of 1 lb. wt. act on it in order to double its momentum? Calculate the increase in kinetic energy of the body.

(*b*) A truck weighing 10 tons, moving with a velocity of 6 ft./sec., collides with and remains fixed to another truck at rest weighing 2 tons. Find the final speed of the trucks. (*O.*)

5. Distinguish between momentum and kinetic energy.

A railway wagon weighing 5 tons and moving at 4 m.p.h. collides with a stationary wagon weighing 2 tons. If the two then move on together, find in foot-pounds the kinetic energy lost. (*G.S.*)

6. Distinguish between momentum and kinetic energy.

A gun of mass 5 tons fires a shot of mass 112 lb. horizontally with a velocity of 1,600 ft./sec. Calculate (*a*) the initial velocity of the gun, (*b*) its initial momentum, (*c*) its initial kinetic energy.

Also determine the distance the gun will move horizontally before coming to rest if its recoil is resisted by a steady horizontal force of 10 tons wt. (*G.S.*)

7. A truck of mass 10 tons travelling at 8 ft./sec. collides with a second truck travelling in the same direction with a speed of 4 ft./sec. The common speed of the trucks after impact is 7 ft./sec. What is the mass of the second truck? If the two trucks experience a resistance of 30 lb. wt. per ton to motion after impact, how far will they travel before coming to rest?

8. A truck of 5 tons mass runs at 10 m.p.h. into a stationary truck, and the two move on together at 4 m.p.h. Find the mass of the second truck, and find what is the loss per cent. in kinetic energy due to the impact. (*O.C.*)

9. A gun weighing 1 ton fires a shot weighing 2½ lb. horizontally with a velocity of 1,200 ft./sec. relative to the ground. Find the kinetic energy of the shot and of the gun at the recoil, each to the nearest foot-pound. (*O.C.*)

10. A railway truck weighing 12 tons travelling at 9 ft./sec. collides with an equal truck travelling at 3 ft./sec. in the same direction. Find their common velocity when the buffer springs are at their greatest compression. Find also the energy in foot tons which is stored in the springs at this instant, assuming that no energy is wasted. (*O.C.*)

11. Define momentum, force.

A body of mass 10 lb. moving with a velocity of 24 ft./sec. overtakes and collides with a body of mass 6 lb. moving with a velocity of 12 ft./sec. in the same direction. After collision both bodies move on together. Calculate the velocity after collision, and find the change in kinetic energy of the system. What would have been the velocity after collision if the bodies had moved originally in opposite directions? (*L.M.*)

12. A bullet having a mass of ½ oz. possesses 100 ft. lb. of "striking" energy. What velocity corresponds to this? What average force will be needed to stop the bullet in a distance of half an inch? (*O.C.*)

13. Distinguish between momentum and force.

A pile driver of mass 2 cwt. falls from rest a vertical distance of 16 ft. and strikes a pile of mass 10 cwt. which it drives a distance of 3 in. into the ground. Assuming that after the blow the pile driver and the pile move together, calculate the resistance, supposed uniform, which the ground offers in tons weight. (*G.S.*)

14. What is meant by an impulsive force? How is such a force measured?

A heavy pendulum of mass 10 lb. hangs at rest supported by a light string so that its centre of gravity is 6 ft. below the point of support. It is given an impulse and is observed to swing through an angle of 60° from the vertical. Determine the magnitude of the impulse. (*L.M.*)

15. Define the terms momentum, potential energy and kinetic energy.

A gun of mass 1 ton fires a projectile which weighs 28 lb. with a speed of 840 ft./sec. What is the velocity of recoil of the gun? If the gun recoils a distance of 21 in., what is the average force opposing the motion? (*L.M.*)

16. State the principle known as the conservation of momentum, and describe a simple laboratory demonstration of the principle.

A ball of mass 6 lb. moving at 20 ft./sec. meets another moving in the opposite direction along the same line at 15 ft./sec. The former is brought to rest, and the latter has its velocity reversed but unaltered in magnitude. Find the mass of the second ball, and state the magnitude of the change of momentum of each. (*J.*)

17. A bullet of mass $\frac{1}{2}$ oz. fired horizontally buries itself into a freely suspended block of wood of mass 20 lb., and the block swings to a position $\frac{1}{8}$ in. higher than its lowest position. Show how to calculate (*a*) the maximum velocity of the swinging block, (*b*) the initial velocity of the bullet to the nearest foot-second. (*O.C.*)

18. Distinguish between momentum and kinetic energy.

Which is conserved during a collision and what happens to the other?

A bullet weighing 30 gm. and travelling at 500 metres/sec. embeds itself in a suspended lump of wood weighing 7·47 kgm. How far will this block have risen above its original position when it reaches the end of its swing? If the length of the suspension is 50 cm., how far will the block have swung in a horizontal direction? [*g* may be assumed to be 1,000 cm./sec.^{2}] (*G.S.*)

19. A projectile weighing 0·1 lb. travelling horizontally embeds itself in a mass of 10 lb. suspended at the end of a cord and displaces it so as to raise it 0·2 in. against gravity. Find the velocity of the projectile. State the reasons for each step in your calculations. (*L.M.*)

20. Distinguish between momentum and kinetic energy, and explain how each is related to forces. A pendulum bob suspended by a string 1 m. long is pulled aside until the string makes an angle of 60° with the vertical and then released. Find the speed with which the bob will be moving when it reaches its lowest point. (*L.M.*)

21. Distinguish between momentum and kinetic energy. A projectile weighing 200 gm. and moving with a velocity of 500 metres/sec. embeds itself in the bob of a pendulum weighing 200 kgm. What is the maximum vertical height to which the bob will rise in its swing backwards? Trace out what happens to the momentum and kinetic energy of the projectile. (*L.M.*)

240. Impulsive Tensions.

Suppose that two masses are connected by a light inextensible string and are set in motion on a smooth horizontal table in

opposite directions in the same straight line. When the string becomes taut a large impulsive tension is produced which at every instant of the time it acts exerts equal and opposite forces on the two masses. The motion of the masses is therefore altered, but the change in momentum of one mass is equal and opposite to that in the other. This change is an instantaneous one, and unless the masses are reduced to rest they will afterwards move with the same velocity, and the tension in the string then becomes zero.

If one mass is at rest on a smooth table and the other is pushed over the edge, the string being slack at first, there is again a jerk when the string becomes taut. Although in this case the impulses given to the masses are not in the same straight line, the magnitudes are equal, and therefore the changes in momenta are equal. Since the jerk increases the momentum of one body by an amount equal to that destroyed in the other, the momentum of the system is unaltered by the jerk.

Example (1).—*Two masses of 5 and 7 lb. are connected by a light inextensible string of length 2 ft. The 5-lb. mass rests on a smooth horizontal table 1 ft. from the edge, and the 7-lb. mass rests on the table close to the edge, the line joining the masses being perpendicular to the edge. If the 7-lb. mass is pushed gently over the edge, find the speed with which the 5-lb. mass is jerked into motion and its speed when it reaches the edge of the table.*

The speed of the 7-lb. mass after falling freely through 1 ft. is

$$\sqrt{2g \times 1} = 8 \text{ ft./sec.}$$

and its momentum is $7 \times 8 = 56$ lb. ft./sec. units.

The string then becomes taut, and if v ft./sec. is the common speed of the masses after the jerk

$$12v = 56,$$

since the total momentum is unaltered.

Therefore $$v = 4\tfrac{2}{3} \text{ ft./sec.}$$

The masses then continue to move with equal velocities, the acceleration f being given by

$$7g = 12f,$$

or $$f = \frac{7}{12}g.$$

After moving 1 ft. to the edge, the velocity V of the 5-lb. mass will be given by

$$V^2 = \left(\frac{14}{3}\right)^2 + \frac{14}{12}g \times 1 = \frac{196}{9} + \frac{32 \times 14}{12} = \frac{532}{9},$$

herefore

$$V = \frac{23{\cdot}06}{3} = 7{\cdot}69 \text{ ft./sec.}$$

Example (2).—*Two masses of* 3 *and* 5 *lb. are connected by a light inextensible string which passes over a smooth fixed pulley. The masses are released from rest with the string taut, and after* 1 *sec. the* 3*-lb. mass picks up a mass of* 4 *lb., which is at rest. Find the speed of the system after the jerk.*

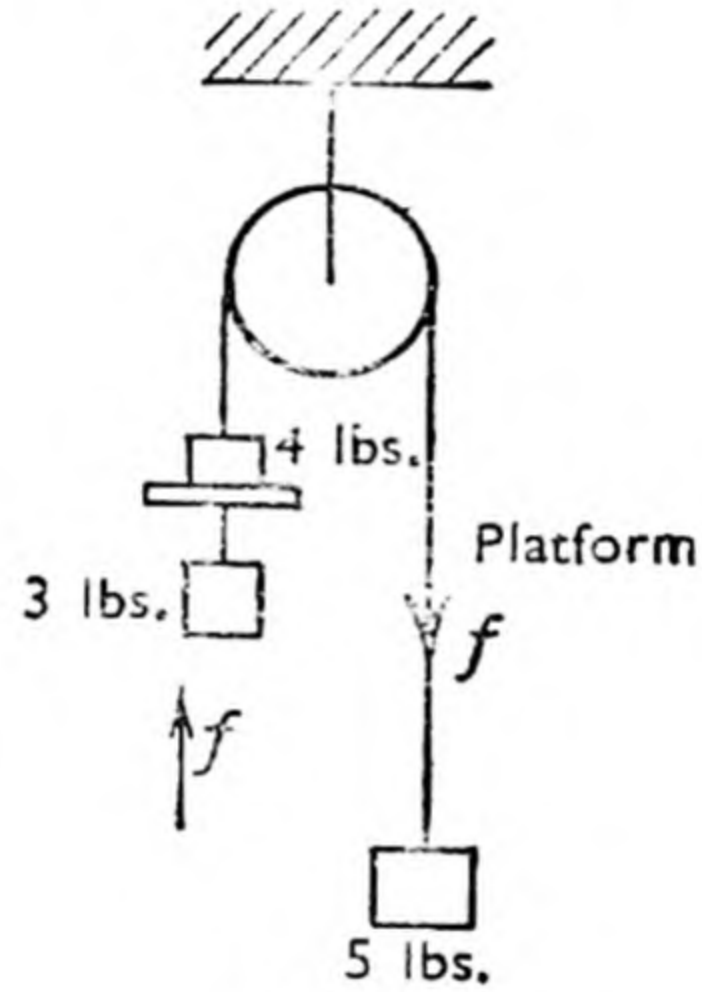


FIG. 399.

Although in this case the masses are not moving in the same straight line, we can still say that the total momentum of the system is unaltered by the jerk.

We must first find the acceleration f of the system when the masses are released.

If T is the tension in the string,

For the 5-lb. mass $5g - T = 5f$,

For the 3-lb. mass $T - 3g = 3f$,

therefore $2g = 8f$, or $f = \frac{1}{4}g = 8$ ft./sec.^{2}

After 1 sec. the velocity acquired is 8 ft./sec.

The momentum of the system is $8 \times 8 = 64$ lb.-ft./sec. units.

After the 4-lb. mass is picked up, the total mass in motion is 12 lb., and if v is the common speed of the system

$$12v = 64,$$

or

$$v = 5\tfrac{1}{3} \text{ ft./sec.}$$

EXAMPLES LXIV

1. Two masses of 1 and 2 lb. are connected by an inelastic string of length 2 ft. The 2-lb. mass rests on a smooth horizontal table and is 1 ft. from the edge. If the 1-lb. mass is let fall from the edge of the table, find the velocity with which the 2-lb. mass is jerked into motion.

2. Two masses, each of 2 lb., connected by a string passing over a smooth pulley, are moving vertically with a velocity of 2 ft./sec. The ascending mass passes through a fixed ring without touching it and removes from the ring a mass of 4 oz. which it carries with it. Find the height to which the 4-oz. mass is carried.

3. Two masses of 3 and 5 lb. are tied to the ends of a string 13 ft. long. The string passes over a smooth peg 8 ft. above a horizontal table, the 5-lb. mass lying on the table, the 3-lb. mass being held close to the peg. If the 3-lb. mass is let fall, show that the 5-lb. mass is jerked into motion with initial velocity $3\sqrt{5}$ ft./sec., and that the 3-lb. mass reaches a point $\frac{5}{16}$ ft. above the table before it comes to rest instantaneously prior to rising again.

4. Two masses of 10 and 8 oz. are connected by a light string passing over a smooth fixed pulley. The system starts from rest, and the 8-oz. mass, after it has risen 3 in., passes through a fixed ring on which rests a bar of mass 2 oz., and so carries the 2-oz. mass on with it. Show that the common velocity of the system just after the 2-oz. mass is picked up is $1\frac{1}{6}$ ft./sec.

241. Collision of Elastic Bodies.

The property by virtue of which a body tends to resume its original shape when being deformed is called Elasticity. The force tending to restore the shape varies considerably for different materials; *e.g.*, it is great in the cases of steel and iron, but almost negligible in the case of lead.

The force of restoration during the collision of two elastic balls, such as billiard balls, increases from zero at the moment of contact to a maximum value when the deformation has reached its maximum. After this the shape is restored and the balls are forced apart. In the case of an elastic ball striking a wall the ball becomes flattened to an extent depending on the speed with which it strikes, and after being reduced to rest the elastic force tending to restore the shape forces it back from the wall.

In both cases the forces acting on the two bodies at any

instant during contact are equal and opposite, and if both are free to move, the changes in their momenta are equal and opposite.

If a ball of mass m strikes a wall in a direction perpendicular to it with velocity u and rebounds with velocity v, the impulse on the ball is measured by the change produced in its momentum. Since momentum mu in one direction has been destroyed and momentum mv generated in the *opposite* direction, the total change is $mu + mv$.

Therefore the impulse given to the ball $= m(u + v)$.

It is clear that for a given value of u the value of v will depend on the materials of which the ball and the wall are made ; *i.e.*, on the degree of their elasticity.

This cannot be calculated theoretically, and we have to fall back on the results of experiments.

In the same way if two bodies of masses m_1 and m_2 moving in the same straight line collide with velocities u_1 and u_2 respectively, we can deduce from the principle of momentum that if v_1 and v_2 are their velocities after impact

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \quad . \quad . \quad . \quad (i)$$

In the cases previously dealt with, the bodies were supposed to keep together after impact so that $v_1 = v_2$, and knowing the values of m_1, m_2, u_1 and u_2, we could calculate the value of v_1 from this one equation.

When the bodies separate we cannot find either v_1 or v_2 from this one equation.

To obtain another equation we must make use of the results of experiments on bodies made of the kind of material in question.

Newton investigated this matter by comparing the relative velocity with which two spheres approached each other with the relative velocity with which they separated. This can be done in the following manner :—

EXPERIMENT (1).—Two balls of different masses are suspended, each by two threads, as shown in Fig. 400. When the balls are pulled aside and then released, they collide when both are moving horizontally. The horizontal displacements of the balls before and after the collision are proportional to the velocities of the balls in their lowest position before

and after impact. The total horizontal distance apart of the balls before they are released and their maximum horizontal distance apart after collision are proportional respectively to the magnitudes of their relative velocities before and after impact.

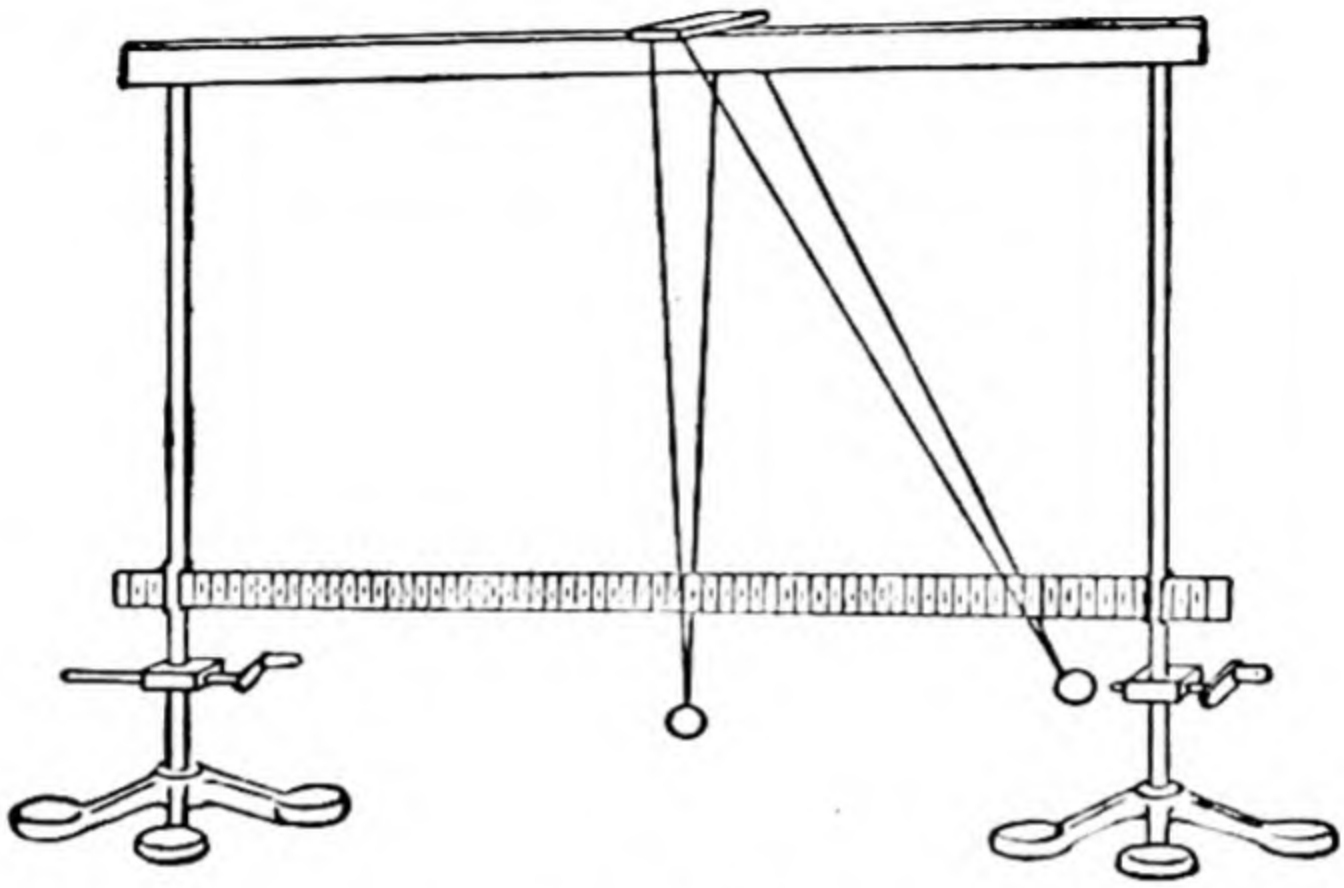


FIG. 400.

Using bodies of the same material it is found that

$$\frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}} = \text{Constant.}$$

This constant depends only upon the materials of which the bodies are made and is independent of their masses.

The numerical value of the constant is called the **Coefficient of Elasticity,** or better, the **Coefficient of Restitution** for the materials of the bodies, and is denoted by e.

Let u_1 and u_2 be the velocities of the bodies before impact, and v_1 and v_2 the velocities after impact, **all measured in the same direction**, the relative velocity of separation is $v_1 - v_2$ and the relative velocity of approach is $u_1 - u_2$. As these velocities are in opposite directions, one of them must be considered as negative and the other positive, we therefore have

$$\frac{v_1 - v_2}{u_1 - u_2} = -e,$$

where e denotes the numerical value of the coefficient of restitution.

This gives us a second equation,

$$v_1 - v_2 = -e(u_1 - u_2) \quad . \quad . \quad . \quad . \quad \textbf{(ii)}$$

And by means of this and equation (i) above we can find the values of v_1 and v_2.

The value of e differs considerably for different bodies; for two glass balls it is about 0·9; for ivory, 0·8; while for lead it is about 0·2.

Bodies for which e is zero are said to be inelastic, whilst for a **perfectly elastic** body the value of e would be unity.

Another method of determining the value of e is described in the following experiment:

EXPERIMENT (2).—A ball is dropped on to a plane horizontal fixed slab of the same material.

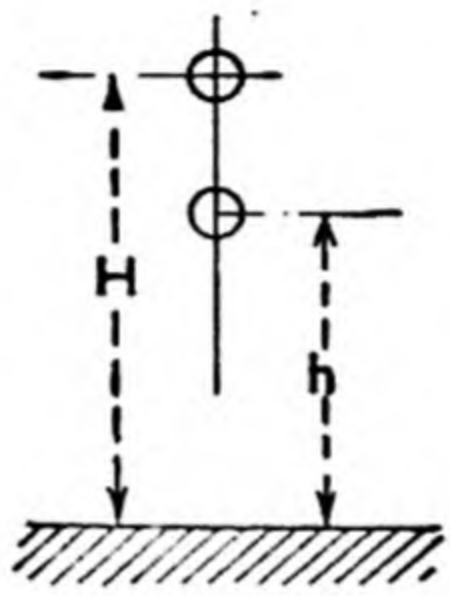


FIG. 401.

The speed at which the ball hits the surface and the speed with which it rebounds are equal respectively to $\sqrt{2gH}$ and $\sqrt{2gh}$, where H and h are the heights from which the ball is dropped and to which it rebounds.

Therefore $$\frac{\text{Speed of separation}}{\text{Speed of approach}} = \frac{\sqrt{2gh}}{\sqrt{2gH}} = \sqrt{\frac{h}{H}} = e.$$

By observing a series of values of H and h an average value of e may be obtained.

242. Direct Impact of Two Spheres.

Impact is said to be *direct* when the direction of motion of each body is along the common normal to the surfaces at the point where they meet. In the case of two spheres this means that they must be moving along the line through their centres.

Let m_1, m_2 be the masses, u_1, u_2 the velocities before impact and v_1, v_2 the velocities after impact and e the coefficient of restitution.

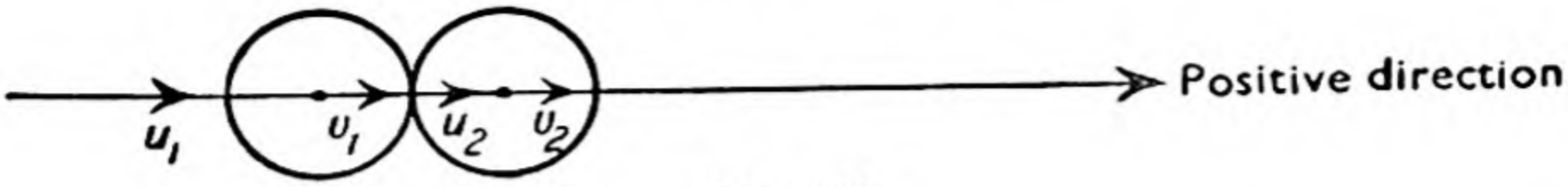


FIG. 402.

By the principle of momentum we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \quad . \quad . \quad . \quad \text{(i)}$$

By Newton's Experimental Law we have

$$v_1 - v_2 = -e(u_1 - u_2) \quad . \quad . \quad . \quad \text{(ii)}$$

v_1 and v_2 can be calculated from these equations when we know the values of m_1, m_2, u_1, u_2 and e.

In all cases such as this the greatest care must be taken to give each velocity its correct sign after deciding on which direction we are going to consider positive. This is usually taken as that in which the body with greater momentum is moving, and we will suppose this is m_1. If the other sphere, m_2, is moving originally in a direction opposite to that of m_1, we must change the sign of u_2 in each of the equations (i) and (ii).

It is most important, however, that we should assume that v_1 and v_2 are in *the same direction*, the positive one. If either of them is really in the opposite direction, the value obtained from the equations will have a negative sign.

In writing down equation (ii) great care must be taken to subtract the velocities in the same order on both sides. It is best to draw a diagram showing clearly the positive direction and the directions of motion of both bodies before impact.

These points are illustrated in the following examples.

Example (1).—*Balls of masses 3 and 5 lb. are moving in the same straight line in the same direction with velocities of 10 and 4 ft./sec. respectively. If the coefficient of restitution between them is 0·8, find their velocities after impact.*

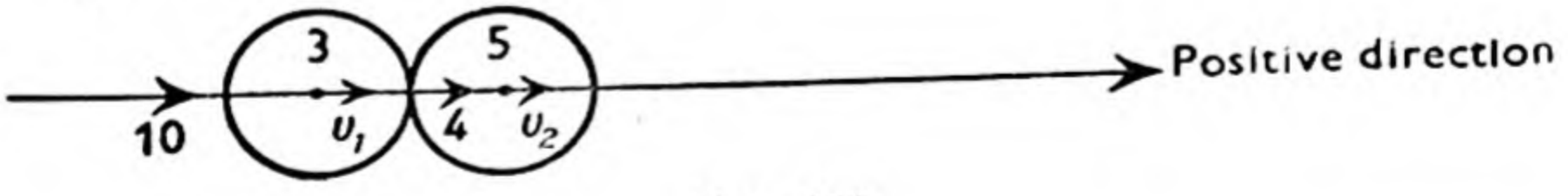


FIG. 403.

Let v_1, v_2 be the velocities of the 3 and 5 lb. spheres respectively after impact. By the principle of momentum

$$3v_1 + 5v_2 = 3 \times 10 + 5 \times 4 = 50 \quad . \quad . \quad . \quad \text{(i)}$$

and by Newton's Law,

$$v_1 - v_2 = -\frac{8}{10}(10 - 4) = -\frac{48}{10} \quad . \quad . \quad . \quad \text{(ii)}$$

therefore

$$5v_1 - 5v_2 = -24 \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Adding equations (i) and (iii),

$$8v_1 = 26, \text{ or } v_1 = 3{\cdot}25 \text{ ft./sec.}$$

Substituting in (ii),

$$v_2 = 3{\cdot}25 + 4{\cdot}8 = 8{\cdot}05 \text{ ft./sec.}$$

Example (2).—*If the balls in the preceding example are moving in opposite directions, find their velocities after impact.*

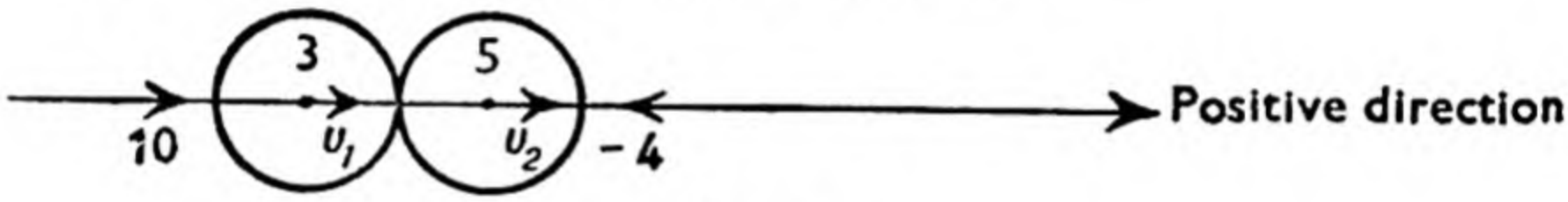


FIG. 404.

Since the 3-lb. ball has the greater momentum we choose its direction of motion as the positive direction, and we assume that both v_1 and v_2 are in the same direction.

The momenta before impact are now in opposite directions and from the principle of momentum

$$3v_1 + 5v_2 = 3 \times 10 - 5 \times 4 = 10 \quad . \quad . \quad . \quad \text{(i)}$$

The relative velocity before impact is now $10 + 4$ or 14 instead of $10 - 4$.

From Newton's Law

$$v_1 - v_2 = -\frac{8}{10}(10 + 4) = -\frac{112}{10} \quad . \quad . \quad . \quad \text{(ii)}$$

therefore

$$5v_1 - 5v_2 = -56 \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Adding equations (i) and (iii)

$$8v_1 = -46, \text{ or } v_1 = -5{\cdot}75 \text{ ft./sec.}$$

This means that after impact the direction of motion of the 3-lb. ball is reversed.

Substituting in (ii),

$$v_2 = -5{\cdot}75 + 11{\cdot}2 = 5{\cdot}45 \text{ ft./sec.}$$

The direction of motion of the 5-lb. ball is also reversed.

Example (3).—*Two balls of masses 2 and 3 lb. moving in the same straight line in opposite directions with velocities of 8 and 2 ft./sec. respec-*

tively collide. If the coefficient of restitution is $\frac{3}{4}$, *find the velocities of the balls after impact.*

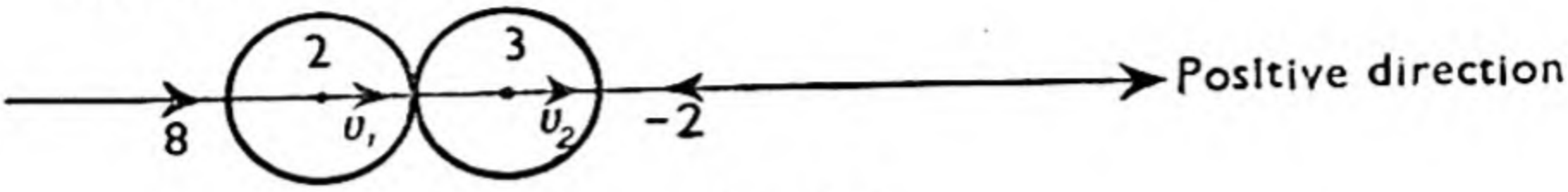


FIG. 405.

Let v_1 and v_2 be the velocities of the 2 and 3 lb. balls respectively after impact.

By the principle of momentum

$$2v_1 + 3v_2 = 16 - 6 = 10 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

By Newton's Law

$$v_1 - v_2 = -\tfrac{3}{4}(8 + 2) = -\frac{30}{4} \quad . \quad . \quad . \quad \text{(ii)}$$

therefore

$$3v_1 - 3v_2 = -\frac{90}{4} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Adding (i) and (iii)

$$5v_1 = 10 - \frac{90}{4} = -\frac{50}{4},$$

therefore

$$v_1 = -\frac{50}{20} = -2{\cdot}5 \text{ ft./sec.}$$

Substitute in (ii)

$$v_2 = -2{\cdot}5 + 7{\cdot}5 = 5 \text{ ft./sec.}$$

The directions of motion of both balls are reversed.

243. Direct Impact of a Body on a Fixed Plane.

Let m be the mass of the body and u the velocity with which it meets the plane and e the coefficient of restitution for the body and the plane.

Since the plane is fixed there is an external force acting on the system during impact and the principle of momentum cannot be applied.

To find the velocity of rebound we must use Newton's Law. From this we know that the relative velocity perpendicular to the plane after impact is $-e$ times the relative velocity before impact, measured in the same direction.

If v is the velocity of rebound,

$$v = -eu.$$

Remembering that the velocity of rebound is in the opposite direction to that of impact, we may say that

$$\text{Velocity of rebound} = e \times \text{velocity of impact.}$$

EXAMPLES LXV

1. A particle falls freely through a height of 16 ft. and strikes a horizontal plane from which it rebounds. If the coefficient of restitution between particle and plane is $\frac{3}{4}$, find the height to which it ascends after rebound.

2. A ball falls through a height of 8 ft. on to a horizontal plane. After rebounding it rises to a height of $4\frac{1}{2}$ ft. above the plane. Find the coefficient of restitution between ball and plane.

3. A small, smooth steel ball of mass 40 gm. slides down a smooth inclined plane from a height of 10 cm. on to a smooth horizontal plane. Just after reaching the horizontal it collides directly with a ball of equal size but mass 20 gm.

(*a*) Find the velocities of the two balls after the collision, taking the coefficient of restitution or elasticity as 0·9.

(*b*) State clearly the principles on which you have based your calculation of what happens at the collision.

(*c*) State in general terms, without calculation, what changes in kinetic energy occur during the collision. (*J.*)

(It may be assumed that the first ball undergoes no impulsive action before striking the second ball.)

4. A sphere of mass 3 lb. moving at 4 ft./sec. overtakes another sphere of mass 2 lb. moving in the same direction with velocity 2 ft./sec. If $e = \frac{1}{4}$, find the velocities after impact.

5. A ball of mass 5 lb. moving at 8 ft./sec. overtakes another of mass 4 lb. moving in the same direction at 5 ft./sec. If $e = \frac{1}{2}$, find the velocities after impact.

6. A ball of mass 10 lb. moving at 8 ft./sec. impinges directly on a ball of mass 8 lb. moving in the opposite direction at 4 ft./sec. If $e = \frac{1}{3}$, find their velocities after impact.

7. A ball of mass m moving at 7 ft./sec. overtakes another of mass $2m$ moving in the same direction at 1 ft./sec. If $e = \frac{3}{4}$, show that the first ball will remain at rest after the impact.

8. Two spheres of masses 2 and 3 oz. are moving in their line of centre towards each other with velocities 24 ft./sec. and 30 ft./sec., and their coefficient of restitution is $\frac{3}{4}$. Find their velocities after impact and the loss in kinetic energy due to the impact.

9. If two perfectly elastic spheres, of equal mass and moving in opposite directions, impinge directly, show that they will exchange velocities.

10. A sphere impinges directly on an equal sphere at rest. If the coefficient of restitution be e, show that their velocities after impact are as $(1 - e) : (1 + e)$.

ANSWERS

EXAMPLES I

1. 31·416 miles ; $\frac{1}{800}$.
2. 150·2 ; 147·7 ; $\frac{3}{127}$.
3. At least 1 in 24.
4. 0·9 per cent.

EXAMPLES II

1. Fraction $= \frac{11}{80}$; $3\frac{9}{32}$ in.
2. $\frac{5}{64}$.
3. 3·14.

EXAMPLES III

1. 4·32 mm.
2. 0·653 cm.
3. 0·6554.

EXAMPLES IV

3. 8·55 lb. wt./in.
4. Extension not proportional to load after 0·4 lb. Spring is then overstretched.

EXAMPLES V

1. 300 lb. wt. ; 300 lb. wt. ; $85\frac{6}{7}$ lb. wt.
2. 1 ton wt.
3. 8 st.
4. 10 lb. wt. ; 5 lb. wt. vertically downwards ; 7 lb. wt. vertically upwards.
5. 4 ton wt.
6. 5 lb. wt. ; 5 lb. wt.
7. $7\frac{1}{2}$ lb. wt. ; the 10 lb. will descend.
8. Tension 3 lb. wt. ; reading of balance 6 lb. wt. ; force on beam 12 lb. wt.
9. $S = 7$ lb. wt. ; $T_1 = 5$ lb. wt. ; $T_2 = 2$ lb. wt.

EXAMPLES VI

1. 2 ft. from C on side of pivot opposite to 20 lb. wt.
2. 5·4 in. on side of pivot opposite to 3 lb. wt. ; 8 lb. wt. vertically downwards.
3. 40 lb. wt.
4. $4\frac{1}{2}$ ft. ; 272 lb. wt. vertically downwards.
5. (*a*) 1 lb. wt. vertically upwards ; (*b*) 1 lb. wt. vertically downwards.
6. Weight 40 lb. ; centre of gravity is 2 ft. 6 in. from A.

EXAMPLES VII

1. $11\frac{3}{8}$ lb. wt. and $18\frac{1}{8}$ lb. wt.
2. 1 ft. 4 in. from B.
3. $T = 10$ lb. wt., 30 lb. wt.
4. $33\frac{1}{3}$, $50\frac{2}{3}$ lb. wt. ; $1\frac{5}{21}$ ft. from midpoint.
5. $5\frac{1}{4}$ lb., $7\frac{3}{4}$ lb.
6. $58\frac{1}{3}$ lb. wt., $31\frac{3}{8}$ lb. wt., $\frac{3}{5}$ ft. from centre of table.
7. 3 ft. 6 in. from this man's end.
8. $1\frac{1}{7}$ lb.
9. $\frac{1}{3}$ ton wt. vertically upwards.
10. $34\frac{7}{8}$ cwt., $28\frac{7}{8}$ cwt.
11. 46 lb., $5\frac{2}{3}\frac{3}{5}$ ft.

EXAMPLES VIII

1. $3\frac{3}{7}$ lb. wt.
2. P=46·2 lb. wt.
3. 120 lb. wt.
4. 250 lb. wt.
5. 70·7, 50 lb. wt. (AB uppermost), 86·6 lb. wt. (CD uppermost).
6. 1,200, 800 lb. wt.
7. 1 ; 0·707 ton wt.
8. 6·93 lb. wt.
9. 34·6 lb. wt.
10. 11·3 lb. wt.
11. At the beginning ; 2·61 cwt.
12. 520 ; 41·9 lb. wt.
13. 93 lb.
14. 63·6 lb. wt.
15. 10·6 lb. wt., 42·4 lb. wt.
16. 0·26 lb. wt.
17. 2 ft. from lower end.
18. $\frac{7}{15}$ ton wt.
19. (*a*) 245 ; (*b*) 212·2 ; (*c*) 122·5 lb. wt.
20. 1,375 lb. wt.

EXAMPLES IX

1. (*a*) 32 ; (*b*) $2\frac{2}{5}$; (*c*) $2\frac{10}{11}$; (*d*) $13\frac{1}{3}$; (*e*) $21\frac{3}{5}$ lb. wt.
2. $13\frac{1}{3}$ lb. wt.
3. 27 in.
4. 4,320 lb. wt.
5. $4\frac{4}{5}$ lb. wt.
6. 14 lb. wt.
7. $45\frac{4}{7}$ lb. wt.

EXAMPLES X

1. $84\frac{1}{2}$ lb.
2. 40 in.
3. 10·5 lb., 5 lb. ; $\frac{1}{2}$ inch.
4. 2·2 in. from scale pan end ; 1·1 in.
5. 4·8 in. from scale pan end.
6. 2·5 lb., 27 in.
9. 2 in.
10. 17 in.
11. 13·2 lb.
14. $\frac{11}{10}$.

EXAMPLES XI

1. 1·76 in. from AB.
2. $8\frac{4}{11}$ in.
3. 5·43 in. from CD ; 8·14 in. from BC.
4. $33\frac{1}{2}°$.

EXAMPLES XII

1. 2 ft. 9 in. from A.
2. 10 ft. from A.
3. 15·32 ft.
4. $13\frac{5}{8}$.
5. $13\frac{3}{7}$.
6. 0·448 in. from centre of disc.
7. $3\frac{3}{5}$ in.
8. $1\frac{1}{8}$ in.
9. 8·9 cm. from apex of cone.
10. 1·62 ft.
11. 3·79 in. from end of portion, diameter $1\frac{1}{4}$ in.
12. 10 in.
13. $\frac{3}{13}$ in. from centre of disc.
14. 5·12 in.
15. 2·4 in. from centre.
16. 2·8 ft. from broader end.
17. 10·5 in. from DC on the perpendicular bisecting DC.

EXAMPLES XIII

1. $12\frac{3}{8}$ ft. ; $10\frac{1}{2}$ ft.
2. $1\frac{3}{4}$ ft. from AD ; $1\frac{2}{7}$ ft. from AB.
3. 2 in. from AC ; $2\frac{6}{17}$ in. from AB.
4. $1\frac{1}{2}$ in. from CD ; $2\frac{1}{24}$ in. from BC.
5. $9\frac{7}{12}$ cm. from BC and CD.
6. 2·85 in. from AB ; 4·03 in. from AD.
7. $\frac{6}{13}$ in. from centre of sheet.
8. $1\frac{6}{7}$ in. from 3-ft. rod ; $1\frac{1}{14}$ ft. from 4-ft. rod.
9. $\frac{2\sqrt{3}}{7}$ ft. or 0·495 ft. from AB.
10. $\text{Tan}^{-1}\frac{1}{8}$ or 7° 7′.
11. 10 in. below centre of table-top.
12. 756 lb.
13. $54\frac{9}{16}$ lb.
14. At a point X in AB, $5\frac{1}{2}$ in. from B. Yes ; any point vertically below X and less than $\frac{\sqrt{3}}{2}$ in. from it.
15. $4\frac{2}{3}$ cm. from AD; $3\frac{1}{2}$ cm. from CD.
16. $\frac{3}{16}$ in. from the centre line.
17. 10·71 lb. placed at edge of table on a line joining any leg to centre of table.
18. 0·288 in. from centre of hexagon.
19. $4\frac{1}{2}$ in. from B ; 2 lb.
20. (*a*) 2,102 lb. wt. ; (*b*) 1 ft. 7 in. above the beam and 3 ft. 4 in. from vertical end of wall ; (*c*) 700 lb. wt. at A ; 1,400 lb. wt. at B.

EXAMPLES XIV

8. 17·32 cm.
9. 53° 8 ft′.
11. 25 lb. wt. ; $10\sqrt{5}$ or 22·4 lb. wt. at $\tan^{-1}\frac{1}{2}$ or 26° 34′ with the horizontal to midpoint of one of the top edges.
12. 1·5 in.
13. 7 : 9.
14. No. The centre of gravity is $\frac{8\sqrt{5}}{25}$ of side of square from AD.
15. $\frac{2}{11}$ lb.
17. 4 cm. from the heavy end.

EXAMPLES XV

1. (*a*) 3,600 ft. lb. ; (*b*) $1\frac{17}{25}$ ft. tons.
2. $235\frac{4}{7}$ ft. tons.
3. $16\frac{1}{4}$ ft. tons.
4. 9,240 ft. lb.
5. 600 ft. tons ; 150 ft. tons.
6. 225,000 ft. lb.
7. 10,500 ft. tons.
8. $12\frac{1}{4}$ ft.
9. 16,200 ft. lb.
10. $471\frac{3}{7}$ ft. tons.
11. $288\frac{3}{4}$ ft. tons.
12. 3,360 ft. tons.
13. 72,000 ft. tons.
14. 275,000 ft. lb.

EXAMPLES XVI

1. 3 ; 12 lb.
2. 6 ; 6 ; 6 in.
3. 80 per cent. ; $\frac{1}{4}$.
4. $3\frac{1}{7}$; 5·04 in.
5. 100 lb. ; $\frac{6}{7}$.
6. 75 per cent. ; 80 per cent.
7. 24,000π ft. lb. ; 960 lb.
9. 160 lb.

EXAMPLES XVII

1. 8 lb. ; $9\frac{1}{16}$ lb. ; 25·8 ; 32.
2. 20 lb. ; 21 lb. ; $5\frac{4}{7}$; $\frac{20}{21}$; 153 lb.
3. $3\frac{11}{25}$; 0·87.
5. $1\frac{2}{3}$ lb.
8. 6 ; 7.
10. 80 per cent. ; 14 lb.
11. $33\frac{1}{3}$ lb. ; 3.
12. 0·97.
13. 10 lb. ; 80 per cent.
14. $2\frac{1}{4}$ cwt.
15. 40 lb.
16. 65 per cent. ; 400 lb. wt.
17. 10 st. 3 lb.
18. The 3-sheave block should be attached to the load ; 6 times the effort is then available for moving the load.
19. 4 ; 3·8 ; 95 per cent.
20. $83\frac{1}{3}$ per cent.

EXAMPLES XVIII

1. 18 ; $\frac{2}{3}$.
2. 16.
3. 9 ; 32 ; 64 ; 20 ; 40 ; 80.
4. (*a*) 320 lb. ; 208 lb.
   (*b*) 10·4.
   (*c*) 35 lb. ; 100 lb.
5. 38·9 per cent. ; 61·1 per cent. of input ; it can be useful in preventing overhauling.
6. 5 lb.
7. 40 per cent. ; 2·4 lb.
8. 46·9 per cent.

EXAMPLES XIX

1. 480 ft. tons ; 80 per cent.
2. 720 ft. tons ; $83\frac{1}{3}$ per cent.
3. 56 ft. tons.
4. 97·1 ft. tons ; 96·6 per cent.
5. 30 ft. lb. ; 9 ft. lb. ; 77 per cent.
6. 1,008 ft. lb. ; 168 lb. wt. ; 194 lb. wt.

EXAMPLES XX

1. 108π ; 42·4 lb.
2. 27·8 lb.
3. $\frac{5\pi}{2}$ in.
4. $\frac{1}{5}$ of a turn ; $\frac{7}{10}$ in. ; 3,200.
5. $\frac{4}{5}$; $\frac{4}{9}$.
6. (*a*) $\frac{1}{10}$; (*b*) $\frac{4}{33}$.
7. $33\frac{1}{3}$; 1,000 lb.
8. 0·91 cwt.
9. 2,160 lb.
10. 108 ; 1,350 lb.
11. $13\frac{1}{3}$ revs. per min.
12. 96.
13. 88 ; 25·5 per cent. ; 77·1 lb.
14. 124·4 lb.
15. 6 ; 60 per cent.
16. 10·5 ; 3·5 ; $33\frac{1}{3}$ per cent.
17. $86\frac{2}{13}$ lb. ; 265·6 ft. lb.
18. 2 ; $3\frac{2}{3}$; $5\frac{1}{3}$; 7 lb. wt. ; 15 ; $12\frac{3}{11}$; $11\frac{1}{4}$; $10\frac{2}{7}$.
19. 7.
20. 3·96 lb. wt.
21. 32 lb. per sq. in.

EXAMPLES XXI

2. $E = 0{\cdot}084L + 30$; 47·5 lb. wt.
3. $a = 0{\cdot}0864$; $e = 2{\cdot}83$.
4. 10 per cent. ; 19·2 per cent.
5. $E = 0{\cdot}383L + 3{\cdot}2$.
6. 480 lb. ; 40 per cent.
7. 26 per cent. ; $a = 2{\cdot}2$, $b = 0{\cdot}085$.
8. 21·8 lb. ; 104 lb.

EXAMPLES XXII

2. 10 lb. ; 100 lb. ; 20 lb. ; 15 lb.
3. $F = 0{\cdot}0786L + 1{\cdot}5$.
4. $F = 0{\cdot}033L + 3{\cdot}08$.

EXAMPLES XXIII

1. (*a*) 66 ft./sec. ; (*b*) 45 m.p.h.
2. A = 10 m.p.h., B = 7·5 m.p.h.
4. 3·06 ft./sec.
5. 3·06 ft./sec. ; 175 revs. per min.
6. 49·3 m.p.h. ; 20·0 m.p.h. ; 56·0 m.p.h. ; 42·1 m.p.h.
7. 32·0 ft./sec. ; 31·9 ft./sec.
8. 159·2 m.p.h.
9. 107·5 m.p.h. ; 100·6 m.p.h. ; 91·8 m.p.h.
10. A = 10·8 knots ; B = 12·5 knots.
11. $7\frac{1}{2}$ knots ; $27\frac{1}{2}°$ W. of N.
12. (i) (*a*) 10·2 m.p.h. ; (*b*) $13\frac{7}{11}$ m.p.h. ; (*c*) $20\frac{5}{11}$ m.p.h.
    (ii) (*a*) 105·6 ft./sec. ; (*b*) 28·8 ft./sec. ; (*c*) 116·6 ft./sec.
14. 96·6 km./hour ; 2,682 cm./sec.
15. (*a*) 750 m.p.h. ; (*b*) 892,800 times as fast.
16. (*a*) 18·6 m.p.h. ; (*b*) 16·7 m.p.h. ; (*c*) 13·8 m.p.h. ; (*d*) 13·7 m.p.h.
17. A beats C by 3 yds. 1 ft.

EXAMPLES XXIV

1. 60 ; 40 ; 32 m.p.h.
2. 1 hr. 45 min., 3 hrs. 30 min., 5 hrs. 15 min., 0, 2 hrs. 20 min., 4 hrs. 40 min. ; farthest apart at 1 hr. and 6 hrs., distance 2 miles.
3. After $12\frac{1}{2}$ min. ; (i) $1\frac{1}{3}$ miles ; (ii) $\frac{2}{3}$ mile.
4. Pedestrian passes cyclist after 1 hr. 20 min., 2 hrs. 40 min., 4 hrs., 5 hrs. 20 min., 6 hrs. 40 min. ; cyclist passes pedestrian after 1 hr. 36 min., 3 hrs. 12 min. ; 4 hrs. 48 min., 6 hrs. 24 min. ; after 8 hrs.
5. (i) 20 m.p.h. ; (ii) 30 m.p.h. ; (iii) $1\frac{1}{2}$ to $3\frac{1}{2}$ min.
6. After 1 hr. 10 min.
7. 18 ft./sec. ; 16·8 ft./sec.
8. 160 ft./sec. after 5 sec. ; 16 ft./sec.
9. 48 ft./sec. ; 80 ft./sec. ; 113 ft./sec. ; 145 ft./sec.
10. 12·5 in./sec.
11. 7·4 ft./sec. ; 13·1 ft./sec. ; 15·5 sec.
12. 5·36 ft./sec. ; 0·257 sec.

EXAMPLES XXV

1. $56\frac{1}{4}$ ft.
2. 1,050 ft.
3. $\frac{1}{10}$ mile ; total distance 1·5 miles.
4. 0·37 mile.
5. 0·636 rad./sec.
6. 1·88 ft. ; 3·13 ft./sec.
7. (*b*) 22 m.p.h. ; (*c*) 2 miles ; (*d*) average speed 24·6 m.p.h.

EXAMPLES XXVI

1. (*a*) 10·9 ; (*b*) 76·97.
2. (*a*) 3,600° ; (*b*) 1,146° ; (*c*) 14,400°.
3. (*a*) 10·47 ; (*b*) 0·698.
4. 0·00014 ; 0·00174.
5. 10·47 rad./sec. ; 2·62, 4·36, 10·47 ft./sec.
6. (*a*) $\frac{1}{30}$; (*b*) $\frac{\pi}{12}$; (*c*) 1,047 m.p.h.
7. 0·0172 rad./day ; 1,076.
8. 3 for rollers ; $\frac{2}{3}$ for pulley.

EXAMPLES XXVII

1. 30 ft./sec.
2. $1\frac{1}{3}$ ft./sec.^{2}
3. 39 min. $6\frac{2}{3}$ sec.
4. $183\frac{1}{3}$ sec. ; $36\frac{2}{3}$ ft./sec.
5. 29 ft./sec. ; greatest 44 ft./sec.
6. (*a*) 0·733 ft./sec.^{2} ; (*b*) $1\frac{2}{3}$ ft./sec.^{2} ; (*c*) 242 ft./sec.^{2}
7. No.
8. (*a*) $3\frac{3}{5}$ ft./sec.^{2} ; (*b*) 26·1 cm./sec.^{2}
9. 1181·1 yds./min.^{2}

EXAMPLES XXVIII

1. Yes ; 0·65 ft./sec.^{2} ; 495 yds.
2. 0·2 ft./sec.^{2} ; 2·18 miles.
3. Acceleration uniform for first second, and then speed becomes approx. constant.
4. 4,275 yds.
5. $2{,}951\frac{3}{5}$ ft.
6. 48 m.p.h.
7. $127\frac{3}{11}$ sec.
8. 877 sec.
9. 6 in./sec.
10. 300 m./hr.^{2} ; $2\frac{1}{8}$ miles.
11. 21 and 10·5 ft./sec.
12. 0·4 ft./sec.^{2}

EXAMPLES XXIX

1. 36 ft./sec. ; 540 ft.
2. 4·5 ft./sec.^{2} ; 225 ft.
3. 1·5 ft./sec.^{2} ; 3 ft./sec.^{2} ; 66 ft./sec. ; 132 ft./sec.
4. 8 ft./sec.^{2} ; 121 ft.
5. 2·87 ft./sec.^{2} ; $20\frac{6}{11}$ sec.
6. 7,575 ft.
7. 693 ft. ; 3·23 ft./sec.^{2}
8. 5 ft./sec. at the top ; 20 ft./sec. at the bottom ; 1162·5 ft.
9. 1,760 ft.
10. 420 yds. ; 60·9 m.p.h.
11. 26 cm./sec. ; $2\frac{1}{2}$ sec. ; 112 cm.
12. 17·15 ft.
15. (*a*) $v=23$, $s=65$; (*b*) $f=-5\frac{5}{8}$; $t=1\frac{3}{5}$.
16. 16 min. $6\frac{1}{2}$ sec.
17. 60 m.p.h. ; A, 8 min. 48 sec. past 12 ; B, 17 min. 36 sec. past 12.
18. $7\frac{1}{3}$ ft./sec.^{2} ; 91·6 ft./sec.
19. $75\frac{3}{4}$ sec.
20. 8 ft./sec. ; 6·4 sec. ; 153·6 ft.
21. 74·8 ft./sec.
22. 2·87 ft./sec.^{2}
23. 82·5 ft. ; 5·62 sec.
24. $8\frac{1}{2}$ sec.
25. 90·2 sec.
26. $91\frac{2}{3}$ ft.
27. $29\frac{1}{3}$ sec. ; $645\frac{1}{3}$ ft.

EXAMPLES XXX

1. 36 ft.
2. 144 ft. ; 96 ft./sec.
3. 1 sec. ; 28 ft.
4. 320 ft./sec. ; 1,600 ft.
5. 16, 48, 80, 112, 144 ft. ; $g(n-\frac{1}{2})$ ft.
6. $2\frac{1}{2}$ sec.
7. 3 sec. ; 64 ft./sec.
8. 16 ft., 48 ft., 8 ft. ; 112 ft./sec.
9. $2\frac{1}{2}$ sec. ; 100 ft. ; 125 ft.
10. 340 ft./sec. ; 1,800 ft.
12. 2·47 sec.
13. 68·4 ft./sec. ; 59·04 ft.
14. 2 sec. ; $2\frac{3}{35}$ sec.
15. $1\frac{17}{88}$ sec.
16. $1\frac{1}{7}$ sec. after the first stone is released ; after they have fallen 39·5 ft.
17. 256 ft. ; 128 ft./sec. ; 96 ft./sec.
18. 3·75 sec. ; 56·25 ft.
19. 0·0104 sec.
21. $u=10$ ft./sec. ; $g=32$.
22. 1,120 ft./sec.

EXAMPLES XXXI

1. 32 ft./sec. ; 32 ft./sec. down ; 128 ft. in each.
2. 76·8 ft./sec. ; 92·16 ft.
3. 36 ft./sec. up ; $156\frac{1}{4}$ ft.
4. $2\frac{1}{2}$ sec. ; $3\frac{1}{8}$ sec.
5. 16 ft./sec. down ; 16 ft./sec. upwards.
6. 144 ft.
7. 56 ft./sec. ; 4·5 sec.
8. 6 sec.
9. 80 ft.
10. (*a*) $44\frac{4}{9}$ sec. longer ; (*b*) $1{,}333\frac{1}{3}$ ft. higher.
11. 2452·5 cm./sec. ; $3{,}065\frac{5}{8}$ cm.
12. $6\frac{2}{21}$; $109\frac{5}{7}$ ft.
13. 10 ft./sec. ; $1{,}501\frac{9}{16}$ ft.
14. 96 ft./sec.
15. $6\frac{1}{4}$ sec. ; 112 ft./sec.

EXAMPLES XXXII

1. 11·6 lb. wt., 17° 34′ ; 232 lb. wt., 27° 14′ ; 13·3 tons wt., 44° 58′ ; 4·87 cwt., 35° 21′ ; 1·27 tons wt., 43° 12′.
2. (*a*) 14 lb. wt. ; (*b*) 12·95 lb. wt. ; (*c*) 10 lb. wt. ; (*d*) 5·67 lb. wt. ; (*e*) 2 lb. wt.
3. 4·47 lb. wt. ; 26° 34′.
4. 7·1 lb. wt. ; 68° 12′.
5. 347·2 lb. wt.
6. 57·1 lb. wt.
7. (*a*) 180°, 2 lb. wt. ; (*b*) 0°, 12 lb. wt. ; (*c*) 135° 35′, 4·9 lb. wt.

EXAMPLES XXXII—*continued.*

8. 173·2 lb. wt. ; resultant bisects angle between parts of rope.
9. 13 lb. wt. at 67° 14′ to 5 lb. wt. force.
10. 104° 29′.
12. 27·5 lb. wt. ; passes through point $1\frac{1}{2}$ ft. from B and $2\frac{1}{2}$ ft. from A ; inclined at 85° to BA.

EXAMPLES XXXIII

1. (*a*) Each component is 5·77 lb. wt. ; (*b*) 8·66 lb. wt., 5 lb. wt. (*c*) 11·1 lb. wt., 8·4 lb. wt.
2. 22·9 lb. ; 15·1 lb.
3. 1·55 cwt. at C ; 2·20 cwt. at B.
4. 2,400 lb. wt. in BC ; 2,078 lb. wt. in AB.
5. 3,864 lb. wt. in piston rod ; 1,035 lb. wt. on guide bar.
6. 3 cwt.
7. 0·78 tons ; 120°.
8. 854·4 lb. wt. at C ; 943·4 lb. wt. at A.
9. 20 lb. wt. ; 11·57 lb. wt.

EXAMPLES XXXIV

1. 8·79 lb. wt. in OA ; 6·53 lb. wt. in OB.
2. 57° 54′ ; 46° 34′.
3. 3·66 cwt, tension in BC ; 4·48 cwt. thrust in AB.
4. 67·1 kilos ; 26° 34′.
5. 50 lb. wt. in AB ; 86·6 lb. wt. in BC.
6. (*a*) 13·9 lb. wt. at 25° 52′ with 9 lb. wt. ; (*b*) 8·19 lb. wt. at 47° 47′ with 9 lb. wt.
7. Reaction = 4·33 lb. wt. ; tension = 2·5 lb. wt.
8. 21° 48′ ; reaction = 2·69 kgm. wt.
9. 38·5 lb. wt. tension in AB ; 28·2 lb. wt. tension in BC ; 35·5 lb. wt. tension in CD ; 39° 45′.
10. 25·2 tons wt. in BA ; 28·6 tons wt. in BC.
11. 313·7 tons wt.
12. At right angles to the first ; 14 and $14\sqrt{3}$ lb. wt.
13. 7·94 and 5·05 lb. wt.
14. 8·22 cm. ; 1·12 kgm. wt. in lower and 3·04 kgm, wt. in upper string.
15. (*a*) M.A. = 2, reaction = $\sqrt{\frac{3}{2}}$W.

    (*b*) M.A. = $\sqrt{3}$, reaction = $\frac{2}{\sqrt{3}}$W.
16. 9·49 lb. wt. ; 6·32 lb. wt.
17. AB, $10\sqrt{2}$ lb. wt. ; BC, $10\sqrt{2}$ lb. wt. ; CD, 20 lb. wt. ; M = 7·32 lb.
18. 259 ; 366 gm. wt. ; 69° 54′.
20. 7·32 lb. or $10(\sqrt{3}-1)$.
21. 0·41 and 1·21 tons wt.
22. CB, 5·77 cwt. ; AB, 2·89 cwt. ; AC, 5·77 cwt. ; CD, 5·77 cwt.
23. Least force is 12·5 gm. wt. ; tension, $12{\cdot}5\sqrt{3} = 21{\cdot}65$ gm. wt.
24. 10 gm. wt. ; W = $\frac{20\sqrt{3}}{3}$ or 11·55 gm.

EXAMPLES XXXV

1. 10·56 lb. wt. ; 39° 20′.
2. 337·6 lb. wt. ; 8° 39′ to vertical on side of OC.
3. 4·25 kgm. ; perpendicular to 3 kgm. force ; 54° with 5 kgm. force.
4. 112·8 lb. wt. ; 20° W. of S.
5. $333\frac{1}{3}$ lb. wt. along BA ; $166\frac{2}{3}$ lb. wt. along BC.
6. $583\frac{1}{3}$ lb. wt. along BA ; $83\frac{1}{3}$ lb. wt. along CB.

EXAMPLES XXXV—*continued.*

7. 4·24 tons wt. in CD ; 0·88 tons wt. in AD.
8. 0·65 tons wt. in OD ; 2·25 tons wt. in OC.
9. BA, 58·5 ; BC, 42 ; BD, 13·5 lb. wt.
10. P=4·85 lb. wt. ; Q=4·43 lb. wt.
11. 3·21 lb. wt. at 158° 12′ to A ; equilibrant is equal but opposite to this.

EXAMPLES XXXVI

1. (*a*) 8·66, 5 lb. wt. ; (*b*) 0·866X, 0·5X ; (*c*) 10·61, 10·61 lb. wt. ; (*d*) $\frac{X}{\sqrt{2}}$, $\frac{X}{\sqrt{2}}$; (*e*) 1·60 ; 1·20 tons wt.
2. (*a*) 9·66 ; −0·259 lb. wt.
   (*b*) −11·49, 9·64 lb. wt.
   (*c*) −1·60, −1·20 tons wt.
   (*d*) −2·5, 4·33 kgm. wt.
   (*e*) −9·40, 3·42 gm. wt.
   (*f*) −2, 3·46 cwt.
3. 3·064, 2·570 lb. wt.
4. θ=48° 35′.
5. 53·6 lb. wt. horizontal ; 107·3 lb. wt. vertical.
6. 6·44 lb. wt. per ton.
7. Reaction, 47·0 lb. wt. ; tension, 17·1 lb. wt.
8. 9·47 lb. wt. ; 8·58 lb. wt.
10. 58·0 lb. wt.
11. F=0·728 ; T=2·128 lb. wt.
12. 7·45 lb. wt.
13. 62½ lb. wt.
14. 23·5 lb. wt. ; W=66·46 lb.
15. $4\sqrt{6} : 9$.
16. 14·14, 10, 11·53 lb. wt. tension in AB, BC, CD respectively ; W=5·76 lb.
17. 7·32 and 5·18 lb. wt.

EXAMPLES XXXVII

1. 8·7 lb. wt ; 60° 53′.
2. 9·07 lb. wt. ; 53° 48′ N. of E.
3. 6 units in direction of fifth string.
4. 13·1 lb. wt. ; 127° 10′.
5. 4·24 lb. wt. ; perpendicular to 4 lb. wt. force, 54° with 6 lb. wt. force.
6. 315·6 lb. wt. ; between B and C ; 32° 32′ with 196 lb. wt. force.
7. 8 lb. wt. along line of 6 lb. wt. force.
8. 1 ton wt. along EO ; 3 tons wt. along OF.

EXAMPLES XXXVIII

1. T=133⅓ lb. wt. ; R=240·4 lb. wt. ; 33° 41′ with vertical.
2. $\frac{3\sqrt{5}}{2}$ tons wt. ; $\frac{\sqrt{37}}{2}$ tons wt. ; 9° 28′ with horizontal.
3. P=3 cwt. ; $3\sqrt{3}$=5·20 cwt.
4. 46 lb. wt. ; 19·2 lb. wt. ; cos *b* =$\frac{11}{12}$.
5. 25·8 st. ; 23·2 st. at 1° to horizontal ; position of load on edge of tailboard.
6. $10\sqrt{5}$=22·4 lb. wt. at $\tan^{-1}\frac{1}{2}$ to vertical ; 20 lb.
7. 5 lb. wt.
8. 8⅓ lb. wt. ; increase to 25 lb. wt.
9. (i) $\frac{\sqrt{89}}{5}$ cwt. at upper, 1·6 cwt. at lower ; (ii) above reversed.
10. $20\sqrt{3}$=34·64 lb. wt. ; $10\sqrt{7}$ =26·46 lb. wt. ; 40° 54′ with vertical.
11. 1,131 gm. wt. ; 894 gm. wt.
12. 2·6 cwt. at 32° to vertical ; 1·6 cwt.
13. 1·6 cwt. ; 2·6 cwt. at 32° to vertical.

EXAMPLES XXXVIII—*continued.*

14. $37\frac{11}{16}$ lb. wt.; 186 lb. wt. at 11° 45′ to vertical.
15. $Q=\frac{5}{4}W$ at 53° 8′ to horizontal; $P=\frac{3}{4}W$; $S=2W$; $R=\frac{3}{4}W$.
16. 150 lb. wt.; $50\sqrt{19}=218$ lb. wt.; 66° 35′ with AB.
17. In CD, 28 lb. wt.; at A, 18 lb. wt.
18. 12·32 in.
19. $3\frac{1}{2}$ lb. wt.; 17·33 lb. wt.; 83° to horizontal.
20. $8\sqrt{3}=13{\cdot}86$ in.
22. 160 lb. wt.; 40 lb. wt.

EXAMPLES XXXIX

1. (*a*) 6 lb. wt., 2 ft.; (*b*) 70 lb. wt., 3 in.; (*c*) 4 kgm. wt.; 200 cm.; (*d*) −80 gm. wt., 17·5 cm.
2. 9 lb. wt., 2 in. from A; 1 lb. wt., 8 in. from A; 9·08 lb. wt., 6° 20′.
3. $2\sqrt{10}P$; $\frac{a}{3}$ from B.

EXAMPLES XL

1. 6 lb. wt.; $4\frac{1}{8}$ in. from A.
2. 1 gm. wt.; 4 cm. from 5 gm. wt. force.
3. 25 lb. wt.; 8·44 in.
4. 13 gm. wt.; 1 cm. from 14 gm. wt. force.
5. 152·7 lb. wt.
6. At A, 18 lb. wt.; at B, 42 lb. wt.
7. $1\frac{7}{8}$ in.
8. A couple of moment $\frac{5\sqrt{11}}{2}$.
9. $5\frac{3}{16}$ ft.; $7\frac{7}{16}$ ft.; $6\frac{3}{16}$ ft. from A.
10. 300 gm. wt.; 400 gm. wt.; 4 in. from end A.
11. 700 gm.
12. 18 lb.
13. 18 dynes; $8\frac{1}{3}$ cm. from 4-dyne force; 5 cm. from 50 gm. and 200 gm., 7 cm. from 50 gm. and 100 gm.
15. $32\frac{1}{2}$ lb. wt. at A; $87\frac{1}{2}$ lb. wt. at bearing near B.
16. Couple of moment 24,000 cm. gm. units.
18. 1 lb.; 1 ft. 6 in. from A.

EXAMPLES XLI

1. 7 lb. wt.; 0·7.
2. (*a*) 8 lb. wt.; (*b*) 12 lb. wt.
3. 0·25.
4. $83\frac{1}{3}$ lb. wt.
5. $62\frac{1}{2}$ lb.
6. 80 lb. wt.
7. 50 lb. wt.; 50 lb. wt. making 143° 4′ with the 40 lb. wt. force.
8. 0·0625.

EXAMPLES XLII

1. 0·4663; 4·226 lb. wt.
2. 34·72 lb. wt.; 44·44 lb. wt.
3. 15 lb. wt.
4. 21·8 lb. wt.; $\tan\lambda=0{\cdot}828$ or $\lambda=39°\ 37'$.
5. 6·21 lb. wt.

EXAMPLES XLIII

2. 23° 25′.
3. 19·1 lb. wt.; 29·1 lb. wt.
4. (i) 8 lb. wt.; (ii) 9·43 lb. wt.; (iii) 7·86 lb. wt.
5. 36·64 lb.

EXAMPLES XLIV

1. 10 lb. wt.
2. 2,500 lb. wt.
3. 1·7 cwt.
4. 149 lb. wt.
5. 5,184 lb. wt.
6. 50 lb. wt.
7. 11 lb. wt. ; 10 lb. wt. ; 1 lb. in.
8. 14 st. ; vertically upwards.
9. 4 lb. wt. ; $\frac{\sqrt{3}}{3}$ or 0·5774.
10. 0·25.
11. W=0·639 lb.
12. 2·5 lb. ft.

EXAMPLES XLV

1. 0·720.
2. 1,120.
3. 8.
4. $\frac{33}{110}$.
5. 40 ft./sec.
6. 75 lb. wt.
7. 716·8 H.P.
8. 125 lb.
9. $3\frac{13}{33}$ H.P. ; $8\frac{8}{55}$ H.P.
10. (*a*) 24·4 sec. ; (*b*) $16\frac{2}{3}$ ft./sec.
11. 110 ft. lb./sec.
12. 13·9 H.P.
13. $\frac{165}{256}$=0·64.
14. 7,920 gals. ; 79,250 lb. wt.
15. 296·5 H.P.
16. $37\frac{1}{2}$ lb. wt.
17. $\frac{1}{33}$.
18. 754·3 ft./min. ; $1\frac{1}{7}$ H.P.
19. 485 lb. wt.
20. $\frac{88}{35}$ H.P.
21. (*a*) $\frac{33}{55}$ H.P. ; (*b*) 61·6 lb. wt. ; (*c*) 770 lb. in.
22. 180 lb. wt.
23. 6·07 H.P.
24. 500,000 lb. wt. ; 20,480 H.P.
25. (*a*) 0·2 H.P. ; (*b*) 1,650 ft. lb.

EXAMPLES XLVI

1. Reading corresponding to 10 lb. is wrong ; 2·15 ft. lb.
2. 4,950 ft. lb.
3. 0·822 in. lb.
4. 147 lb. wt. ; 121 lb. wt. ; 19,300 ft. lb.
5. 57·8 lb. wt./sq. in. ; 142,400 ft. lb. per stroke ; 518 I.H.P.
6. 48·9 B.H.P. ; 61·1 I.H.P.
8. 2·18 ft.
9. 1·41 ft.
10. 157·1 lb. per sq. in.
11. $205\frac{1}{3}$ H.P.
12. 0·533.
13. 88·6 lb. per sq. in. ; 3,225 ft. lb.
14. 11·7 H.P. ; 8·8 B.H.P.

EXAMPLES XLVII

1. 5 m.p.h. ; 36° 52′ N. of E.
2. 2·83 m.p.h.
3. 46·4 m.p.h. ; 62° 46′ N. of E.
4. 73 m.p.h. ; 28° 40′ E. of S.
5. $\sqrt{3}$ m.p.h. ; 60° with the up-stream direction.
6. 2·77 knots ; 41° 6′ S. of W.
7. 8·66 ft./sec.
8. 27·4 ft./sec. vertical ; 75·2 ft./sec. horizontal.
9. 0·985 min.
10. 68·4 m.p.h. ; 47° 2′ N. of E.
11. 25 m.p.h. ; 0·8 hr. ; 34·6 miles.
12. 69 min.
13. 11° 32′ W. of N. ; 98 m.p.h.
14. 8·66 m.p.h.
15. 8·9 min.

EXAMPLES XLVIII

1. (*a*) 50 ft. horizontal ; 16 ft. vertical ; (*b*) 100 ft. ; 64 ft.
2. 1,000 ft.
3. 144 ft. ; 110 yds.
4. $2\frac{3}{4}$ sec. ; 275 ft.
5. 2 sec. ; 110 ft./sec.
6. $13\frac{1}{3}$ ft./sec.
7. 4,100 ft.
8. 110·9 ft./sec.
9. 12 ft./sec. ; 22 ft./sec. ; $\tan^{-1}\frac{6}{11}$ or 28° 37′ with horizontal.
10. $\sqrt{5}=2{\cdot}236$ sec. ; 137·5 ft.
11. (i) 17·32 sec. ; (ii) 25,540 ft. ; (iii) 390 m.p.h. at 75° 11′ with horizontal.
12. 449·4 ft. high ; 53 ft.

EXAMPLES XLIX

1. 16 ft. ; 1 sec. ; 80 ft.
2. 270·6 ft.
3. 5,470 ft. ; 387·1 ft./sec, vertically, 707·1 ft./sec. horizontally.
4. $1\frac{1}{2}$ sec. ; 9 ft.
5. $64\sqrt{2}$ ft./sec. ; at 45° to horizontal.
6. $V^2=2gh$; $48\sqrt{2}$ ft./sec. ; 72 ft.
7. 1,320 ft. ; 346 ft./sec.
8. 14·180 yd. ; 651 ft. for 15′ more, 639 ft. for 15′ less.
9. 150 ft.
10. $8\sqrt{3}$ ft.
11. $60\sqrt{10}$ ft./sec. ; $3\frac{3}{4}$ sec. ; $56\frac{1}{4}$ ft.
12. 47·5 ft.
13. 100 ft. ; 80·9 ft./sec. at $\tan^{-1}\frac{20}{3}$ with horizontal.
14. $\frac{5}{8}$ sec. ; 60 ft. ; to $9\frac{13}{16}$ ft.
16. 41·8 ft./sec. ; $\tan^{-1}\frac{10}{3}$ with horizontal.
17. 28° 39′ ; 72° 40′.
18. 2,400 ft./sec.
19. 28·9 ft.
20. 98·0 ft./sec. ; 75 ft. ; a worse result.

EXAMPLES L

1. 85 m.p.h. ; $\tan^{-1}\frac{15}{8}$ or 61° 56′ N. of E.
2. 6·3 m.p.h.
3. 17 min.
4. 10 m.p.h. ; at 45° to direction of A's motion ; 12·3 m.p.h. at 35° 6′ with downstream direction.
5. $60\sqrt{3}$ m.p.h. ; 120 m.p.h.
6. From 10° 54′ W. of S.
7. 120 m.p.h. ; 104 m.p.h. ; 108·2 m.p.h.
8. 5·92 m.p.h. ; 28° 42′ N. of W. ; 12·8 m.p.h. ; 51° 20′ S. of E.
9. $10\sqrt{7}$ m.p.h. ; $\tan^{-1}\frac{\sqrt{3}}{2}$ W. of N.
10. 50 m.p.h. ; $\tan^{-1}\frac{4}{3}$ E. of N. ; 40 m.p.h. due W. ; 85·4 m.p.h. at $\tan^{-1}\frac{8}{3}$ W. of S.
11. 50 m.p.h. at $\cos^{-1}\frac{4}{5}$ W. of N. ; 31·62 m.p.h. at $\tan^{-1}\frac{1}{3}$ N. of E.
12. S. at 6 m.p.h. ; 31·73 ft./sec. at 33° 42′ with vertical.
13. 31·43 ft./sec. at $\tan^{-1}\frac{4}{7}$; 27·46 ft./sec. at $\tan^{-1}\frac{4}{1}$.
14. (i) $\frac{5}{11}$; (ii) $\frac{1}{1}$; (iii) $\frac{5{\cdot}6}{9{\cdot}85}$.
15. Equal to original velocity of ball at 60° to AB.
16. 3·23 ft./sec. at 38° 16′ to direction of initial velocity reversed.

EXAMPLES LI

1. 15 m.p.h. ; 36° 52′ E. of N. ; 6 miles.
2. 19·6 min. past noon ; 2·35 sea miles.
3. 15·3 m.p.h. at 22° S. of E. ; 5·86 miles S. of cross-roads.
4. 7·81 sea miles.
5. After 0·447 hrs. ; 41° 48′ E. of N.
6. 1·11 miles ; 80° 36′ W. of N.
7. 32 m.p.h. at 51° 20′ to direction of second train ; 458 yds.
9. $15\sqrt{5}$ m.p.h. ; from $\tan^{-1}\frac{1}{2}$ W. of N.
10. $15\sqrt{2}$ m.p.h.
11. $2\frac{15}{17}$ hrs.
12. 1·13 hrs. ; 0·38 hr.

EXAMPLES LII

1. 8 ft./sec.^{2}
2. 235·2 cm./sec.^{2}
3. (*a*) 128 pdls. ; (*b*) 9,800 dynes ; (*c*) 100 lb. wt. ; (*d*) 10 kgm. wt.
4. 50 lb. wt.
5. 3 lb. wt.
6. 2·2 cwt.
7. $9\frac{7}{9}$ ft./sec.^{2} ; 410·7 lb. wt.
8. 80 ft./sec. ; 200 ft.
9. 1,440,000 ft./sec.^{2} ; 2812·5 lb. wt.
10. 0·1 ft./sec.^{2}
11. g sin 30° or 16 ft./sec.^{2} ; 10 lb. wt. the resolved part of the weight parallel to the plane.
12. $25\frac{2}{3}$ lb. wt.
13. $70\frac{5}{13}$ lb. wt.
14. 1,050 lb. wt.
15. $8\frac{1}{3}$ pdls.
16. (i) 85·6 lb. wt. ; (ii) 7·13 lb. wt.
17. 88 pdls. ; $2\frac{3}{4}$ lb. wt.
18. 0·336 tons wt.
19. (*a*) $\frac{100}{g}\times$ weight ; (*b*) $\frac{1000}{2g}\times$ weight.
20. 2,200 ft.
21. $1\frac{3}{8}$ cwt. ; (*a*) 726 ft. cwt. ; (*b*) $3\frac{7}{8}$ cwt ; 2,046 ft. cwt.
22. 15·4 pdls.
23. (*a*) $373\frac{1}{3}$ lb. wt. ; (*b*) 420 lb. wt.
24. $\frac{32}{121}$ ft./sec.^{2}
25. 10 lb. ; 10 lb. 15 oz. ; 9 lb. 1 oz.
26. $6\frac{1}{8}$, 7, and $7\frac{7}{8}$ lb. wt.
27. At 26′ to the vertical.

EXAMPLES LIII

1. $6{\cdot}15\times10^7$ dynes, 62·75 kgm. wt. ; $4{\cdot}9\times10^7$ dynes, 50 kgm. wt.
2. 128 pdls. or 4 lb. wt. ; 5 lb.
3. 11·58 sec.
4. 900 ft. from position of rest ; $112\frac{1}{2}$ ft. ; $11\frac{1}{4}$ sec.
5. 122·5 cm./sec. ; 143·5 gm. wt. ; 135·5 gm. wt.
6. 600 gm. ; 20 gm.
7. $46\frac{2}{3}$ cm./sec.^{2}
8. (1) $\frac{m(a+1)}{a-1}$ gm. ; $\frac{mg(a+1)}{a}$ dynes ; (2) $\frac{m(a-1)}{a+1}$ gm. ; $\frac{mg(a-1)}{a}$ dynes
9. 931 gm.
10. 28 pdls. ; 8 ft.
11. 10 gm.
12. $\frac{15}{17}g$ ft./sec.^{2} ; $\frac{1}{2}\sqrt{\frac{15}{17}}$ sec.
13. 882 cm./sec.^{2}
14. $\frac{99}{119}$ ft./sec.^{2} ; 15·5 ft.

EXAMPLES LIV

1. (*a*) 250 ft. pdls. ; (*b*) 15,625 ft. pdls. ; (*c*) 1,024 cm. dynes ; (*d*) 484 ft. tonals.
2. (*a*) 50 pdls. ; (*b*) $4\frac{1}{2}$ ft.
3. 937·5 lb. wt.
4. (*a*) 6,400 ft. pdls. ; (*b*) 200 ft. lb.
5. 20 pdls.
6. 0·15 tons wt.
7. 937·5 lb. wt.
8. 187,500 lb. wt.
9. 22,500 ft. pdls. ; $\frac{32}{15}$.
10. 122·5 cm.

EXAMPLES LIV—*continued.*

11. 2,009·2 ft. lb. ; 803·7 ft. lb.
13. $\frac{\sqrt{322}}{20}v$ ft./sec. ; 11 planks.
14. 125 tons wt.
15. 17,580 ft. pdls. ; 2·29 ft.
16. 57·037 lb. wt.
17. 100 ft. lb. ; 100 ft. lb.
18. 815 lb. wt.
20. 160, 480, 800 ft. lb.
21. 1,512½ lb. wt.

EXAMPLES LV

1. 810 lb. wt.
2. 950 lb. wt.
3. 0·17 H.P.
4. 35·9 ft.
5. 132·3 ft.
6. 5,800 lb. wt. ; 928 H.P.
7. 11·2 H.P.
8. 268·8 H.P. ; 7528·9 yds.
9. 8·4 H.P.
10. 13⅓ H.P.
11. 160 H.P.
12. (*a*) 198,000 ft. lb. ; 10 H.P. ; (*b*) 321,200 ft. lb. ; 16⅔ H.P.
13. 9·49 H.P.
14. 3,536 ft. tons ; 30 m.p.h.
15. 18,000 ft. lb. loss ; 5·740 ft. lb. gain.
16. 152,000 ft. pdls.
17. 60 per cent. of work done by gravity.
18. 815 H.P.

EXAMPLES LVI

1. 6,000,000 ft. lb. ; 3,600,000 ft. lb.
2. $7\frac{2}{11}$ H.P. ; $9\frac{3}{11}$ H.P.
3. 7,920 gals.
4. 25×10^8 ft. lb. ; $1{\cdot}22\times10^8$ ft. lb. ; 6·62 hrs.
6. 255·9 gals.
6. $8\sqrt{330}=145{\cdot}3$ ft./sec.

EXAMPLES LVII

1. 4 ft./sec.
2. $6\frac{3}{4}$ in.
4. 1·57 sec.
5. (*a*) 0·8 sec. ; (*b*) $l=256$ cm.
6. 982 cm./sec.^{2}

EXAMPLES LVIII

1. 3·86° F.
2. 2·33° C.
3. (i) 128·4 B.Th.U. ; (ii) 37·4 B.Th.U.
4. 20 per cent.
5. 11·79 C.H.U./sec.
6. 16·8 B.H.P.
7. 9 per cent. ; 22·9 B.H.P.
8. 20·2 H.P.

EXAMPLES LIX

1. 465 ft. lb. ; 15 ft. lb.
2. $63\frac{1}{2}$ ft. pdls. ; $3\sqrt{2}$ ft./sec.
3. $196\frac{3}{5}$ ft. lb. ; $3\frac{19}{32}$ ft. lb. ; $39\frac{7}{8}$ lb. wt.
4. 110,000 ft. lb.

EXAMPLES LX

1. (*a*) 320 pdls. sec. ; (*b*) 10 lb. wt. sec. ; (*c*) 5 ft./sec.
2. 24 lb. wt. sec. ; 64 ft./sec. ; (*a*) 32 ft./sec. ; (*b*) 128 ft./sec.
3. 110 pdls.
4. 3⅛ lb. wt.
5. 72 ft./sec. ; 22·5 lb. wt.
6. 3½ lb. wt. ; 1⅓ sec. longer.
7. 1·93 tons wt.
8. 400 pdls. ; 22⅔ ft./sec.
9. 1,260 ft. lb.
10. 48 lb. wt. ; $\frac{1}{16}$ sec.

EXAMPLES LXI

1. 1,540 lb. wt.
2. 260 lb. wt.
3. 119·6 lb. wt.
4. 50 ft./sec. ; 200 lb. ft. sec. ; 200 ft. pdls. ; 129 lb. wt.
5. 69⅓ lb. wt./sq. in.
6. 175·7 pdls.

EXAMPLES LXII

1. 750 lb. ft. sec. units ; $23\frac{5}{16}$ lb. wt.
2. 65·1 lb. wt.
3. $\frac{25}{256}$ pdls.
4. 210 lb. wt.
5. 1,069 lb. ft. sec. units. 1,069 pdls. at 60° to original direction of pipe.
6. 18·9 lb. wt./sq. ft.
7. 35 lb. wt.

EXAMPLES LXIII

1. $\frac{1600}{321}$ ft./sec.
2. $1\frac{5}{7}$ ft./sec. ; $\frac{4}{7}$ ft. tons.
3. $\frac{50}{27}$ ft./sec. ; 1111·1 ft. pdls.
4. 100 lb. ft. sec. units ; 500 ft. pdls. ; $3\frac{1}{8}$ sec. ; 1,500 ft. pdls. ; 5 ft./sec.
5. 1,721 ft. lb.
6. (*a*) 16 ft./sec. ; (*b*) −80 ft. ton sec. units ; (*c*) 640 ft. tonals, 2 ft.
7. $3\frac{1}{3}$ tons ; $57\frac{1}{6}$ ft.
8. $7\frac{1}{2}$ tons ; 60 per cent.
9. 52,250 ft. lb. ; 62·7 ft. lb.
10. 6 ft./sec. ; $3\frac{1}{8}$ ft. tons.
11. 19·5 ft./sec. ; 270 ft. pdls. ; 10·5 ft./sec.
12. $320\sqrt{2}$ ft./sec. ; 2,400 lb. wt.
13. $1\frac{2}{3}$ tons wt.
14. 138·56 lb. ft. sec. units.
15. $10\frac{1}{2}$ ft./sec. ; 2.205 lb. wt.
16. 4 lb. ; 120 lb. ft. sec. units each.
17. 0·816 ft./sec. ; 523 ft./sec.
18. Height 20 cm. ; horizontal distance 40 cm.
19. 104·3 ft./sec.
20. 313·2 cm./sec.
21. 1·3 cm.

EXAMPLES LXIV

1. $2\frac{2}{3}$ ft./sec.
2. $\frac{18}{17}$ ft.

EXAMPLES LXV

1. 9 ft.
2. $\frac{3}{4}$.
3. 177 cm./sec. ; 51·3 cm./sec.
4. 3 ft./sec. ; $3\frac{1}{2}$ ft./sec.
5. 6 ft./sec. ; $7\frac{1}{2}$ ft./sec.
6. Zero ; 6 ft./sec. reversed.
8. 32·7, 7·8 ft./sec., both reversed ; 44·54 ft. pdls.